GENERALIZED BURNSIDE ALGEBRA OF TYPE $B_n$

HASAN ARSLAN AND HIMMET CAN

Abstract. In this paper, we firstly give an alternative method to determine the size of $C(S_n)$ which is the set of elements of type $S_n$ in a finite Coxeter system $(W_n, S_n)$ of type $B_n$. We also show that all cuspidal classes of $W_n$ are actually the conjugacy classes $K_{\lambda}$ for every $\lambda \in DP^+(n)$. We then define the generalized Burnside algebra $HB(W_n)$ for $W_n$ and construct a surjective algebra morphism between $HB(W_n)$ and Mantaci-Reutenauer algebra $MR(W_n)$. We obtain a set of orthogonal primitive idempotents $e_{\lambda}$, $\lambda \in DP(n)$ of $HB(W_n)$, that is, all the characteristic class functions of $W_n$. Finally, we give an effective formula to compute the number of elements of all the conjugacy classes $K_{\lambda}$, $\lambda \in DP(n)$ of $W_n$.

1. Introduction

Solomon’s descent algebra of a finite Coxeter system $(W, S)$ was introduced by Solomon in 1976 in [11]. In 1992, Bergeron, Bergeron, Howlett and Taylor elegantly reconstructed the Solomon’s descent algebra for a finite Coxeter system by using the group structure of Coxeter group and also they introduced a family of orthogonal primitive idempotents of the Solomon’s descent algebra by lifting orthogonal primitive idempotents of parabolic Burnside algebra in [1].

Let $W_n$ be the Coxeter group of type $B_n$. As a convention, throughout this paper, we denote by $HB(W_n)$, $MR(W_n)$, $SC(n)$ and $DP(n)$ the generalized Burnside algebra of type $B_n$, the Mantaci-Reutenauer algebra, the set of all signed compositions of $n$ and the set of all double partitions of $n$, respectively.

Mantaci-Reutenauer algebra $MR(W_n)$, which is a subalgebra of the group algebra $QW_n$ and contains the Solomon’s descent algebras of type $A_n$ and $B_n$, was firstly constructed in [10]. In [2], Bonnafé and Hohlweg reconstructed $MR(W_n)$ by the methods which depend more on the structure of $W_n$ as a Coxeter group. Bonnafé studied the representation theory of Mantaci-Reutenauer algebra in [3].

Received by the editors: July 31, 2019; Accepted: October 18, 2019.

2010 Mathematics Subject Classification. Primary 20F55; Secondary 19A22.

Key words and phrases. Cuspidal class, Mantaci-Reutenauer algebra, Burnside algebra, orthogonal primitive idempotents.

©2020 Ankara University
Communications Faculty of Sciences University of Ankara-Series A1 Mathematics and Statistics
252
In Section 3, we prove that for every positive signed composition \( A \) of \( n \), the parabolic closure of the reflection subgroup \( W_A \) is \( W_n \). As a result of this, we obtain that the number of all elements of type \( S_n \) is equal to \( \sum_{\lambda \in \mathcal{DP}^+(n)} |\mathcal{K}_\lambda| \) and realize that all cuspidal classes of \( W_n \) are the conjugacy classes \( \mathcal{K}_\lambda \) for \( \lambda \in \mathcal{DP}^+(n) \).

In Section 4, we introduce the Burnside algebra \( HB(W_n) \) generated by isomorphism classes of reflection subgroups of \( W_n \) corresponding to signed compositions of \( n \). We call \( HB(W_n) \) the \textit{generalized Burnside algebra} of type \( B_n \). Generalized Burnside algebra \( HB(W_n) \) is isomorphic to the algebra \( Q\text{Irr}W_n \) generated by the irreducible characters of \( W_n \). Then we construct a set of orthogonal primitive idempotents of \( HB(W_n) \). These orthogonal primitive idempotents are actually all the characteristic class functions of the Coxeter group \( W_n \). We determine the coefficient of the sign character \( \varepsilon_n \) of \( W_n \) in the expression of the each orthogonal primitive idempotent of \( HB(W_n) \) in terms of irreducible characters of \( W_n \). We get a formula to compute the number of elements of all the conjugacy classes \( \mathcal{K}_\lambda \), \( \lambda \in \mathcal{DP}(n) \) of \( W_n \).

