MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES
7 (2) 191-194 (2019) ©MSAEN

Relations Among Bell Polynomials, Central Factorial Numbers, and Central Bell Polynomials

Feng Qi and Bai-Ni Guo*

Abstract
In the note, by virtue of the Faà di Bruno formula and two identities for the Bell polynomials of the second kind, the authors derive three relations among the Bell polynomials, central factorial numbers of the second kind, and central Bell polynomials.

Keywords: *Bell polynomial; central factorial number of the second kind; central Bell polynomial; Bell polynomial of the second kind; Faà di Bruno formula.*

AMS Subject Classification (2010): *Primary 11B83; Secondary 11B75, 33B10.*

****Corresponding author***

1. Preliminaries

The Bell numbers B_k for $k \ge 0$ can be generated [4, 7, 12] by

$$e^{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = 1 + t + t^2 + \frac{5}{6}t^3 + \frac{5}{8}t^4 + \frac{13}{30}t^5 + \frac{203}{720}t^6 + \frac{877}{5040}t^7 + \cdots$$

As a generalization of the Bell numbers B_k for $k \ge 0$, the Bell polynomials $T_k(x)$ for $k \ge 0$ can be generated [8–10, 15, 17] by

$$\begin{aligned} e^{x(e^t-1)} = \sum_{k=0}^{\infty} T_k(x)\frac{t^k}{k!} &= 1 + xt + \frac{1}{2}x(x+1)t^2 + \frac{1}{6}x\big(x^2+3x+1\big)t^3 \\ &+ \frac{1}{24}x\big(x^3+6x^2+7x+1\big)t^4 + \frac{1}{120}x\big(x^4+10x^3+25x^2+15x+1\big)t^5 + \cdots \end{aligned} \tag{1.1}$$

The polynomials $T_k(x)$ for $k \ge 0$ are also called [11, 18] the Touchard polynomials or the exponential polynomials. It is clear that $T_k(1) = B_k$.

The central factorial numbers of the second kind $T(n,k)$ for $n \ge k \ge 0$ can be generated [1, 6] by

$$\frac{1}{k!}\left(2\sinh\frac{t}{2}\right)^k = \sum_{n=k}^{\infty} T(n,k)\frac{t^n}{n!},$$

where

$$\sinh t = \frac{e^t - e^{-t}}{2} \tag{1.2}$$

is the hyperbolic sine function.

The central Bell polynomials $B_k^{(c)}(x)$ for $k \ge 0$ can be generated [5] by

$$\exp\left(2x\sinh\frac{t}{2}\right) = \sum_{k=0}^{\infty} B_k^{(c)}(x)\frac{t^k}{k!}. \tag{1.3}$$

Received : *16–05–2019,* ***Accepted :*** *04–10–2019*

In this note, by virtue of the Faà di Bruno formula and two identities for the Bell polynomials of the second kind, we will discuss relations among the Bell polynomials $T_k(x)$, central factorial numbers of the second kind $T(n,k)$, and central Bell polynomials $B_k^{(c)}(x)$.

2. Lemmas

The Bell polynomials of the second kind, denoted by $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ for $n\ge k\ge 0$, are defined [2, 3, 11, 13, 15–17] by

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{1\le i\le n-k+1\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n-k+1}i\ell_i=n\\ \sum_{i=1}^{n-k+1}\ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}.$$

For proving our main results, we need the following lemmas.

Lemma 2.1 ([2, 3, 14, 19–21])**.** *The Faà di Bruno formula can be described in terms of* $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ *by*

$$\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}f\circ h(x)=\sum_{k=0}^{n}f^{(k)}(h(x))\,\mathrm{B}_{n,k}\big(h'(x),h''(x),\dots,h^{(n-k+1)}(x)\big). \tag{2.1}$$

For $n\ge k\ge 0$*, the Bell polynomials of the second kind* $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ *satisfy the identity*

$$\mathrm{B}_{n,k}\big(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\big)=a^kb^n\,\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}) \tag{2.2}$$

is valid, where $a,b\in\mathbb{C}$.

Lemma 2.2 ([14, 19, 21])**.** *For* $n\ge k\ge 0$*, the Bell polynomials of the second kind* $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ *satisfy the closed formula*

$$\mathrm{B}_{n,k}\left(1,0,1,\dots,\frac{1-(-1)^{n-k+1}}{2}\right)=\frac{1}{2^kk!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}(k-2\ell)^n. \tag{2.3}$$

3. Main results and their proofs

Now we are in a position to state and prove our main results.

