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Recurrence Relations of the Hypergeometric-type functions on the quadratic-type lattices

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Abstract

The central idea of this article is to present a systematic approach to construct some recurrence relations for the solutions of the second-order linear difference equation of hypergeometric-type defined on the quadratic-type lattices. We introduce some recurrence relations for such solutions by also considering their applications to polynomials on the quadratic-type lattices.

Keywords: Hypergeometric function on q-quadratic lattices; Second-order linear difference equation of hypergeometric-type on the q-quadratic lattices; Recurrence relations; q-Racah polynomials; dual Hahn polynomials; TTRR.
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1. Introduction

   Hypergeometric functions have been studied by many researchers [18, 19, 20], with special interest previously in such functions defined on different type lattices such as uniform lattice like linear-type and non-uniform lattices, like quadratic, q-linear and q-quadratic types. In 1983, these functions were studied by Nikiforov and Uvarov who started from the second-order linear difference equation of hypergeometric-type satisfied by such functions, thereby paving the way for this theory to be developed by several other authors (see e.g. [3, 8, 12, 17, 18, 19, 20]). Discrete polynomials are in the special class of these kind of hypergeometric functions and used in many problems [9, 10, 11, 17, 18, 19, 20].

   In particular, q-polynomials on the q-quadratic lattices have been of particular interest in recent studies (see e.g. [9, 10, 11, 17, 18, 19, 20]) since they are the most general discrete orthogonal families, from which

all the other hypergeometric orthogonal polynomials can be obtained. Such polynomials are the solutions of the second-order difference equation of hypergeometric-type defined on the q-quadratic lattices.

In this work, we introduce an approach to construct recurrence relations for the hypergeometric functions on the q-quadratic lattices $x(s) = c_1 q^s + c_2 q^{-s} + c_3$ which, also cover the hypergeometric-type functions on the quadratic lattices $x(s) = c_1 s^2 + c_2 s + c_3$ as a limit case when $q \longrightarrow 1$ (see e.g. [1]). Here, we also apply q-Racah and dual Hahn polynomials on the q-quadratic and quadratic lattices.

Since, in several quantum-mechanical models, the wave functions can be expressed in terms of some hypergeometric-type functions, such recurrence relations give more information about the physical systems modelled by such functions. In fact, the recurrence relations are more useful for the evaluation of these functions than the direct method (see e.g., [7, 14, 15] and the references therein).

This paper is motivated by the work done by R. Álvarez-Nodarse et al. [4, 5, 6]. In fact, in [5], the authors considered the continuous case and obtained some recurrence relations for the Jacobi, Laguerre and Hermite polynomials in addition to the difference analogues of hypergeometric functions on the linear lattices $x(s) = s$ to apply the theory for the Hahn, Meixner, Charlier and Kravchuk polynomials. In [6], the authors studied the difference analogues of hypergeometric functions on the linear-type lattices, and later applied the theory to the q-polynomials on q-linear lattices $x(s) = c_1 q^s + c_2$ while considering the big q-Jacobi, Alternative q-Charlier polynomials as applications. For the quadratic case, there are only a few known recurrence relations (see the results by Suslov in [19, 20]). As such, the main aim of the present paper is to extend the results of [6] to the general quadratic-type lattice and develope constructive approach for the recurrence relations of such functions. Here we go further and consider the recurrence relations for the functions on the quadratic-type lattices and apply the theory to the q-Racah and dual Hahn polynomials, thus expanding the results of the papers [4, 5, 6].

Notice that since the lattice considered in this paper is not linear-type, the general results of [6] may not be applied. In particular, some of the representative examples considered in [6] cannot be obtained for the quadratic-type lattices.

The structure of the paper is as follows: In section 2, the preliminary results are introduced. In section 3 and 4, the general theorems for recurrence relations are given. Finally, the last section concludes the paper with some representative examples.

2. Preliminaries

We include some useful information (see e.g. [1, 18]) on the q-hypergeometric functions needed for the rest of the paper.

The hypergeometric functions on the non-uniform lattices satisfy the following second-order difference equation of hypergeometric-type on the non-uniform lattices

$$\sigma(s) \frac{\Delta}{\Delta x(s - \frac{1}{2})} \left[\frac{\nabla y(s)}{\nabla x(s)} \right] + \tau(s) \frac{\Delta y(s)}{\Delta x(s)} + \lambda y(s) = 0, \tag{1}$$

where

$$\sigma(s) = \widetilde{\sigma}(x(s)) - \frac{1}{2} \widetilde{\tau}(x(s)) \Delta x(s - \frac{1}{2}), \quad \tau(s) = \widetilde{\tau}(x(s)). \tag{2}$$

Here, $\Delta y(s) = y(s+1) - y(s)$ and $\nabla y(s) = y(s) - y(s-1)$ are the forward and backward difference operators, respectively, where

$$\Delta y(s) = \nabla y(s + 1), \tag{3}$$

and the coefficients $\widetilde{\sigma}(x(s))$ and $\widetilde{\tau}(x(s))$ are polynomials in $x(s)$ of degree at most 2 and 1, respectively, and λ is a constant.

In this paper, we study the quadratic-type lattices: the so-called quadratic lattice

$$x(s) = c_1 s^2 + c_2 s + c_3, \tag{4}$$

and the q-quadratic lattice,

$$x(s) = c_1(q)q^s + c_2(q)q^{-s} + c_3(q) = c_1(q)\left[q^s + q^{-s-\mu}\right] + c_3(q), \quad q^{-\mu} = \frac{c_2(q)}{c_1(q)} \tag{5}$$

with $c_1 \neq 0, c_1(q) \neq 0$.

Remark 2.1. *Quadratic-type lattices have the following properties:*

$$\frac{x(s+k) + x(s)}{2} = \alpha_k x_k(s) + \beta_k, \tag{6}$$

$$x(s+k) - x(s) = \gamma_k \Delta x_k(s - \frac{1}{2}) = \gamma_k \nabla x_k(s + \frac{1}{2}) \tag{7}$$

where

$$x_k(s) = x(s + \frac{k}{2}) \tag{8}$$

and

$$\alpha_k = \frac{q^{\frac{k}{2}} + q^{-\frac{k}{2}}}{2}, \quad \beta_k = -\frac{c_3}{2}\left(q^{\frac{k}{4}} - q^{-\frac{k}{4}}\right)^2, \quad \gamma_k = [k]_q. \tag{9}$$

Here, $[k]_q$ is the symmetric q-number defined by

$$[k]_q = \frac{q^{\frac{k}{2}} - q^{-\frac{k}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}. \tag{10}$$

Theorem 2.2. *[18, 19] The difference equation (1) has a particular solution*

$$y_\nu(z) = \frac{C_\nu}{\rho(z)} \sum_{s=a}^{b-1} \frac{\rho_\nu(s)\nabla x_{\nu+1}(s)}{\left[x_\nu(s) - x_\nu(z)\right]^{\nu+1}}, \tag{11}$$

provided that the condition

$$\left. \frac{\sigma(s)\rho_\nu(s)\nabla x_{\nu+1}(s)}{\left[x_{\nu-1}(s) - x_{\nu-1}(z+1)\right]^{\nu+1}} \right|_{s=a}^{b} = 0 \tag{12}$$

is satisfied. Here, C_ν is a constant.

Notice that $\rho(s)$ and $\rho_\nu(s)$ satisfy the following Pearson equations

$$\frac{\rho(s+1)}{\rho(s)} = \frac{\sigma(s) + \tau(s)\Delta x(s - \frac{1}{2})}{\sigma(s+1)} = \frac{\phi(s)}{\sigma(s+1)},$$

$$\frac{\rho_\nu(s+1)}{\rho_\nu(s)} = \frac{\sigma(s) + \tau_\nu(s)\Delta x_\nu(s - \frac{1}{2})}{\sigma(s+1)} = \frac{\phi_\nu(s)}{\sigma(s+1)}, \tag{13}$$

where

$$\tau_\nu(s) = \frac{\sigma(s+\nu) - \sigma(s) + \tau(s+\nu)\Delta x(s + \nu - \frac{1}{2})}{\Delta x_{\nu-1}(s)}, \tag{14}$$

and, therefore

$$\begin{aligned}
\phi_\nu(s) &= \sigma(s) + \tau_\nu(s)\Delta x_\nu(s - \frac{1}{2}) \\
&= \sigma(s+1) + \tau_{\nu-1}(s+1)\Delta x_{\nu-1}(s + \frac{1}{2}) \\
&= \dots \\
&= \sigma(s+\nu) + \tau(s+\nu)\Delta x(s + \nu - \frac{1}{2}).
\end{aligned} \tag{15}$$

Notice that

$$\phi_\nu(s) = \phi(s+\nu) = \sigma(s+\nu) + \tau(s+\nu)\Delta x(s+\nu-\frac{1}{2}), \tag{16}$$

where, $\nu \in C$ is the solution of

$$\lambda_\nu + [\nu]_q\Big\{\alpha_{\nu-1}\widetilde{\tau}' + [\nu-1]_q\frac{\widetilde{\sigma}''}{2}\Big\} = 0,$$

with $[x]_q$ and α_k defined by (10) and (9), respectively.