2. Preliminaries

2.1. Hyperoctahedral group. Let \((W_n, S_n)\) denote a Coxeter group of type \( B_n \) and write its generating set as \( S_n = \{t, s_1, \cdots, s_{n-1}\} \). Any element \( w \) of \( W_n \) acts by the permutation on the set \( X_n = \{-n, \cdots, -1, 1, \cdots, n\} \) such that for every \( i \in I_n \), \( w(-i) = -w(i) \). The Dynkin diagram of \( W_n \) is as follows:

\[
B_n : \begin{array}{c}
\circ \\
& \circ \\
& & \circ \\
& & & \circ \\
& & & & \circ \\
& & & & & \circ \\
& & & & & & \circ \\
& & & & & & & \circ \\
& & & & & & & & \circ \\
& & & & & & & & & \circ \\
& & & & & & & & & & \circ \\
& & & & & & & & & & & \circ \\
& & & & & & & & & & & & \circ \\
& & & & & & & & & & & & & \circ \\
\end{array}
\]

If \( J \subset S_n \), the subgroup \( W_J \) generated by \( J \) is called a \textit{standard parabolic subgroup} of \( W_n \). A \textit{parabolic subgroup} of \( W_n \) is a subgroup of \( W_n \) conjugate to \( W_J \) for some \( J \subset S_n \). Let \( t_1 := t \) and \( t_i := s_{i-1}t_{i-1}s_{i-1} \) for each \( i, 2 \leq i \leq n \). Put \( T_n := \{t_1, \cdots, t_n\} \). It is well-known that there are the following relations between the elements of \( S_n \) and \( T_n \):

\begin{itemize}
  \item[(1)] \( t_i^2 = 1, s_i^2 = 1 \) for all \( i, j \), \( 1 \leq i \leq n, \ 1 \leq j \leq n - 1 \);
  \item[(2)] \( ts_1t_1 = s_1ts_1t_1 \);
  \item[(3)] \( s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \) for all \( i \), \( 1 \leq i \leq n - 2 \);
  \item[(4)] \( ts_i = s_it_i, \ 1 < i \leq n - 1 \);
  \item[(5)] \( s_is_j = s_js_i \) for \( |i - j| > 1 \);
  \item[(6)] \( t_it_j = t_jt_i \) for \( 1 \leq i, j \leq n \).
\end{itemize}

We denote by \( l : W_n \to \mathbb{N} \) the length function attached to \( S_n \). Let \( T_n \) denote the reflection subgroup of \( W_n \) generated by \( T_n \). It is also clear that \( T_n \) is a normal subgroup of \( W_n \). Now let \( S_n = \{s_1, \cdots, s_{n-1}\} \) and let \( W_n \) denote the reflection subgroup of \( W_n \) generated by \( S_n \), where \( W_n \) is isomorphic to the symmetric group \( \Xi_n \) of degree \( n \). Thus \( W_n = W_n \rtimes T_n \).

Let \( \{e_1, \cdots, e_n\} \) be the canonical basis of the Euclidian space \( \mathbb{R}^n \) over \( \mathbb{R} \). Let \[
\Psi_n^+ = \{e_i : 1 \leq i \leq n\} \cup \{e_j + \lambda e_i : \lambda \in \{-1, 1\} \text{ and } 1 \leq i < j \leq n\}.
\]
Then $\Psi_n$ is a root system of type $B_n$. For further information about the Coxeter groups of type $B_n$, see [3], [9].

A signed composition of $n$ is an expression of $n$ as a finite sequence $A = (a_1, \ldots, a_k)$ whose each part consists of non-zero integers such that $\sum_{i=1}^{k} |a_i| = n$. Put $|A| = \sum_{i=1}^{k} |a_i|$. We write $\mathcal{SC}(n)$ to denote the set of all signed compositions of $n$.

Let $A = (a_1, \ldots, a_k) \in \mathcal{SC}(n)$. $A$ is said to be positive (resp. negative) if $a_i > 0$ (resp. $a_i < 0$) for every $i \geq 1$. If $a_i < 0$ for every $i \geq 2$, then $A$ is called parabolic. Let define $A^+ = (|a_1|, \ldots, |a_r|)$. Then $A^+$ is a positive signed composition of $n$.

The set of positive signed compositions of $n$ is denoted by $\mathcal{SC}^+(n)$.