Theorem 3.1. *For* $k\ge 0$*, we have*

$$B_k^{(c)}(x)=\frac{(-1)^k}{2^k}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}T_\ell(x)T_{k-\ell}(-x), \tag{3.1}$$

$$B_k^{(c)}(x)=\sum_{\ell=0}^{k}T(k,\ell)x^\ell, \tag{3.2}$$

and

$$\sum_{\ell=0}^{k}T(k,\ell)x^\ell=\left(-\frac{1}{2}\right)^k\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}T_\ell(x)T_{k-\ell}(-x). \tag{3.3}$$

Proof. Utilizing (1.2) and (1.1) in sequence, it follows that

$$\begin{aligned}\exp\left(2x\sinh\frac{t}{2}\right)&=e^{x(e^{t/2}-1)}e^{-x(e^{-t/2}-1)}=\left[\sum_{k=0}^{\infty}\frac{T_k(x)}{2^k}\frac{t^k}{k!}\right]\left[\sum_{k=0}^{\infty}\frac{T_k(-x)}{(-2)^k}\frac{t^k}{k!}\right]\\&=\sum_{k=0}^{\infty}\left[\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}T_\ell(x)T_{k-\ell}(-x)\right]\left(-\frac{1}{2}\right)^k\frac{t^k}{k!}.\end{aligned}$$

Comparing this with (1.3) yields (3.1).

From (1.3), (2.1), (2.2), and (2.3) in sequence, it follows that

$$\begin{aligned}
B_k^{(c)}(x) &= \lim_{t\to0}\frac{\mathrm{d}^k}{\mathrm{d}\,t^k}\left[\exp\left(2x\sinh\frac{t}{2}\right)\right]\\
&= \lim_{t\to0}\sum_{\ell=0}^{k}\left(e^{2xu}\right)^{(\ell)}\mathrm{B}_{k,\ell}\left(\left(\sinh\frac{t}{2}\right)',\left(\sinh\frac{t}{2}\right)'',\dots,\left(\sinh\frac{t}{2}\right)^{(n-k+1)}\right)\\
&= \lim_{t\to0}\sum_{\ell=0}^{k}(2x)^\ell e^{2xu}\,\mathrm{B}_{k,\ell}\left(\frac{1}{2}\cosh\frac{t}{2},\frac{1}{2^2}\sinh\frac{t}{2},\dots,\frac{1}{2^{n-k+1}}\sinh^{(n-k+1)}\frac{t}{2}\right)\\
&= \sum_{\ell=0}^{k}\frac{(2x)^\ell}{2^k}\mathrm{B}_{k,\ell}\left(1,0,1,0,\dots,\frac{1-(-1)^{n-k+1}}{2}\right)\\
&= \sum_{\ell=0}^{k}\frac{(2x)^\ell}{2^k}\frac{1}{2^\ell\ell!}\sum_{m=0}^{\ell}(-1)^m\binom{\ell}{m}(\ell-2m)^k\\
&= \sum_{\ell=0}^{k}\frac{x^\ell}{\ell!}\sum_{m=0}^{\ell}(-1)^m\binom{\ell}{m}\left(\frac{\ell}{2}-m\right)^k\\
&= \sum_{\ell=0}^{k}T(k,\ell)x^\ell,
\end{aligned}$$

where $u=u(t)=\sinh\frac{t}{2}$. Consequently, we derive the relation (3.2).

Combining the relation (3.1) with the equation (3.2) leads to the equality (3.3). The proof of Theorem 3.1 is complete. ☐

References

[1] Butzer, P. L., Schmidt, M., Stark, E. L. and Vogt, L., *Central factorial numbers; their main properties and some applications*, Numer. Funct. Anal. Optim. **10** (1989), no. 5-6, 419–488; available online at `https://doi.org/10.1080/01630568908816313`.

[2] Charalambides, C. A., *Enumerative Combinatorics*, CRC Press Series on Discrete Mathematics and its Applications. Chapman & Hall/CRC, Boca Raton, FL, 2002.

[3] Comtet, L., *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Revised and Enlarged Edition, D. Reidel Publishing Co., 1974; available online at `https://doi.org/10.1007/978-94-010-2196-8`.

[4] Guo, B.N. and Qi, F., *An explicit formula for Bell numbers in terms of Stirling numbers and hypergeometric functions*, Glob. J. Math. Anal. **2** (2014), no. 4, 243–248; available online at `http://dx.doi.org/10.14419/gjma.v2i4.3310`.

[5] Kim, D. S., Jang, G.W., Dolgy, D. V. and Kim, T., *An expression for central Bell polynomials*, Adv. Stud. Contemp. Math. **29** (2019), no. 2, 257–262; available online at `http://dx.doi.org/10.17777/ascm2019.29.2.257`.