In the following, we will use the function $\widetilde{\sigma}_\nu(s)$ defined as

$$\widetilde{\sigma}_\nu(s) = \sigma(s) + \frac{1}{2}\tau_\nu(s)\Delta x_\nu(s-\frac{1}{2}). \tag{17}$$

By (15) and (17),

$$\phi_\nu(s) + \sigma(s) = 2\widetilde{\sigma}_\nu(s), \tag{18}$$

$$\phi_\nu(s) - \sigma(s) = \tau_\nu(s)\Delta x_\nu(s-\frac{1}{2}). \tag{19}$$

The generalized power of the lattices $x_m(s)$, given in (8), are defined as [1]

$$\Big[x_m(s) - x_m(z)\Big]^{(k)} = \prod_{i=0}^{k-1}(x_m(s) - x_m(z-i)), \quad k \in \mathbb{N}$$

$$\Big[x_m(s) - x_m(z)\Big]^{(0)} = 1.$$

The generalized power for the lattices (4) and (5) are obtained as follows:
For the quadratic lattice of the form (4)

$$\Big[x_\nu(s) - x_\nu(z)\Big]^{(\alpha)} = c_1^\alpha\frac{\Gamma(s-z+\alpha)\Gamma(s+z+\nu+\mu+1)}{\Gamma(s-z)\Gamma(s+z+\nu-\alpha+\mu+1)}, \quad \mu = \frac{c_2}{c_1}. \tag{20}$$

For the q-quadratic lattice of the form (5)

$$\Big[x_\nu(s) - x_\nu(z)\Big]^{(\alpha)} = \frac{\Gamma_q(s-z+\alpha)\Gamma_q(s+z+\nu+C+1)}{\Gamma_q(s-z)\Gamma_q(s+z+\nu-\alpha+C+1)}q^{-\alpha(s+\frac{\nu}{2})}$$

$$\times\Big[c_1(q)(1-q)^2\Big]^\alpha = c_1^\alpha(q)q^{-\alpha(s+\frac{\nu}{2})}\frac{(q^{s-z};q)_\infty(\eta q^{s+z+\nu-\alpha+1};q)_\infty}{(q^{s-z+\alpha};q)_\infty(\eta q^{s+z+\nu+1};q)_\infty}, \tag{21}$$

where $C = \frac{\log(c_2(q)/c_1(q))}{\log q}$, $\eta = \frac{c_2(q)}{c_1(q)}$ and classical q-Gamma function, Γ_q, is related to the infinite q-product [13] by formula

$$\Gamma_q(s) = (1-q)^{1-s}\frac{(q;q)_\infty}{(q^s;q)_\infty}, \quad 0 < q < 1.$$

Here, the infinite q-product [13] is defined by $(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$.

Proposition 2.3. *[1, 6, 19] Let ν be a complex number with m, k as positive integers with $m \geq k$. For the quadratic-type lattice of the form (4) and (5), we have*

$$\frac{\left[x_\nu(s) - x_\nu(z)\right]^{(m)}}{\left[x_\nu(s) - x_\nu(z)\right]^{(k)}} = \left[x_\nu(s) - x_\nu(z - k)\right]^{(m-k)}, \tag{22}$$

$$\frac{\left[x_\nu(s) - x_\nu(z)\right]^{(m+1)}}{\left[x_{\nu-1}(s) - x_{\nu-1}(z)\right]^{(m)}} = \left[x_{\nu-m}(s + m) - x_{\nu-m}(z)\right], \tag{23}$$

$$\frac{\left[x_\nu(s) - x_\nu(z)\right]^{(m+1)}}{\left[x_{\nu-1}(s + 1) - x_{\nu-1}(z)\right]^{(m)}} = \left[x_{\nu-m}(s) - x_{\nu-m}(z)\right]. \tag{24}$$

The proof is straightforward using (20) and (21), hence it is omitted. The generalization of the above expressions can be written with the following lemma.

Lemma 2.4. *Let μ_i and ν_i, $i = 1, 2, 3$ be complex numbers such that the differences $\nu_i - \nu_j$ and $\mu_i - \mu_j$ $i, j = 1, 2, 3$ are integers and*

$$\mu_0 - \mu_i \geq \nu_0 - \nu_i \tag{25}$$

where ν_0 is the ν_i, $i = 1, 2, 3$ with the largest real part, and μ_0 is the μ_i, $i = 1, 2, 3$ with the largest real part. Then, the ratio of the generalized power can be calculated with the following formulas:
 1. *If $\nu_i = \nu_0$*

$$\frac{\left[x_{\nu_0}(s) - x_{\nu_0}(z)\right]^{(\mu_0+1)}}{\left[x_{\nu_i}(s) - x_{\nu_i}(z)\right]^{(\mu_i+1)}} = \left[x_{\nu_0}(s) - x_{\nu_0}(z - \mu_i - 1)\right]^{(\mu_0 - \mu_i)}.$$

 2. *If $\nu_0 - \nu_i > 0$ and $\mu_0 - \mu_i = n$, $\nu_0 - \nu_i = n$*

$$\frac{\left[x_{\nu_0}(s) - x_{\nu_0}(z)\right]^{(\mu_0+1)}}{\left[x_{\nu_i}(s) - x_{\nu_i}(z)\right]^{(\mu_i+1)}} = \prod_{i=0}^{n-1} \left[x_{\nu_0-\mu_0}(s + \mu_0 - i) - x_{\nu_0-\mu_0}(z)\right].$$

 3. *If $\nu_0 - \nu_i > 0$ and $\mu_0 - \mu_i = n$, $\nu_0 - \nu_i = n - k$, $(\nu_0 - \nu_i < n)$*

$$\frac{\left[x_{\nu_0}(s) - x_{\nu_0}(z)\right]^{(\mu_0+1)}}{\left[x_{\nu_i}(s) - x_{\nu_i}(z)\right]^{(\mu_i+1)}} = \prod_{l=0}^{n-k-1} \left[x_{\nu_0-\mu_0}(s + \mu_0 - l) - x_{\nu_0-\mu_0}(z)\right]$$

$$\times \prod_{j=0}^{k-1} \left[x_{\nu_i}(s) - x_{\nu_i}(z - \mu_0 + n - 1 - j)\right].$$

Proof. At this stage, we only sketch the proof for the 3rd case, and the others can be done in an analogous way. One can write the ratio of the generalized power in the 3rd case by

$$\frac{\left[x_{\nu_0}(s) - x_{\nu_0}(z)\right]^{(\mu_0+1)}}{\left[x_{\nu_i}(s) - x_{\nu_i}(z)\right]^{(\mu_i+1)}} = \frac{\left[x_{\nu_0}(s) - x_{\nu_0}(z)\right]^{(\mu_0+1)}}{\left[x_{\nu_0-1}(s) - x_{\nu_0-1}(z)\right]^{(\mu_0)}}$$

$$\times \frac{\left[x_{\nu_0-1}(s) - x_{\nu_0-1}(z)\right]^{(\mu_0)}}{\left[x_{\nu_0-2}(s) - x_{\nu_0-2}(z)\right]^{(\mu_0-1)}} \cdots \frac{\left[x_{\nu_0-n+k}(s) - x_{\nu_0-n+k}(z)\right]^{(\mu_0-n+k+1)}}{\left[x_{\nu_0-n+k-1}(s) - x_{\nu_0-n+k-1}(z)\right]^{(\mu_0-n+k)}}$$

$$\times \cdots \frac{\left[x_{\nu_0-n+k}(s) - x_{\nu_0-n+k}(z)\right]^{(\mu_0-n+2)}}{\left[x_{\nu_0-n+k}(s) - x_{\nu_0-n+k}(z)\right]^{(\mu_0-n+1)}},$$

where $\mu_0 - \mu_i = n$, $\nu_0 - \nu_i = n - k$. Then, from the hypothesis and the formulas (22) and (23) of Proposition 2.3, the results in the lemma follow.

□

3. Recurrence relation on the quadratic-type lattices

Here, we obtain the general recurrence relation for the functions on the quadratic-type lattices defined by (4) and (5). To do so, we generalize the idea used for the linear-type lattices in the recent papers [4, 5, 6]. Next, we prove the following lemma as a generalization of the linear-type lattices considered in Lemma 3.2. of [6, page 4] for the quadratic-type lattices.