A double partition $\mu = (\mu^+, \mu^-)$ of $n$ consists of a pair of partitions $\mu^+$ and $\mu^-$ such that $|\mu| = |\mu^+| + |\mu^-| = n$. If the number of positive parts of $n$ (resp. negative parts of $n$) is equal to zero, then we write $\emptyset$ instead of $\mu^+$ (resp. $\mu^-$). We denote the set of all double partitions of $n$ by $\mathcal{DP}(n)$. We define $\mathcal{DP}^+(n) = \{\mu = (\mu^+, \mu^-) \in \mathcal{DP}(n) : \mu^- = \emptyset\}$. For $\mu = (\mu^+, \mu^-) \in \mathcal{DP}(n)$, $\tilde{\mu} := \mu^+ - \mu^-$ is the signed composition obtained by appending the sequence of components of $\mu^+$ to that of $-\mu^-$ [2].

Now let $A \in \mathcal{SC}(n)$. If $\mu^+$ (resp. $\mu^-$) is rearrangement of the positive parts (resp. absolute value of negative parts) of $A$ in decreasing order, then $\lambda(A) := (\mu^+; \mu^-)$ is a double partition of $n$ and also $\lambda(\tilde{\mu}) = \mu$ for every $\mu \in \mathcal{DP}(n)$ [2]. In [2], Bonnafé and Hohlweg constructed some reflection subgroups of $W_n$ corresponding to signed compositions of $n$ as an analogue to $\Xi_n$ as follows: For each $A = (a_1, \ldots, a_k) \in \mathcal{SC}(n)$, the reflection subgroup $W_A$ of $W_n$ is generated by $S_A$, which is given by

$$
S_A = \{s_p \in W_{-n} : |a_1| + \cdots + |a_{i-1}| + 1 \leq p \leq |a_1| + \cdots + |a_i| - 1\} \\
\cup \{t_{|a_1|+\cdots+|a_{j-1}|+1} \in T_n \mid a_j > 0\} \subset S_n'
$$

where $S_n' = \{s_1 \cdots s_{n-1}, t_1, t_2, \ldots, t_n\}$. By the definition of $S_A$, there exists an isomorphism $W_A \cong W_{a_1} \times \cdots \times W_{a_r}$ [2]. By taking into account the definition of the generating set $S_A$ and the isomorphism $W_A \cong W_{a_1} \times \cdots \times W_{a_r}$, for $i$, $1 \leq i \leq r$ if $a_i > 0$ then we have $\text{rank } W_{a_i} = a_i$ and if $a_i < 0$ then we have $\text{rank } W_{a_i} = |a_i| - 1$.

Therefore, we get

$$
\text{rank } W_A = \text{rank } S_A = n - \text{neg}(A),
$$

where $\text{neg}(A)$ denotes the number of negative parts of $A$. Because of $\sum_{i=1}^{r} |a_i| = n$, we obtain $\text{rank } W_A = |S_A| \leq n$.

For $A, B \in \mathcal{SC}(n)$, we write $A \subset B$ if $W_A \subset W_B$, where $\subset$ is a partial ordering relation on $\mathcal{SC}(n)$ [2]. For $A \in \mathcal{SC}(n)$ let $\text{cox}_A$ be a Coxeter element of $W_A$ in terms of generating set $S_A$. For $B, B' \subset A$, we write $B \equiv_A B'$ if $W_B$ is conjugate to $W_{B'}$ under $W_A$ and also $\text{cox}_B$ and $\text{cox}_{B'}$ are conjugate to each other in $W_A$ if and only if $B \equiv_A B'$ [3]. We write $B \equiv_n B'$ if $W_B$ is conjugate to $W_{B'}$ under $W_n$. This equivalence is a special case for these kind of reflection subgroups of $W_n$, because this statement is not true for every reflection subgroup of $W_n$. Although some two
reflection subgroups $R$ and $R'$ of $W_n$ contain $W_n$-conjugate Coxeter elements $\text{cox}_R$ and $\text{cox}_{R'}$ respectively, these subgroups are not able to $W_n$-conjugate to each other \cite{6}. For every element $w$ of $W_n$, there exists a unique $\lambda \in \mathcal{D}\mathcal{P}(n)$ such that $w$ is $W_n$-conjugate to $\text{cox}_\lambda$ \cite{3}. Let $K_\lambda$ be the conjugacy class of $\lambda$ corresponding to $\lambda \in \mathcal{D}\mathcal{P}(n)$. Since the number of conjugacy classes of $W_n$ is equal to $|\mathcal{D}\mathcal{P}(n)|$, thus we may split up $W_n$ into $|\mathcal{D}\mathcal{P}(n)|$ conjugacy classes. In \cite{3}, Bonnafé showed that for $A, B \in \mathcal{S}\mathcal{C}(n)$, $W_n$ is conjugate to $W_B$ in $W_n$ if and only if $\lambda(A) = \lambda(B)$.