[6] Merca, M., *Connections between central factorial numbers and Bernoulli polynomials*, Period. Math. Hungar. **73** (2016), no. 2, 259–264; available online at `https://doi.org/10.1007/s10998-016-0140-5`.

[7] Qi, F., *An explicit formula for the Bell numbers in terms of the Lah and Stirling numbers*, Mediterr. J. Math. **13** (2016), no. 5, 2795–2800; available online at `https://doi.org/10.1007/s00009-015-0655-7`.

[8] Qi, F., *Integral representations for multivariate logarithmic polynomials*, J. Comput. Appl. Math. **336** (2018), 54–62; available online at `https://doi.org/10.1016/j.cam.2017.11.047`.

[9] Qi, F., *On multivariate logarithmic polynomials and their properties*, Indag. Math. (N.S.) **29** (2018), no. 5, 1179–1192; available online at `https://doi.org/10.1016/j.indag.2018.04.002`.

[10] Qi, F., *Simplifying coefficients in differential equations related to generating functions of reverse Bessel and partially degenerate Bell polynomials*, Bol. Soc. Paran. Mat. **39** (2021), no. 4, in press; available online at `http://dx.doi.org/10.5269/bspm.41758`.

[11] Qi, F., *Some inequalities and an application of exponential polynomials*, Math. Inequal. Appl. **22** (2019), in press; available online at `https://doi.org/10.20944/preprints201708.0079.v2`.

[12] Qi, F., *Some inequalities for the Bell numbers*, Proc. Indian Acad. Sci. Math. Sci. **127** (2017), no. 4, 551–564; available online at `https://doi.org/10.1007/s12044-017-0355-2`.

[13] Qi, F., Čerňanová, V. and Semenov, Y. S., *Some tridiagonal determinants related to central Delannoy numbers, the Chebyshev polynomials, and the Fibonacci polynomials*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **81** (2019), no. 1, 123–136.

[14] Qi, F. and Guo, B.N., *Explicit formulas for special values of the Bell polynomials of the second kind and for the Euler numbers and polynomials*, Mediterr. J. Math. **14** (2017), no. 3, Article 140, 14 pages; available online at `https://doi.org/10.1007/s00009-017-0939-1`.

[15] Qi, F., Lim, D. and Guo, B.N., *Explicit formulas and identities for the Bell polynomials and a sequence of polynomials applied to differential equations*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113** (2019), no. 1, 1–9; available online at `https://doi.org/10.1007/s13398-017-0427-2`.

[16] Qi, F., Lim, D. and Yao, Y.H., *Notes on two kinds of special values for the Bell polynomials of the second kind*, Miskolc Math. Notes **20** (2019), no. 1, 465–474; available online at `https://doi.org/10.18514/MMN.2019.2635`.

[17] Qi, F., Niu, D.W. and Guo, B.N., *Some identities for a sequence of unnamed polynomials connected with the Bell polynomials*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **113** (2019), no. 2, 557–567; available online at `https://doi.org/10.1007/s13398-018-0494-z`.

[18] Qi, F., Niu, D.W., Lim, D. and Guo, B.N., *Some properties and an application of multivariate exponential polynomials*, HAL archives (2018), available online at `https://hal.archives-ouvertes.fr/hal-01745173`.

[19] Qi, F., Niu, D.W., Lim, D. and Yao, Y.H., *Special values of the Bell polynomials of the second kind for some sequences and functions*, HAL archives (2018), available online at `https://hal.archives-ouvertes.fr/hal-01766566`.

[20] Qi, F. and Zheng, M.M., *Explicit expressions for a family of the Bell polynomials and applications*, Appl. Math. Comput. **258** (2015), 597–607; available online at `https://doi.org/10.1016/j.amc.2015.02.027`.

[21] Wei, C.F. and Qi, F., *Several closed expressions for the Euler numbers*, J. Inequal. Appl. 2015, **2015**:219, 8 pages; available online at `https://doi.org/10.1186/s13660-015-0738-9`.

Affiliations

FENG QI
ADDRESS: Inner Mongolia University for Nationalities, College of Mathematics, Tongliao 028043, Inner Mongolia, China; Tianjin Polytechnic University, School of Mathematical Sciences, Tianjin 300387, China.
E-MAIL: qifeng618@gmail.com
ORCID ID: https://orcid.org/0000-0001-6239-2968

BAI-NI GUO (CORRESPONDING AUTHOR)
ADDRESS: Henan Polytechnic University, School of Mathematics and Informatics, Jiaozuo 454010, Henan, China.
E-MAIL: bai.ni.guo@gmail.com
ORCID ID: https://orcid.org/0000-0001-6156-2590