Lemma 3.1. *Let $x(z)$ be quadratic-type lattices of the form (4) and (5). Then, the following linear relation holds*

$$\sum_{i=1}^{3} A_i(z) \Psi_{\nu_i,\mu_i}(z) = 0, \tag{26}$$

where the coefficients $A_i(z)$ are non-zero polynomial functions in $x(z)$ and

$$\Psi_{\nu,\mu}(z) = \sum_{s=a}^{b-1} \frac{\rho_\nu(s) \nabla x_{\nu+1}(s)}{\left[x_\nu(s) - x_\nu(z)\right]^{(\mu+1)}} \tag{27}$$

provided that the differences $\nu_i - \nu_j$ and $\mu_i - \mu_j$ $i, j = 1, 2, 3$ are integers such that $\mu_0 - \mu_i \geq \nu_0 - \nu_i$, $i = 1, 2, 3$, and the following condition holds

$$\left. \frac{\sigma(s) \rho_{\nu_*}(s) x^k(s)}{\left[x_{\nu_0-1}(s) - x_{\nu_0-1}(z)\right]^{(\mu_0)}} \right|_{s=a}^{b} = 0, \quad k = 0, 1, 2, \dots. \tag{28}$$

Where, ν_* is the ν_i, $i = 1, 2, 3$ with the smallest real part, ν_0 is the ν_i, $i = 1, 2, 3$ with the largest real part and μ_0 is the μ_i, $i = 1, 2, 3$ with the largest real part.

Proof. By adding the function Ψ_{ν_i,μ_i} defined by (27) into the sum, we have

$$
\sum_{i=1}^{3} A_i(z)\Psi_{\nu_i,\mu_i}(z) = \sum_{i=1}^{3} A_i(z) \sum_{s=a}^{b-1} \frac{\rho_{\nu_i}(s)\nabla x_{\nu_i+1}(s)}{\left[x_{\nu_i}(s) - x_{\nu_i}(z)\right]^{(\mu_i+1)}}
$$

$$
= \sum_{s=a}^{b-1}\sum_{i=1}^{3} A_i(z)\frac{\rho_{\nu_i}(s)\nabla x_{\nu_i+1}(s)}{\left[x_{\nu_i}(s) - x_{\nu_i}(z)\right]^{(\mu_i+1)}} = \sum_{s=a}^{b-1} \frac{\rho_{\nu_*}(s)}{\left[x_{\nu_0}(s) - x_{\nu_0}(z)\right]^{(\mu_0+1)}}
$$

$$
\times \left(\sum_{i=1}^{3} A_i(z)\frac{\rho_{\nu_i}(s)}{\rho_{\nu_*}(s)}\nabla x_{\nu_i+1}(s)\frac{\left[x_{\nu_0}(s) - x_{\nu_0}(z)\right]^{(\mu_0+1)}}{\left[x_{\nu_i}(s) - x_{\nu_i}(z)\right]^{(\mu_i+1)}} \right)
$$

where

$$
\rho_{\nu_i}(s) = \phi(s + \nu_*)\phi(s + \nu_* + 1)...\phi(s + \nu_i - 1)\rho_{\nu_*}(s) \tag{29}
$$

by the Pearson equation (13). Thus, we have

$$
\sum_{i=1}^{3} A_i(z)\Psi_{\nu_i,\mu_i}(z) = \sum_{s=a}^{b-1} \frac{\rho_{\nu_*}(s)}{\left[x_{\nu_0}(s) - x_{\nu_0}(z)\right]^{(\mu_0+1)}}\Pi(s)
$$

where

$$
\Pi(s) = \sum_{i=1}^{3} A_i(z)\nabla x_{\nu_i+1}(s)\frac{\left[x_{\nu_0}(s) - x_{\nu_0}(z)\right]^{(\mu_0+1)}}{\left[x_{\nu_i}(s) - x_{\nu_i}(z)\right]^{(\mu_i+1)}} \tag{30}
$$

$$
\times \phi(s + \nu_*)\phi(s + \nu_* + 1)...\phi(s + \nu_i - 1)
$$

where the ratio of the generalized power can be computed using Lemma 2.4.

We need to show that there exists a polynomial $Q(s)$ in the linear space $\Lambda = Span\{z^n\}_{n\in\mathbb{Z}}$ with $z = q^s, s \in \mathbb{Z}$ such that

$$
\frac{\rho_{\nu_*}(s)}{\left[x_{\nu_0}(s) - x_{\nu_0}(z)\right]^{(\mu_0+1)}}\Pi(s) = \Delta\left[\frac{\sigma(s)\rho_{\nu_*}(s)}{\left[x_{\nu_0-1}(s) - x_{\nu_0-1}(z)\right]^{(\mu_0)}}Q(s)\right]. \tag{31}
$$

If $Q(s)$ exists, then the sum in s over $s = a$ to $b-1$, together with the boundary condition (28) lead to the relation (26). By substituting the q-quadratic lattice $x(s) = c_1(q)q^s + c_2(q)q^{-s} + c_3(q)$ in each factors of $\Pi(s)$ in (30), one can rewrite it as a polynomial in $z = q^s$ and $1/z = q^{-s}$, which is a special class of Laurent polynomials [16],

$$
\Lambda_{2n} = \{R \in {}_{-n}\Lambda_n | \text{ the coefficent of } z^n \text{ is nonzero}\}
$$

whose basis is $\{1, z^{-1}, z, z^{-2}, z^2, z^{-3}, z^3, ...\}$, where $z = q^s, s \in \mathbb{Z}$ and its L-degree is $2n$.

In order to prove the existence of the polynomial $Q(s)$, we rewrite the right hand-side of (31)

$$
\frac{\sigma(s+1)\rho_{\nu_*}(s+1)}{\left[x_{\nu_0-1}(s+1) - x_{\nu_0-1}(z)\right]^{(\mu_0)}}Q(s+1) - \frac{\sigma(s)\rho_{\nu_*}(s)}{\left[x_{\nu_0-1}(s) - x_{\nu_0-1}(z)\right]^{(\mu_0)}}Q(s) =
$$

$$
\frac{\rho_{\nu_*}(s)}{\left[x_{\nu_0}(s) - x_{\nu_0}(z)\right]^{(\mu_0+1)}}\left(\sigma(s+1)\frac{\rho_{\nu_*}(s+1)}{\rho_{\nu_*}(s)}\frac{\left[x_{\nu_0}(s) - x_{\nu_0}(z)\right]^{(\mu_0+1)}}{\left[x_{\nu_0-1}(s+1) - x_{\nu_0-1}(z)\right]^{(\mu_0)}} \right.
$$

$$
\times Q(s+1) - \sigma(s)\frac{\left[x_{\nu_0}(s) - x_{\nu_0}(z)\right]^{(\mu_0+1)}}{\left[x_{\nu_0-1}(s) - x_{\nu_0-1}(z)\right]^{(\mu_0)}}Q(s) \left. \right).
$$

By using the Pearson equation (13) and formulas (24) and (23) of Proposition 2.3, respectively, one gets

$$\frac{\rho_{\nu_*}(s)}{\left[x_{\nu_0}(s) - x_{\nu_0}(z)\right]^{(\mu_0+1)}} \Pi(s) = \frac{\rho_{\nu_*}(s)}{\left[x_{\nu_0}(s) - x_{\nu_0}(z)\right]^{(\mu_0+1)}} \left\{ \phi_{\nu_*}(s) \left[x_{\nu_0-\mu_0}(s)\right.\right.$$

$$\left.\left. -x_{\nu_0-\mu_0}(z)\right] Q(s+1) - \sigma(s) \left[x_{\nu_0-\mu_0}(s+\mu_0) - x_{\nu_0-\mu_0}(z)\right] Q(s)\right\}.$$

Therefore,

$$\begin{aligned}
\Pi(s) &= \phi_{\nu_*}(s) \left[x_{\nu_0-\mu_0}(s) - x_{\nu_0-\mu_0}(z)\right] Q(s+1) \\
&\quad - \sigma(s) \left[x_{\nu_0-\mu_0}(s+\mu_0) - x_{\nu_0-\mu_0}(z)\right] Q(s),
\end{aligned} \tag{32}$$

where $\phi_{\nu_*}(s) = \sigma(s) + \tau_{\nu_*}(s) \nabla x_{\nu_*+1}(s)$.

Recall that $\Pi(s)$ is a Laurent polynomial belongs to $_{-n}\Lambda_n$, where $z = q^s$. Notice that, $\sigma(s)$ is a polynomial of the degree at most two in $x(s)$ and also a Laurent polynomial belongs to $_{-2}\Lambda_2$, where $z = q^s$, of L-degree at most four. Moreover, $\tau_{\nu_*}(s)$ is a polynomial of degree one in $x_{\nu_*}(s)$ and also a Laurent polynomial belongs to $_{-1}\Lambda_1$, where $z = q^s$, of L-degree two. In addition, $x_k(s)$ is a Laurent polynomial belongs to $_{-1}\Lambda_1$, where $z = q^s$, of L-degree two.

Therefore, by substituting the q-quadratic lattice (5) and taking into account property (8), one can see that $Q(s)$ is also a Laurent polynomial, whose L-degree is at least six less than the L-degree of $\Pi(s)$.