For a subset $X$ of $W_n$, we denote by $\text{Fix}(X) = \{v \in \mathbb{R}^n : \forall x \in X, x(v) = v\}$ the subspace of $\mathbb{R}^n$ fixed by $X$ and let write $W_{\text{Fix}(X)} = \{w \in W_n : \forall v \in \text{Fix}(X), w(v) = v\}$ for the stabilizer of $\text{Fix}(X)$ in $W_n$. By \cite{6}, the set $W_{\text{Fix}(X)}$ is called the parabolic closure of $X$ and it is denoted by $A(X)$. For any $w \in W_n$, if we take $X = \{w\}$ then we write $\text{Fix}(w)$ and $A(w)$ instead of $\text{Fix}(\{w\})$ and $A(\{w\})$, respectively. By \cite{15}, $w$ is said to be an element of type $J$ if there exists a $J \subseteq S_n$ such that $A(w)$ is conjugate to $W_J$ under $W_n$.

2.2. Mantaci-Reutenauer algebra. For any $A \in \mathcal{S}\mathcal{C}(n)$, we set

$$D_A = \{x \in W_n : \forall s \in S_A, l(xs) > l(x)\}.$$  

By \cite{2} and \cite{7}, $D_A$ is the set of distinguished coset representatives of $W_A$ in $W_n$. Let

$$d_A = \sum_{w \in D_A} w \in \mathbb{Q}W_n,$$

and let

$$\mathcal{M}\mathcal{R}(W_n) = \bigoplus_{A \in \mathcal{S}\mathcal{C}(n)} \mathbb{Q}d_A.$$  

For every $A \in \mathcal{S}\mathcal{C}(n)$, from \cite{2} $\Phi_n : \mathcal{M}\mathcal{R}(W_n) \to \mathbb{Q}\text{Irr}W_n$ is a surjective algebra morphism such that $\Phi_n(d_A) = \text{Ind}_{W_A}^{W_n} 1_A$, where $1_A$ stands for the trivial character of $W_A$. It is well-known from \cite{2} that the radical of $\mathcal{M}\mathcal{R}(W_n)$ is $\text{Ker}\Phi_n = \bigoplus_{A,B \in \mathcal{S}\mathcal{C}(n), A \equiv B} \mathbb{Q}(d_A - d_B)$.

By \cite{2}, for $A, B \in \mathcal{S}\mathcal{C}(n)$, the set of distinguished double coset representatives is defined as $D_{AB} = D_A^{-1} \cap D_B$ and for any $x \in D_{AB}$,

$$W_A \cap xW_B = W_{A \cap B}.$$

For $A, B \in \mathcal{S}\mathcal{C}(n)$, let define \cite{3} the sets $D_{AB}^c = \{x \in D_{AB} : x^{-1}W_A \subseteq W_B\}$ and $D_{AB}^c = \{x \in D_{AB} : W_A = xW_B\}$.

The following proposition proved by Bonnafé in \cite{3} gives the ring multiplication structure in $\mathcal{M}\mathcal{R}(W_n)$.

**Proposition 1 (\cite{3}).** Let $A$ and $B$ be any two signed composition of $n$. Then,

i. There is a map $f_{AB} : D_{AB} \to \mathcal{S}\mathcal{C}(n)$ satisfying the following conditions:

- For every $x \in D_{AB}$, $f_{AB}(x) \subseteq B$ and $f_{AB}(x) \equiv_B x^{-1}A \cap B$.
- $d_A d_B - \sum_{x \in D_{AB}} d_{f_{AB}(x)} \in \mathcal{M}\mathcal{R}(W_n) \cap \mathcal{M}\mathcal{R}(W_n) \cap \text{Ker}\Phi_n$.  


ii. If $A$ parabolic or $B$ is semi-positive, then $f_{AB}(x) = x^{-1} A \cap B$ for $x \in D_{AB}$ and $d_A d_B = \sum_{x \in D_{AB}} d_{x^{-1} A \cap B}$.

iii. $\tau(A)(d_B) = |D_{AB}^c|$.

iv. $D_{AB} = \{ x \in W_n : S_A = S_B \}$.

v. $W(B) = \{ w \in W_n : s_B = S_B \}$.

vi. $W_n(W_B) = W(B) \times W_B$.