Note that two Laurent polynomials are equal if their coefficients are the same just like the case with the ordinary polynomials. Then, one can use the equality of the coefficents of the Laurent polynomials in order to find $A_i(z)$. This completes the proof.

$\square$

In the limit case as $q \to 1$, one can also get the results of Lemma 3.1 for the quadratic lattice $x(s) = c_1 s^2 + c_2 s + c_3$.

3.1. Examples

In this part, we construct several recurrence relations in order to show how Lemma 3.1 works.

Example 3.2. *The functions $\Psi_{\nu,\nu}$, $\Psi_{\nu,\nu-1}$ and $\Psi_{\nu,\nu-2}$ are connected by the following relation*

$$A_1(z)\Psi_{\nu,\nu}(z) + A_2(z)\Psi_{\nu,\nu-1}(z) + A_3(z)\Psi_{\nu,\nu-2}(z) = 0$$

where the coefficients $A_1(z)$, $A_2(z)$ and $A_3(z)$ are the functions in z

$$\begin{aligned}
A_1(z) &= \tau_\nu(0)\beta_\nu - \gamma_\nu\widetilde{\sigma}_\nu(0) - \tau_\nu(0)x(z) \\
&\quad + \left[\tau_\nu'\beta_\nu + \alpha_\nu\tau_\nu(0) - \gamma_\nu\widetilde{\sigma}_\nu'(0) - \tau_\nu'x(z)\right]x_\nu(z-\nu) \\
&\quad + \left[\tau_\nu'\alpha_\nu - \gamma_\nu\frac{\widetilde{\sigma}_\nu''}{2}\right]\left[2x_\nu(z-\nu)x_\nu(z-\nu+1) - x_\nu^2(z-\nu+1)\right], \\
A_2(z) &= \tau_\nu'\beta_\nu + \alpha_\nu\tau_\nu(0) - \gamma_\nu\widetilde{\sigma}_\nu'(0) - \tau_\nu'x(z) \\
&\quad + \left[\tau_\nu'\alpha_\nu - \gamma_\nu\frac{\widetilde{\sigma}_\nu''}{2}\right]2x_\nu(z-\nu+1), \\
A_3(z) &= \tau_\nu'\alpha_\nu - \gamma_\nu\frac{\widetilde{\sigma}_\nu''}{2}
\end{aligned}$$

where α_ν, β_ν and γ_ν are defined by (9) and

$$\widetilde{\sigma}_\nu(s) = \frac{\widetilde{\sigma}_\nu''}{2}x_\nu^2(s) + \widetilde{\sigma}_\nu'(0)x_\nu(s) + \widetilde{\sigma}_\nu(0), \tag{33}$$

$$\tau_\nu(s) = \tau_\nu' x_\nu(s) + \tau_\nu(0) \tag{34}$$

are the Taylor polynomial expansion of the functions $\widetilde{\sigma}_\nu(s)$ and $\tau_\nu(s)$ defined by (17) and (14), respectively.

Proof. By Lemma 3.1, we have $\nu_1 = \nu$, $\nu_2 = \nu$, $\nu_3 = \nu$ and $\mu_1 = \nu$, $\mu_2 = \nu - 1$, $\mu_3 = \nu - 2$. By formula (30)

$$\Pi(s) = \Delta x_\nu(s - \tfrac{1}{2})\Big\{ A_3(z)x_\nu^2(s) + \Big[A_2(z) - 2A_3(z)x_\nu(z - \nu + 1)\Big]x_\nu(s)$$

$$+ \Big[A_1(z) - A_2(z)x_\nu(z - \nu) + A_3(z)x_\nu^2(z - \nu + 1)\Big]\Big\} \tag{35}$$

and by (32)

$$\Pi(s) = \phi_\nu(s)\big[x(s) - x(z)\big]Q(s + 1) - \sigma(s)\big[x(s + \nu) - x(z)\big]Q(s). \tag{36}$$

Notice that if we add q-quadratic lattice (5) into (35) and use property (8), then $\Pi(s)$ in (35) becomes a Laurent polynomial belongs to $_{-3}\Lambda_3$, where $z = q^s$. Note that the L-degree of $\Pi(s)$ is six. Since the L-degree of $Q(s)$ is at least six less than $\Pi(s)$, then the degree of $Q(s)$ becomes zero, i.e. $Q(s) = k$, where k is a constant. Then, (36) can be rewritten as the following:

$$\Pi(s) = k\Big\{ \phi_\nu(s)x(s) - \sigma(s)x(s + \nu) - \big[\phi_\nu(s) - \sigma(s)\big]x(z)\Big\}$$

$$= k\Big\{ \frac{1}{2}\phi_\nu(s)x(s) - \frac{1}{2}\sigma(s)x(s + \nu) + \frac{1}{2}\phi_\nu(s)x(s) - \frac{1}{2}\sigma(s)x(s + \nu)$$

$$+ \frac{1}{2}\phi_\nu(s)x(s + \nu) - \frac{1}{2}\phi_\nu(s)x(s + \nu) + \frac{1}{2}\sigma(s)x(s) - \frac{1}{2}\sigma(s)x(s)$$

$$- \big[\phi_\nu(s) - \sigma(s)\big]x(z)\Big\}.$$

Choosing $k = 1$, the above expression becomes

$$\Pi(s) = \phi_\nu(s)\frac{x(s + \nu) + x(s)}{2} - \sigma(s)\frac{x(s + \nu) + x(s)}{2}$$

$$- \phi_\nu(s)\frac{x(s + \nu) - x(s)}{2} - \sigma(s)\frac{x(s + \nu) - x(s)}{2}$$

$$- \big[\phi_\nu(s) - \sigma(s)\big]x(z).$$

Then, we have

$$\Pi(s) = \Big[\phi_\nu(s) - \sigma(s)\Big]\frac{x(s + \nu) + x(s)}{2}$$

$$- \Big[\phi_\nu(s) + \sigma(s)\Big]\frac{x(s + \nu) - x(s)}{2}$$

$$- \big[\phi_\nu(s) - \sigma(s)\big]x(z).$$

By using expressions (18), (19), (6) and (7), we get

$$\Pi(s) = \Delta x_\nu(s - \frac{1}{2})\Big\{ \tau_\nu(s)\Big[\alpha_\nu x_\nu(s) + \beta_\nu\Big] - \gamma_\nu\widetilde{\sigma}_\nu(s) - \tau_\nu(s)x(z)\Big\}.$$

Using $\widetilde{\sigma}_\nu(s)$ and $\tau_\nu(s)$ from (33) and (34), it follows

$$\Pi(s) = \Delta x_\nu(s - \tfrac{1}{2}) \quad \left\{ \left[\tau_\nu' \alpha_\nu - \gamma_\nu \frac{\widetilde{\sigma}_{\nu''}}{2} \right] x_\nu^2(s) + \left[\tau_\nu' \beta_\nu + \alpha_\nu \tau_\nu(0) - \gamma_\nu \widetilde{\sigma}_{\nu'}(0) \right. \right.$$
$$\left. \left. - \tau_\nu' x(z) \right] x_\nu(s) + \left[\tau_\nu(0) \beta_\nu - \gamma_\nu \widetilde{\sigma}_\nu(0) - \tau_\nu(0) x(z) \right] \right\}. \tag{37}$$

Now, by equating the polynomials $\Pi(s)$ in (35) and (37), we obtain the following system of equations

$$A_3(z) = \tau_\nu' \alpha_\nu - \gamma_\nu \frac{\widetilde{\sigma}_{\nu''}}{2}$$
$$A_2(z) - 2A_3(z) x_\nu(z - \nu + 1) = \tau_\nu' \beta_\nu + \alpha_\nu \tau_\nu(0) - \gamma_\nu \widetilde{\sigma}_{\nu'}(0) - \tau_\nu' x(z)$$
$$A_1(z) - A_2(z) x_\nu(z - \nu) + A_3(z) x_\nu^2(z - \nu + 1) \;=\; \tau_\nu(0) \beta_\nu - \gamma_\nu \widetilde{\sigma}_\nu(0)$$
$$-\; \tau_\nu(0) x(z).$$

By solving this system, one can obtain the coefficients $A_1(z)$, $A_2(z)$ and $A_3(z)$. $\qquad\square$

Example 3.3. *The following relation holds*

$$A_1(z)\Psi_{\nu,\nu-1}(z) + A_2(z)\Psi_{\nu,\nu-2}(z) + A_3(z)\Psi_{\nu+1,\nu}(z) = 0$$

where $A_1(z)$, $A_2(z)$ and $A_3(z)$ are the functions in z

$$A_1(z) \;=\; -\frac{\sigma(z - \nu + 1)}{\nabla x_{\nu+1}(z - \nu + 1)}$$
$$A_2(z) \;=\; \frac{1}{\gamma_{\nu-1}} \frac{1}{\Delta x(z)} \left[\tau_\nu(z) - \frac{\sigma(z - \nu + 1)}{\nabla x_{\nu+1}(z - \nu + 1)} \right]$$
$$A_3(z) \;=\; -\gamma_\nu.$$

Here, γ_ν is defined by (9).

Proof. By Lemma 3.1, we have $\nu_1 = \nu$, $\nu_2 = \nu$, $\nu_3 = \nu + 1$ and $\mu_1 = \nu - 1$, $\mu_2 = \nu - 2$, $\mu_3 = \nu$. By formula (30),

$$\Pi(s) \;=\; A_1(z)\nabla x_{\nu+1}(s)\left[x_1(s + \nu) - x_1(z)\right]$$
$$+ A_2(z)\nabla x_{\nu+1}(s)\left[x_1(s + \nu) - x_1(z)\right]\left[x_\nu(s) - x_\nu(z - \nu + 1)\right] \tag{38}$$
$$+ A_3(z)\nabla x_{\nu+2}(s)\phi(s + \nu)$$

and by (32),

$$\Pi(s) = \phi_\nu(s)\left[x_1(s) - x_1(z)\right]Q(s + 1) - \sigma(s)\left[x_1(s + \nu) - x_1(z)\right]Q(s). \tag{39}$$

Notice that if we use q-quadratic lattice (5) and property (8), then $\Pi(s)$ in (38) becomes a Laurent polynomial belongs to $_{-3}\Lambda_3$, where $z = q^s$. The L-degree of $\Pi(s)$ is six. Since the L-degree of $Q(s)$ is at least six less than $\Pi(s)$, then degree of $Q(s)$ becomes zero, i.e. $Q(s) = k$, where k is a constant. Let us choose $k = 1$.

We remark here that since $\Pi(s)$ in (38) and (39) are Laurent polynomials belong to $_{-3}\Lambda_3$, where $z = q^s$, one can find the coefficients $A_i(z)$ by equating them. Here, we consider giving particular values to make some terms of $\Pi(s)$ in (38) or (39) zero. Therefore, it becomes simpler to determine coefficients $A_i(z)$. Firstly, let $s = z - \nu$ in $\Pi(s)$ be defined by (38) and (39). Notice that the first two terms in (38) and the second term of (39) vanish, leading to

$$\Pi(z - \nu) = A_3(z)\phi(z)\nabla x_{\nu+2}(z - \nu) = \phi_\nu(z - \nu)\left[x_1(z - \nu) - x_1(z)\right],$$

where $\phi_\nu(z-\nu)=\phi(z)$ by (16) and $x_1(z-\nu)-x_1(z)=-\gamma_\nu\nabla x_{\nu+2}(z-\nu)$ by (7) with (8). Then, one gets

$$A_3(z)=-\gamma_\nu.$$

In order to find $A_1(z)$ let $s=z-\nu+1$ in $\Pi(s)$ defined by (38) and (39). Notice that the second term of (38) vanishes and gives

$$
\begin{aligned}
\Pi(z-\nu+1) &= A_1(z)\nabla x_{\nu+1}(z-\nu+1)\big[x_1(z+1)-x_1(z)\big] \\
&+ A_3(z)\phi(z+1)\nabla x_{\nu+2}(z-\nu+1) \\
&= \phi_\nu(z-\nu+1)\big[x_1(z-\nu+1)-x_1(z)\big] \\
&- \sigma(z-\nu+1)\big[x_1(z+1)-x_1(z)\big]
\end{aligned}
$$

where $\phi_\nu(z-\nu+1)=\phi(z+1)$ by (16) and $x_1(z+1)-x_1(z)=\Delta x_1(z)=\Delta x(z+\frac{1}{2})$ by the forward operator with (8). Moreover, $x_1(z-\nu+1)-x_1(z)=-\gamma_\nu\nabla x_{\nu+2}(z-\nu+1)$ by (7). Replacing $A_3(z)=-\gamma_\nu$, one has

$$A_1(z)=-\frac{\sigma(z-\nu+1)}{\nabla x_{\nu+1}(z-\nu+1)}.$$

Finally, to find $A_2(z)$ we set $s=z$ in $\Pi(s)$ defined by (38) and (39). Notice that first term of (39) disappears and leads to

$$
\begin{aligned}
\Pi(z) &= A_1(z)\nabla x_{\nu+1}(z)\big[x_1(z+\nu)-x_1(z)\big] \\
&+ A_2(z)\nabla x_{\nu+1}(z)\big[x_1(z+\nu)-x_1(z)\big]\big[x_\nu(z)-x_\nu(z-\nu+1)\big] \\
&+ A_3(z)\nabla x_{\nu+2}(z)\phi(z+\nu)=-\sigma(z)\big[x_1(z+\nu)-x_1(z)\big],
\end{aligned}
$$

where $\nabla x_{\nu+2}(z)=\nabla x_\nu(z+1)=\Delta x_\nu(z)$ by (3) with (8) and $x_1(z+\nu)-x_1(z)=\gamma_\nu\Delta x_\nu(z)$, $x_\nu(z)-x_\nu(z-\nu+1)=\gamma_{\nu-1}\Delta x(z)$ by (7), (3) with (8). Then, with the help of (16) together with (19) and (3), one can have

$$A_2(z)=\frac{1}{\gamma_{\nu-1}}\frac{1}{\Delta x(z)}\left[\tau_\nu(z)-\frac{\sigma(z-\nu+1)}{\nabla x_{\nu+1}(z-\nu+1)}\right],$$

which completes the proof. The other proofs can be made by using the same method. Thus, we do not include them here.

□

Example 3.4. *The functions $\Psi_{\nu,\nu}$, $\Psi_{\nu,\nu-1}$ and $\Psi_{\nu+1,\nu+1}$ have the following relation*

$$A_1(z)\Psi_{\nu,\nu}(z)+A_2(z)\Psi_{\nu,\nu-1}(z)+A_3(z)\Psi_{\nu+1,\nu+1}(z)=0$$

where coefficients $A_1(z)$, $A_2(z)$ and $A_3(z)$ are the functions in z

$$
\begin{aligned}
A_1(z) &= \frac{\phi(z)}{\Delta x(z)}\left[-\gamma_\nu+\gamma_{\nu+1}\frac{\nabla x_{\nu+2}(z-\nu)}{\nabla x_{\nu+1}(z-\nu)}\right]-\frac{\sigma(z-\nu)}{\nabla x_{\nu+1}(z-\nu)} \\
A_2(z) &= \frac{1}{\gamma_\nu}\frac{\tau_\nu(z)-A_1(z)}{\Delta x(z-\frac{1}{2})} \\
A_3(z) &= -\gamma_{\nu+1}
\end{aligned}
$$

where γ_ν is defined by (9).

Example 3.5. *The functions $\Psi_{\nu,\nu+1}$, $\Psi_{\nu-1,\nu}$ and $\Psi_{\nu-1,\nu-1}$ hold the relation that follows*

$$A_1(z)\Psi_{\nu,\nu+1}(z)+A_2(z)\Psi_{\nu-1,\nu}(z)+A_3(z)\Psi_{\nu-1,\nu-1}(z)=0$$

where coefficients $A_1(z)$, $A_2(z)$ and $A_3(z)$ are the functions in z

$$A_1(z) = -\gamma_{\nu+1}$$

$$A_2(z) = -\gamma_\nu \frac{\phi(z-1)}{\Delta x(z-\frac{1}{2})} - \frac{\sigma(z-\nu)}{\nabla x_\nu(z-\nu)}$$

$$+ \gamma_{\nu+1} \frac{\phi(z-1)\nabla x_{\nu+1}(z-\nu)}{\Delta x(z-\frac{1}{2})\nabla x_\nu(z-\nu)}$$

$$A_3(z) = \frac{1}{\gamma_\nu} \frac{\tau_{\nu-1}(z) - A_2(z)}{\nabla x(z)}.$$

Here, γ_ν is defined by (9).