In the Proposition[1] the symbols $\subset$ and $\sim$ denote a pre-order and an ordering defined on $\mathcal{S}C(n)$, respectively. If $A \equiv_n B$, then it is clear $D_{AB}^c = D_{AB}^c$ and $W(A) = D_{AB}^c$. Thus $\mathcal{M}R(W_n)$ is called Mantaci-Reutenauer algebra of $W_n$.

For $\lambda \in D\mathcal{P}(n)$, the map $\tau_\lambda : \mathcal{M}R(W_n) \rightarrow \mathbb{Q}$, $x \mapsto \Phi_n(x)(\text{cox}_\lambda)$ is independent of the choice of $\text{cox}_\lambda \in \mathcal{K}_\lambda$ and it is also an algebra morphism [2].

3. Some Properties of Coxeter Group of Type $B_n$

Let $A \in \mathcal{S}C(n)$ and let $l_A : W_A \rightarrow \mathbb{N}$ be the length function of $W_A$ in terms of its generating set $S_A$. When $A$ is not a parabolic signed composition of $n$, the value $l_A(w)$ is not equal to $l(w)$ for some $w \in W_A$. The following lemma gives a relation between these two length functions. The proof of the following lemma is clear from the fact that $l(t_i) = 2i - 1$ for $i, 1 \leq i \leq n$.

**Lemma 2.** Let $A \in \mathcal{S}C(n)$. Then for every $w \in W_A$

$$l(w) \equiv l_A(w) \pmod{2}.$$  

Let $\varepsilon_n$ and $\varepsilon_A$ be the sign character of $W_n$ and $W_A$, respectively. As a result of the previous lemma, we get

$$\varepsilon_n(w) = (-1)^{l(w)} = (-1)^{l_A(w)} = \varepsilon_A(w).$$

Since the restriction of $\varepsilon_n$ to $W_A$, that is $\text{res}_{W_A}^W \varepsilon_n$, is an irreducible character of $W_A$ for every $A \in \mathcal{S}C(n)$ and Lemma[2], then we have $\text{res}_{W_A}^W \varepsilon_n = \varepsilon_A$.

**Example 3.** For a concrete example, let $A = (-2, 3, -1, -3, 1) \in \mathcal{S}C(10)$. Then $S_A = \{ s_1 \} \cup \{ t_3, s_3, s_4 \} \cup \{ s_7, s_8 \} \cup \{ t_{10} \} \subset S_{10}$ and $S'_A = W_A \cap S'_{10} = \{ s_1 \} \cup \{ t_3, s_3, s_4, t_4, t_5 \} \cup \{ s_7, s_8 \} \cup \{ t_{10} \}$. Thus $W_A \cong W_{-2} \times W_3 \times W_{-1} \times W_{-3} \times W_1$. For $w = s_7 t_3 s_3 s_1 t_{10} = s_7 s_2 s_1 t_1 s_2 s_3 s_1 s_9 s_8 s_7 s_6 s_5 s_4 s_3 s_2 s_1 t_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9 \in W_{10}$, so $l(w) = 27$. It follows that $l(w) \equiv l_A(w) \equiv 1 \pmod{2}$.

**Proposition 4.** If $B \in \mathcal{S}C^+(n)$, then the parabolic closure of $W_B$ is $A(W_B) = W_n$.

**Proof.** Since $B \in \mathcal{S}C^+(n)$, we have $T_0 \leq W_B$ and so $w_n \in W_B$. By considering $w_n$ as a linear map $-id_{\mathbb{R}^n} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, we obtain $\text{Fix}(w_n) = \{ 0 \}$. Thus, the parabolic closure of $w_n$ is $A(w_n) = W_{\text{Fix}(w_n)} = W_n$. Because of the relation $w_n \in W_B \subset A(\text{cox}_B) = A(W_B)$, we get $w_n \in A(\text{cox}_B)$. By [11], the inclusion $A(w_n) \subset$
Lemma 5. Let $W_A = A(W_B)$ holds. If we take into account the fact that $A(w_n) = W_n$, then we have $A(W_B) = W_n$. This completes the proof. 

As a consequence of Proposition 4 if $B \in \mathcal{SC}^+(n)$, then the parabolic closure of $W_B$ is $W_n$ and each element of $\mathcal{K}_A(W)$ is of type $S_n$.

**Lemma 5.** Let $A$ be a signed composition of $n$. Then $w_n$ belongs to $W_A$ if and only if $A \in \mathcal{SC}^+(n)$.