Example 3.6. *The following relation holds*

$$A_1(z)\Psi_{\nu-1,\nu-1}(z) + A_2(z)\Psi_{\nu,\nu}(z) + A_3(z)\Psi_{\nu,\nu+1}(z) = 0$$

where $A_1(z)$, $A_2(z)$ *and* $A_3(z)$ *are the functions in* z

$$A_1(z) = \frac{1}{\gamma_\nu}\left\{ -\frac{\sigma(z)}{\nabla x(z)} + \frac{\phi(z+\nu-1)\nabla x_\nu(z-\nu)}{\nabla x_\nu(z)\nabla x(z)} \right.$$

$$+\gamma_\nu \frac{\phi(z+\nu-1)\nabla x_\nu(z-\nu+1)}{\nabla x_\nu(z)\nabla x_{\nu+1}(z-\nu)} + \frac{\phi(z+\nu-1)\sigma(z-\nu)\Delta x(z-\frac{1}{2})}{\phi(z-1)\nabla x_\nu(z)\nabla x_{\nu+1}(z-\nu)}$$

$$\left. -\gamma_{\nu+1}\frac{\phi(z+\nu-1)}{\nabla x_\nu(z)} \right\}$$

$$A_2(z) = \gamma_{\nu+1} - \gamma_\nu \frac{\nabla x_\nu(z-\nu+1)}{\nabla x_{\nu+1}(z-\nu)} - \frac{\sigma(z-\nu)\Delta x(z-\frac{1}{2})}{\phi(z-1)\nabla x_{\nu+1}(z-\nu)}$$

$$A_3(z) = -\gamma_{\nu+1}\nabla x_\nu(z-\nu)$$

where γ_ν *is defined by (9).*

Example 3.7. $\Psi_{\nu,\nu}$, $\Psi_{\nu,\nu-1}$ *and* $\Psi_{\nu-1,\nu-1}$ *are connected by*

$$A_1(z)\Psi_{\nu,\nu}(z) + A_2(z)\Psi_{\nu,\nu-1}(z) + A_3(z)\Psi_{\nu-1,\nu-1}(z) = 0$$

where coefficients $A_1(z)$, $A_2(z)$ *and* $A_3(z)$ *are the functions in* z

$$A_1(z) = -\gamma_\nu Q(z-\nu+1)$$

$$A_2(z) = \frac{C(z)}{D(z)}$$

$$A_3(z) = -\sigma(z) + \frac{\phi(z+\nu-1)Q(z-\nu+1)}{\nabla x_\nu(z)} - \frac{\phi(z+\nu-1)\nabla x(z+\frac{1}{2})}{\nabla x_\nu(z)}$$

$$\times \frac{C(z)}{D(z)},$$

where

$$
\begin{aligned}
C(z) &= \frac{1}{\gamma_{\nu+1}}\phi(z+\nu)\Delta x(z)\nabla x_\nu(z)Q(z+2) \\
&\quad - \sigma(z+1)\nabla x_\nu(z)\nabla x_\nu(z+1)Q(z+1) \\
&\quad + \frac{\gamma_\nu}{\gamma_{\nu+1}}\phi(z+\nu)\nabla x_\nu(z)\nabla x_{\nu+1}(z+1)Q(z-\nu+1) \\
&\quad + \sigma(z)\nabla x_\nu(z+1)\nabla x_\nu(z+1)Q(z) \\
&\quad - \phi(z+\nu-1)\nabla x_\nu(z+1)\nabla x_\nu(z+1)Q(z-\nu+1),
\end{aligned}
$$

$$
\begin{aligned}
D(z) &= \phi(z+\nu)\nabla x_\nu(z)\nabla x_{\nu+1}(z+1)\nabla x(z+1) \\
&\quad - \phi(z+\nu-1)\nabla x_\nu(z+1)\nabla x_\nu(z+1)\nabla x(z+\tfrac{1}{2})
\end{aligned}
$$

and γ_ν is defined by (9). Here, $Q(z)$ is a first-degree polynomial in $x(z)$. In particular, considering $Q(z) = \nabla x_\nu(z)$ leads to the following formulas:

$$
A_1(z) = -\gamma_\nu \nabla x_\nu(z-\nu+1)
$$
$$
A_2(z) = \frac{C(z)}{D(z)}
$$

$$
\begin{aligned}
A_3(z) &= -\sigma(z) + \frac{\phi(z+\nu-1)\nabla x_\nu(z-\nu+1)}{\nabla x_\nu(z)} - \frac{\phi(z+\nu-1)\nabla x(z+\frac{1}{2})}{\nabla x_\nu(z)} \\
&\quad \times \frac{C(z)}{D(z)},
\end{aligned}
$$

where

$$
\begin{aligned}
C(z) &= \frac{1}{\gamma_{\nu+1}}\phi(z+\nu)\Delta x(z)\nabla x_\nu(z)\nabla x_\nu(z+2) \\
&\quad - \sigma(z+1)\nabla x_\nu(z)\nabla x_\nu(z+1)\nabla x_\nu(z+1) \\
&\quad + \frac{\gamma_\nu}{\gamma_{\nu+1}}\phi(z+\nu)\nabla x_\nu(z)\nabla x_{\nu+1}(z+1)\nabla x_\nu(z-\nu+1) \\
&\quad + \sigma(z)\nabla x_\nu(z+1)\nabla x_\nu(z+1)\nabla x_\nu(z) \\
&\quad - \phi(z+\nu-1)\nabla x_\nu(z+1)\nabla x_\nu(z+1)\nabla x_\nu(z-\nu+1),
\end{aligned}
$$

$$
\begin{aligned}
D(z) &= \phi(z+\nu)\nabla x_\nu(z)\nabla x_{\nu+1}(z+1)\nabla x(z+1) \\
&\quad - \phi(z+\nu-1)\nabla x_\nu(z+1)\nabla x_\nu(z+1)\nabla x(z+\tfrac{1}{2}).
\end{aligned}
$$

Next section includes the recurrence relations for the solutions of the second-order linear difference equation of hypergeometric type (1) using Lemma 3.1.

4. Recurrence relations of the solutions of the second-order linear difference equation of hypergeometric type

We first remark that the solution y_ν of the difference equation (1) can be rewritten using the function $\Psi_{\nu,\mu}$ as

$$
y_\nu(z) = \frac{C_\nu}{\rho(z)}\Psi_{\nu,\nu}(z). \tag{40}
$$

Here, we include the recurrence relations related with solutions y_ν and their difference derivatives as defined in [19, 20] by

$$y_\nu^{(k)} := \Delta^{(k)} y_\nu(s) = \frac{C_\nu^{(k)}}{\rho_k(s)} \Psi_{\nu,\nu-k}(s) \tag{41}$$

where

$$\Delta^{(k)} = \left(\frac{\Delta}{\Delta x_{k-1}}\right) \cdots \left(\frac{\Delta}{\Delta x_1}\right) \left(\frac{\Delta}{\Delta x_0}\right),$$

$$\rho_k(s) = \sigma(s+1)\rho_{k-1}(s+1) = \rho(s+k) \prod_{i=1}^{k} \sigma(s+i), \tag{42}$$

$$C_\nu^{(k)} = \left[\alpha_{\nu-k}\widetilde{\tau}'_{k-1} + \gamma_{\nu-k}\frac{\widetilde{\sigma}''_{k-1}}{2}\right] C_\nu^{(k-1)} = \kappa_{\nu+k-1}C_\nu^{(k-1)} = \prod_{i=0}^{k-1} \kappa_{\nu+i}C_\nu,$$

$$\kappa_\nu = \alpha_{\nu-1}\widetilde{\tau}' + \gamma_{\nu-1}\frac{\widetilde{\sigma}''}{2}, \qquad C_\nu^{(0)} = C_\nu.$$

The following theorem has been proved for the linear-type lattices in [4, 5, 6]. It is also valid for the quadratic-type lattices but together with the condition of Lemma 3.1.

Theorem 4.1. *The following linear relation holds*

$$\sum_{i=1}^{3} B_i(s) y_{\nu_i}^{(k_i)}(s) = 0, \tag{43}$$

by the conditions of Lemma 3.1, where

$$B_i(s) = A_i(s)(C_{\nu_i}^{(k_i)})^{-1}\phi(s+k_*)...\phi(s+k_i-1).$$

Here, $A_i(s)$ are the coefficient functions of the recurrence relations defined in Lemma 3.1.

Proof. By Lemma 3.1, there exist the functions $A_i(z)$, $i = 1, 2, 3$ such that the following linear relation holds

$$\sum_{i=1}^{3} A_i(s) \Psi_{\nu_i,\nu_i-k_i}(s) = 0.$$

Therefore, by the definition of the difference derivative (41), we have

$$\sum_{i=1}^{3} A_i(s)(C_{\nu_i}^{(k_i)})^{-1}\rho_{k_i}(s) y_{\nu_i}^{(k_i)}(s) = 0,$$

which can be rewritten as the following

$$\sum_{i=1}^{3} B_i(s) y_{\nu_i}^{(k_i)}(s) = 0, \quad B_i(s) = A_i(s)(C_{\nu_i}^{(k_i)})^{-1}\phi(s+k_*)...\phi(s+k_i-1),$$

by dividing the equality with $\rho_{k_*}(s)$, where $k_* = \min\{k_1, k_2, k_3\}$ and then, using expression (29), which completes the proof. □

From (41), the examples 3.2, 3.3, 3.4 and 3.7 lead to the following relations

$$B_1(s)y_\nu(s) + B_2(s)y_\nu^{(1)}(s) + B_3(s)y_\nu^{(2)}(s) = 0,$$
$$B_1(s)y_\nu^{(1)}(s) + B_2(s)y_\nu^{(2)}(s) + B_3(s)y_{\nu+1}^{(1)}(s) = 0,$$
$$B_1(s)y_\nu(s) + B_2(s)y_\nu^{(1)}(s) + B_3(s)y_{\nu+1}(s) = 0,$$
$$B_1(s)y_\nu(s) + B_2(s)y_\nu^{(1)}(s) + B_3(s)y_{\nu-1}(s) = 0.$$

Notice that the last two relations are the so-called raising and lowering relations, respectively, which are equivalent to the following Δ-ladder-type recurrence relations, respectively,

$$B_1(s)y_\nu(s) + B_2(s)\frac{\Delta y_\nu(s)}{\Delta x(s)} + B_3(s)y_{\nu+1}(s) = 0, \tag{44}$$

$$\widetilde{B}_1(s)y_\nu(s) + \widetilde{B}_2(s)\frac{\Delta y_\nu(s)}{\Delta x(s)} + \widetilde{B}_3(s)y_{\nu-1}(s) = 0. \tag{45}$$

Note that in order to get a Δ-ladder-type recurrence relation, it is sufficient to put $k_1 = 0$, $k_2 = 1$, $k_3 = 0$ and $\nu_1 = \nu$, $\nu_2 = \nu$, $\nu_3 = \nu + m$ into (43), where $m = \mp 1$.