**Proof.** When $A$ is a positive signed composition of $n$, we can easily see from the proof of Proposition 4 that $w_n$ is an element of $W_A$. Conversely, let $w_n$ be in $W_A$. We suppose that $A = (a_1, \cdots, a_t)$ is not a positive signed composition of $n$. Then there exists $a_i < 0$ for some $i$, $1 \leq i \leq r$. Thus from the definition of $W_A$, we obtain $t_{\{a_1, \cdots, a_i, \cdots, a_r\}} \not\in S'_A = W_A \cap S'_n$. Hence for any $x \in W_A$ and $e_{a_1, \cdots, a_i, \cdots, a_r} + \cdots + e_{a_1, \cdots, a_i, \cdots, a_r} \in \mathbb{R}^n$, we have $x(e_{a_1, \cdots, a_i, \cdots, a_r} + \cdots + e_{a_1, \cdots, a_i, \cdots, a_r}) = e_{a_1, \cdots, a_i, \cdots, a_r} + \cdots + e_{a_1, \cdots, a_i, \cdots, a_r}$ and so $e_{a_1, \cdots, a_i, \cdots, a_r} \in \text{Fix}(W_A)$. This is a contradiction, because the subspace $\text{Fix}(W_A)$ consists of only $0$. Therefore, we get $A \in \mathcal{SC}^+(n)$. 

**Theorem 6.** If the set $C(S_n)$ denotes the set of all elements of $W_n$ of type $S_n$, then we have 

$$C(S_n) = \bigcap_{\lambda \in \mathcal{DP}^+(n)} \mathcal{K}_\lambda. \tag{1}$$

**Proof.** For each $\lambda \in \mathcal{DP}^+(n)$, we have $\lambda \in \mathcal{SC}^+(n)$. From Proposition 4 for every element of $\mathcal{K}_\lambda$ is of type $S_n$ and so the reverse inclusion holds. Now let $w \in C(S_n)$. Then $w$ is $W_n$-conjugate to $\text{cox}_A$ for some $A \in \mathcal{SC}(n)$. Thus we get $A(w) = A(\text{cox}_A) = A(W_A) = W_n$. From here, for every $x \in W_n$ and every $v \in \text{Fix}(W_A)$ we obtain $x(v) = v$. In particular, if we take $w_n = -id_{K_n} \in W_n$, then it is seen that $\text{Fix}(W_A) \subseteq \{0\}$. Thus $w_n$ is an element of $W_A$. Otherwise, if $A \not\in \mathcal{SC}^+(n)$, then from the proof of Lemma 5 we get $\text{Fix}(W_A) \neq \{0\}$, which is a contradiction. Hence $A \in \mathcal{SC}^+(n)$. By taking the definition of $\lambda$ into account, we get a $\lambda \in \mathcal{DP}^+(n)$ such that $\lambda(A) = \lambda$. Thus $w$ belongs to $\mathcal{K}_\lambda$ and so it is seen that the inclusion $C(S_n) \subseteq \bigcap_{\lambda \in \mathcal{DP}^+(n)} \mathcal{K}_\lambda$ satisfies. It is required. 

Since the exponents of $W_n$ are in turn $1, 3, \cdots, 2n-1$, then from (1) the number of elements of type $S_n$ is equal to the product of exponents of $W_n$ and so $|C(S_n)| = 1 \cdot 3 \cdots 2n-1$. By the equation (1), we obtain the formula 

$$|C(S_n)| = \sum_{\mu \in \mathcal{DP}^+(n)} |\mathcal{K}_\mu|.$$ 

Thus Theorem 6 gives us an alternative method to compute the number of elements of type $S_n$. We will give a formula in Corollary 19 to find the number of elements of every conjugacy class $\mathcal{K}_\lambda$, $\lambda \in \mathcal{DP}(n)$ of $W_n$. Moreover, we will give an example for Theorem 6 in Section 5.
A conjugacy class of a finite Coxeter group $W$ which does not contain an element of a proper standard parabolic subgroup of $W$ is called a *cuspidal class* of $W$ \[8\].

**Corollary 7.** Let $A$ be a positive signed composition of $n$. Then the conjugacy class $K_{\lambda(A)}$ is a cuspidal class of $W_n$.

If we consider the proof of Proposition [4] and Corollary [7] then all cuspidal classes of $W_n$ are the conjugacy classes $K_{\lambda(A)}$ for every $A \in SC^+(n)$. From Theorem [6] the set $C(S_n)$ is disjoint union of cuspidal classes of $W_n$. Therefore, each element of $W_n$ of type $S_n$ belongs to a unique cuspidal class of $W_n$.

**4. Generalized Burnside Algebra of $W_n$**

Let $A, B$ be any two signed compositions of $n$. Then, we have that

$$A \equiv_n B \Leftrightarrow W_A \sim W_n W_B \Leftrightarrow [W/W_A] = [W/W_B]$$

where $[W/W_A]$ represents the isomorphism class of $W_n$-set $W/W_A$. The orbits of $W_n$ on $W/W_A \times W/W_B$ are of the form $(W_A x, W_B)$ where $x \in D_{AB}$. The stabilizer of $(W_A x, W_B)$ in $W_n$ is $x^{-1} W_A \cap W_B = W_{x^{-1} A \cap B}$. Therefore

$$[W/W_A] \cdot [W/W_B] = [W/W_A \times W/W_B] = \sum_{x \in D_{AB}} [W/W_{x^{-1} A \cap B}].$$

Thus, we are now in a position to give the following definition.

**Definition 8.** The generalized Burnside algebra of $W_n$ is $\mathbb{Q}$-spanned by the set $\{[W/W_A] : A \in SC(n)\}$ and it is denoted by $HB(W_n)$.

From part (i) of Proposition [4] and the structure of $\text{Ker}(\Phi_n)$, the ring multiplication rule in $\mathcal{MR}(W_n)$ may be expressed by

$$d_A d_B = \sum_{x \in D_{AB}} d_{f_{AB}(x)} + \sum_{N \equiv n, N' \subset A} a_{NN'} (d_N - d_{N'})$$

where $a_{NN'} \in \mathbb{Z}$; $N, N' \subset A$; $N, N' \sim B$; $f_{AB}(x) \subset B$ and $f_{AB}(x) \equiv_B x^{-1} A \cap B$.

Now we define

$$\psi : \mathcal{MR}(W_n) \to HB(W_n), \ d_A \mapsto [W/W_A].$$

Thus $\psi$ is well-defined and surjective linear map. By considering the structure of $\text{Ker}\Phi_n$ and $f_{AB}(x) \equiv_B x^{-1} A \cap B$ we get

$$\psi(d_{AB}) = \psi\left( \sum_{x \in D_{AB}} d_{f_{AB}(x)} + \sum_{N \equiv n, N' \subset A} a_{NN'} (d_N - d_{N'}) \right)$$

$$= \sum_{x \in D_{AB}} [W/W_{f_{AB}(x)}]$$

$$= \psi(d_A) \psi(d_B).$$
Then the map $\psi$ is an algebra morphism. Since $\dim_{\mathbb{Q}} HB(W_n) = \dim_{\mathbb{Q}} \mathbb{Q} \operatorname{Irr} W_n = |\mathcal{DP}(n)|$, then there is an algebra isomorphism between $HB(W_n)$ and $\mathbb{Q} \operatorname{Irr} W_n$ such that

$$HB(W_n) \to \mathbb{Q} \operatorname{Irr} W_n, \quad [W/W_A] \mapsto \operatorname{Ind}^W_W 1_A.$$  

Now let $\lambda, \mu \in \mathcal{DP}(n)$ and let $\varphi_{\lambda} = \operatorname{Ind}^W_W 1_\lambda$. From part (iii) of Proposition 7, $\varphi_{\lambda}(\cos_{\lambda}) = \tau_{\lambda}(d_{\lambda}) = |D_{\lambda}|^2 \neq 0$ and $\tau_{\lambda}(d_{\mu}) = 0$ if $\lambda \nmid \mu$. Thus the matrix $(\tau_{\lambda}(d_{\mu}))_{\lambda, \mu \in \mathcal{DP}(n)}$ is lower diagonal. Then $(\varphi_{\lambda}(\cos_{\mu}))_{\lambda, \mu}$ is upper diagonal and also has positive diagonal entries. Therefore $(\varphi_{\lambda}(\cos_{\mu}))_{\lambda, \mu}$ is invertible and we write $(u_{\lambda, \mu})_{\lambda, \mu \in \mathcal{DP}(n)}$ for the inverse of $(\varphi_{\lambda}(\cos_{\mu}))_{\lambda, \mu}$. We define

$$e_{\lambda} = \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda, \mu} \varphi_{\mu}.$$  

By definition of $e_{\lambda}$ and $(\varphi_{\lambda}(\cos_{\mu}))^{-1} = (u_{\lambda, \mu})$, we obtain that

$$\varphi_{\lambda}(\cos_{\mu}) = \sum_{\gamma \in \mathcal{DP}(n)} u_{\lambda, \gamma} \varphi_{\gamma}(\cos_{\mu}) = \delta_{\lambda, \mu}.$$  

Hence the set $\{e_{\lambda} : \lambda \in \mathcal{DP}(n)\}$ is a collection of orthogonal primitive idempotents of $HB(W_n)$. Consequently, we have $HB(W_n) = \sum_{\lambda \in \mathcal{DP}(n)} \mathbb{Q} e_{\lambda}$.