Corollary 4.2. *The following three-term recurrence relation holds*

$$B_1(s)y_\nu(s) + B_2(s)y_{\nu+1}(s) + B_3(s)y_{\nu-1}(s) = 0,$$

provided that the conditions in Lemma 3.1 exist. Here, coefficients $B_i(s)$, $i = 1, 2, 3$ are the polynomial functions.

Proof. By substituting $k_1 = 0$, $k_2 = 0$, $k_3 = 0$ and $\nu_1 = \nu$, $\nu_2 = \nu + 1$, $\nu_3 = \nu - 1$ in (43), one can obtain the above relation. $\square$

5. Applications to polynomials on the quadratic-type lattices

In this section, we include the applications of the method to the q-Racah and dual Hahn polynomials which are defined by (11) with $\nu = n$. These polynomials are general and defined on the q-quadratic lattices of the form $x(s) = q^{-s} + \delta\gamma q^{s+1}$ and the quadratic lattices of the form $x(s) = s(s + 1)$, respectively.

One can find a detailed study on these polynomials in [1, 17, 18]. Since the q-Racah and dual Hahn polynomials are defined by (11) where $\nu = n$, then condition (12) is satisfied. Therefore, Lemma 3.1 and Theorem 4.1 hold for such polynomials.

In the following, we include the two types of recurrence relations consisting of the polynomials and their difference-derivatives.

5.1. The application of the method to the q-Racah polynomials

Let $\nu_1 = n$, $\nu_2 = n$, $\nu_3 = n + 1$ and $k_1 = 0$, $k_2 = 1$, $k_3 = 0$ in Theorem 4.1, then we get

$$B_1(s)P_n(s) + B_2(s)\Delta^{(1)}P_n(s) + B_3(s)P_{n+1}(s) = 0.$$

$$B_1(s) = \frac{1}{C_n}\left\{\frac{\phi(s)}{\Delta x(s)}\left[-\gamma_n + \gamma_{n+1}\frac{\nabla x_{n+2}(s-n)}{\nabla x_{n+1}(s-n)}\right] - \frac{\sigma(s-n)}{\nabla x_{n+1}(s-n)}\right\}$$
$$B_2(s) = \frac{\phi(s)}{C_n^{(1)}}\frac{1}{\gamma_n}\frac{\tau_n(s) - C_nB_1(s)}{\Delta x(s-\frac{1}{2})}$$
$$B_3(s) = -\frac{\gamma_{n+1}}{C_{n+1}}$$

where γ_n is defined by (9) and $C_n^{(1)} = (\alpha_{n-1}\widetilde{\tau}' + \gamma_{n-1}\frac{\widetilde{\sigma}''}{2})C_n$. Notice that this case is considered in example 8 with $n = \nu$. Placing the corresponding values of the q-Racah polynomials from Table 1 in coefficients $B_i(s)$, $i = 1, 2, 3$, we have

$$
\begin{aligned}
B_1(s) &= \frac{1}{C_n}\Big\{ \frac{-q^{1/2}(q^{1/2} - q^{-1/2})q^s(q^{-s} - \alpha q)(q^{-s} - \gamma q)(1 - \beta\delta q^{s+1})}{(1 - \delta\gamma q^{2s+2})} \\
&\times \quad (1 - \delta\gamma q^{s+1})\Big[-[n]_q + [n+1]_q \frac{q^{-1/2}(1 - \delta\gamma q^{2s-n+2})}{(1 - \delta\gamma q^{2s-n+1})} \Big] \\
&+ \quad \frac{q^{2-n/2}q^s(q^{1/2} - q^{-1/2})(1 - \delta q^{s-n})(1 - q^{-s+n})(\alpha - \delta\gamma q^{s-n})}{(1 - \delta\gamma q^{2s-n+1})} \\
&\times \quad (\gamma - \beta\gamma q^{-s+n})\Big\}, \\
B_2(s) &= \frac{2(q^{1/2} - q^{-1/2})(q^{-s} - \alpha q)(q^{-s} - \gamma q)(1 - \beta\delta q^{s+1})(1 - \gamma\delta q^{s+1})}{K} \\
&\times \quad \frac{-(1 - \alpha\beta q^{2n+2})x_n(s) + q^{-n/2}(1 - \alpha q^{n+1})(1 - \beta\delta q^{n+1})(1 - \gamma q^{n+1})}{q^{-s}(1 - \delta\gamma q^{2s+2})} \\
&- \quad \frac{(q^{1/2} - q^{-1/2})\Big\{(1 - \delta\gamma q^{n+1})(1 - \alpha\beta q^{2n+2})]\Big\} + C_n B_1(s)}{(q^{1/2} - q^{-1/2})q^{-s}(1 - \delta\gamma q^{2s+2})}, \\
B_3(s) &= -\frac{[n+1]_q}{C_{n+1}}.
\end{aligned}
$$

where $K = C_n[n]_q\Big[(q^{(n-1)/2} - q^{-(n-1)/2})(1 - \beta\gamma q^2) + [n-1]_q q^{-n}(q^{1/2} - q^{-1/2})(1 + \alpha\beta q^{2n+2})\Big]$.

5.2. The application of the method to the dual Hahn polynomials

Let $\nu_1 = n - 1$, $\nu_2 = n$, $\nu_3 = n + 1$ and $k_1 = 0$, $k_2 = 1$, $k_3 = 0$ in Theorem 4.1, then we get

$$
B_1(s)P_{n-1}(s) + B_2(s)\Delta^{(1)}P_n(s) + B_3(s)P_{n+1}(s) = 0.
$$

In order to obtain this relation, we use the following three-term recurrence relation (TTRR)

$$
x(s)P_n(s) = \widetilde{\alpha}_n P_{n+1}(s) + \widetilde{\beta}_n P_n(s) + \widetilde{\gamma}_n P_{n-1}(s), \quad n = 0, 1, 2, \dots, \tag{46}
$$

with the initial conditions $P_0(s) = 1$, $P_{-1}(s) = 0$, and also the differentiation formula [1, Eq. (5.67)] (or [11, Eq. (25)])

$$
\phi(s)\frac{\Delta P_n(s)_q}{\Delta x(s)} = \widehat{\alpha}_n P_{n+1}(s)_q + \widehat{\beta}_n(s)P_n(s)_q, \tag{47}
$$

where $\phi(s) = \sigma(s) + \tau(s)\Delta x(s - \frac{1}{2})$, and

$$
\widehat{\alpha}_n = -\frac{\lambda_n}{[n]_q\widetilde{\tau}_n'}\frac{B_n}{B_{n+1}}, \quad \widehat{\beta}_n(s) = \frac{\lambda_n}{[n]_q}\frac{\tau_n(s)}{\widetilde{\tau}_n'} - \lambda_n\Delta x(s - \frac{1}{2}).
$$

Notice that the above differentiation formula is valid for the q-polynomials on the q-quadratic lattices. In order to obtain a formula for the polynomials on the quadratic lattices, one can consider the limit case when $q \to 1$.

Now, to compute $\Delta^{(1)}P_n(s) = \frac{\Delta P_n(s)}{\Delta x(s)}$, we first multiply the above equality by $\phi(s)$ and then use formula (47), then we reach

$$
B_1(s)\phi(s)P_{n-1}(s) + B_2(s)\Big[\widehat{\alpha}_n P_{n+1}(s) + \widehat{\beta}_n(s)P_n(s)\Big] + B_3(s)\phi(s)P_{n+1}(s) = 0,
$$

which can be rewritten as the following form

$$[B_2(s)\widehat{\alpha}_n + B_3(s)\phi(s)] P_{n+1}(s) + B_2(s)\widehat{\beta}_n(s)P_n(s) + B_1(s)\phi(s)P_{n-1}(s) = 0.$$

By the TTRR, we have the following system of equations

$$B_2(s)\widehat{\alpha}_n + B_3(s)\phi(s) = \widetilde{\alpha}_n, \quad B_2(s)\widehat{\beta}_n(s) = \widetilde{\beta}_n - x(s), \quad B_1(s)\phi(s) = \widetilde{\gamma}_n,$$

which leads to

$$B_1(s) = \frac{\widetilde{\gamma}_n}{\phi(s)}, \; B_2(s) = \frac{\widetilde{\beta}_n - x(s)}{\widehat{\beta}_n(s)}, \; B_3(s) = \frac{1}{\phi(s)} \left[\widetilde{\alpha}_n - \frac{\widehat{\alpha}_n}{\widehat{\beta}_n(s)}(\widetilde{\beta}_n - x(s)) \right].$$

Considering the limit case as $q \to 1$, the above coefficients become

$$B_1(s) = \frac{\gamma_n}{\phi(s)}, \; B_2(s) = \frac{\beta_n - x(s)}{\widehat{\beta}_n(s)}, \; B_3(s) = \frac{1}{\phi(s)} \left[\alpha_n - \frac{\widehat{\alpha}_n}{\widehat{\beta}_n(s)}(\beta_n - x(s)) \right].$$

Then, inserting the corresponding values of the dual Hahn polynomials from Table 2, [18, Table 3.7., Page 109] in coefficients $B_i(s)$, $i = 1, 2, 3$ leads to

$$
\begin{aligned}
B_1(s) &= \frac{(a + c + n)(b - a - n)(b - c - n)}{(s + a + 1)(s + c + 1)(b - s - 1)}, \\
B_2(s) &= -\frac{[ab - ac + bc + (b - a - c - 1)(2n + 1) - 2n^2 - s(s + 1)]}{\kappa_{n+1}}, \\
&\times \; (2s + n + 1) \\
B_3(s) &= \frac{n + 1}{(s + a + 1)(s + c + 1)(b - s - 1)} [1 + B_2(s)],
\end{aligned}
$$

where $\kappa_n = (s + a + n)(s + c + n)(b - s - n) - (s - a)(s + b)(s - c) + (n - 1)(2s + 1)(2s + n)$.

6. Concluding remarks

In the present work, some recurrence relations are developed for the hypergeometric functions on the quadratic-type lattices with applications in the q-Racah and dual Hahn polynomials. To obtain the recurrence relations for the other classes of polynomials, one can use appropriate limit transitions.

7. Acknowledgements

Table 1: Main data of the monic q-Racah polynomials

$P_n(s)$	$R_n(x(s), \alpha, \beta, \gamma, \delta\|q)$, $\quad x(s) = q^{-s} + \delta\gamma q^{s+1}$, $\quad \Delta x(s) = q^{-s}(1 - \delta\gamma q^{2s+2})(q^{-1} - 1)$
$\sigma(s)$	$\delta^2\gamma^2 q^2 (q^{1/2} - q^{-1/2})^2 q^{-2s}(q^s - 1)(q^s - \delta^{-1})(q^s - \beta\gamma^{-1})(q^s - \alpha\delta^{-1}\gamma^{-1})$
$\phi(s)$	$(q^{1/2} - q^{-1/2})^2 q^{-2s}(1 - \alpha q^{s+1})(1 - \beta\delta q^{s+1})(1 - \gamma q^{s+1})(1 - \delta\gamma q^{s+1})$
$\tau(s)$	$\frac{(q^{-1/2} - q^{1/2})q^{-s}}{(1 - \gamma\delta q^{2s+1})}\Big[(1 - \alpha q^{s+1})(1 - \beta\delta q^{s+1})(1 - \gamma q^{s+1})(1 - \delta\gamma q^{s+1}) - (\delta\gamma q)^2(q^s - 1)$ $\times (q^s - \delta^{-1})(q^s - \beta\gamma^{-1})(q^s - \alpha\delta^{-1}\gamma^{-1})\Big]$
τ'	$(q^{1/2} - q^{-1/2})(1 - \beta\gamma q^2)$
$\tau_n(s)$	$-(q^{1/2} - q^{-1/2})\Big\{(1 - \alpha\beta q^{2n+2})x(s + \frac{n}{2}) + q^{-n/2}\Big[(1 - \alpha q^{n+1})(1 - \beta\delta q^{n+1})(1 - \gamma q^{n+1})$ $-(1 + \delta\gamma q^{n+1})(1 - \alpha\beta q^{2n+2})\Big]\Big\}$
λ_n	$-q^{-n+\frac{1}{2}}(1 - q^n)(1 - \alpha\beta q^{n+1})$
$\frac{\widetilde{\sigma}''_\nu}{2}$	$\frac{1}{2}q^{-n}(q^{1/2} - q^{-1/2})^2(1 + \alpha\beta q^{2n+2})$
$\widetilde{\sigma}'_\nu(0)$	$-\frac{1}{2}q^{-\frac{n}{2}}(q^{1/2} - q^{-1/2})^2\Big[\beta\delta q + \alpha q + \delta\gamma q + \gamma q + \alpha\beta\delta q^{n+2} + \alpha\beta q^{n+2} + \beta\delta\gamma q^{n+2} + \alpha\gamma q^{n+2}\Big]$
$\widetilde{\sigma}_\nu(0)$	$\delta\gamma q q^{-n}(q^{1/2} - q^{-1/2})^2\Big[\beta\alpha q^{n+1} + \beta\delta q^{n+1} + \alpha q^{n+1} + \beta q^{n+1} + \alpha\delta^{-1}q^{n+1} + \gamma q - 1 - \alpha\beta q^{2n+2}\Big]$
$\widetilde{\beta}_n$	$1 + \delta\gamma q - \frac{(1 - \alpha q^{n+1})(1 - \alpha\beta q^{n+1})(1 - \beta\delta q^{n+1})(1 - \gamma q^{n+1})}{(1 - \alpha\beta q^{2n+1})(1 - \alpha\beta q^{2n+2})}$ $-\frac{q(1 - q^n)(1 - \beta q^n)(\gamma - \alpha\beta q^n)(\delta - \alpha q^n)}{(1 - \alpha\beta q^{2n})(1 - \alpha\beta q^{2n+1})}$
$\widetilde{\gamma}_n$	$\frac{(1 - \alpha q^n)(1 - \alpha\beta q^n)(1 - \beta\delta q^n)(1 - \gamma q^n)}{(1 - \alpha\beta q^{2n-1})(1 - \alpha\beta q^{2n})} \frac{q(1 - q^n)(1 - \beta q^n)(\gamma - \alpha\beta q^n)(\delta - \alpha q^n)}{(1 - \alpha\beta q^{2n})(1 - \alpha\beta q^{2n+1})}$
$\widehat{\alpha}_n$	$q^{-n+\frac{1}{2}}(q^{-1/2} - q^{1/2})(1 - \alpha\beta q^{2n+1})$
$\overline{\beta}_n(s)$	$-q^{-n/2+1/2}(q^{1/2} - q^{-1/2})\frac{(1 - \alpha\beta q^{n+1})}{(1 - \alpha\beta q^{2n+2})}\Big\{q^{-1}(1 - \alpha\beta q^{2n+2})x(s + \frac{n}{2}) + q^{-n/2}q^{-n-1}\Big[(1 - \alpha q^{n+1})$ $\times(1 - \beta\delta q^{n+1})(1 - \gamma q^{n+1}) - (1 + \delta\gamma q^{n+1})(1 - \alpha\beta q^{2n+2})\Big]\Big\}$
$\widehat{\beta}_n(s)$	$\overline{\beta}_n(s) - q^{-s-n+\frac{1}{2}}(q^{1/2} - q^{-1/2})(1 - q^n)(1 - \alpha\beta q^{n+1})(1 - \delta\gamma q^{2s+1})$

Table 2: Main data of the dual Hahn polynomials

$P_n(s)$	$W_n^{(c)}(x(s))$, $\quad x(s) = s(s+1)$, $\quad \Delta x(s) = 2s + 2$
$\sigma(s)$	$(s - a)(s + b)(s - c)$
$\phi(s)$	$(s + a + 1)(s + c + 1)(b - s - 1)$
$\tau(s)$	$ab - ac + bc - a + b - c - 1 - x(s)$
λ_n	n
$\alpha_n = \widehat{\alpha}_n$	$n + 1$
β_n	$ab - ac + bc + (b - a - c - 1)(2n + 1) - 2n^2$
γ_n	$(a + c + n)(b - a - n)(b - c - n)$
$\widehat{\beta}_n(s)$	$\frac{(s+a+n+1)(s+c+n+1)(b-s-n-1)-(s-a)(s+b)(s-c)+n(2s+1)(2s+n+1)}{2s+n+1}$

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