For each $A \in \mathcal{SC}(n)$,

$$s_A : HB(W_n) \to \mathbb{Q}, \quad s_A([X]) = [W_A]X$$

is an algebra map, where $[W_A]X = \{x \in X : W_A x = x\}$. Since $HB(W_n)$ is semisimple and commutative algebra, then all algebra maps $HB(W_n) \to \mathbb{Q}$ are of the form $s_A$ for every $A \in \mathcal{SC}(n)$. The proof of the following lemma is immediately seen from [5].

**Lemma 9.** For $A, B \in \mathcal{SC}(n)$, we have that

$$s_A = s_B \iff \lambda(A) = \lambda(B).$$

Thus the dual basis of $HB(W_n)$ is $\{s_{\lambda} : \lambda \in \mathcal{DP}(n)\}$. For any $\lambda, \mu \in \mathcal{DP}(n)$, we have the following equality

$$s_{\lambda}(e_{\mu}) = \delta_{\lambda, \mu}, \quad (2)$$

and so any element $x$ in $HB(W_n)$ can be expressed as $x = \sum_{\lambda \in \mathcal{DP}(n)} s_{\lambda}(x)e_{\lambda}$.

Let $A$ be a signed composition of $n$. Induction and restriction of characters give rise to two maps between $HB(W_A)$ and $HB(W_n)$. For any $A, B \in \mathcal{SC}(n)$ such that $B \subset A$, we have $\operatorname{Ind}^W_W ([W_A/W_B]) = [W_n/W_B]$.

**Definition 10.** Let $A, B \in \mathcal{SC}(n)$ be such that $B \subset A$. The **restriction** is a linear map

$$\operatorname{res}^A_B : HB(W_A) \to HB(W_B), \quad \operatorname{res}^A_B ([W_A/W_C]) = \sum_{d \in W_A \cap D_{CB}} [W_B/W_{B \cap d^{-1}C}].$$
Before going into a further discussion of the restriction and induced character theories of generalized Burnside algebra, we shall first mention the number of elements of the conjugacy class of $W_A$ in $W_n$.

**Proposition 11.** Let $A \in \mathcal{S}(n)$ and $\lambda(A) = \lambda$. The number of all reflection subgroups of $W_n$ which are conjugate to $W_A$ is

$$|[W_A]| = |D_A| \cdot u_{\lambda, \lambda}.$$  

**Proof.** Put $[W_A] = \{xW_A : x \in W_n\}$. Now we note that $xW_Ay^{-1} = yW_Ay^{-1}$ if and only if $x \in yN_{W_n}(W_A)$. Thus, the number of distinct conjugates of $W_A$ in $W_n$ is $|W_n : N_{W_n}(W_A)|$. Since also $N_{W_n}(W_A) = W(A) \ltimes W_A$, we have

$$|[W_A]| = \frac{|W_n|}{|W(A)| \cdot |W_A|} = \frac{|D_A|}{|W(A)|}.$$  

Furthermore, from the fact that $\tau_{\lambda(A)}(d_A) = |D_A^\lambda| = |W(A)|$ and $\varphi_\lambda(\text{cox}_\lambda) = \tau_{\lambda(A)}(d_A) = \frac{1}{u_{\lambda, \lambda}}$, as desired. \hfill \square

**Example 12.** We consider the set $D_{(2,1)} = \{1, s_2, s_1, s_2\}$ consisting of the distinguished coset representatives of reflection subgroup $W_{(2,1)}$ in $W_3$. The number of all reflection subgroups conjugate to $W_{(2,1)}$ in $W_3$ is

$$|[W_{(2,1)}]| = |D_{(2,1)}| \cdot u_{(2,1,\emptyset),(2,1,\emptyset)} = 3 \cdot 1 = 3.$$  

These are explicitly $W_{(2,1)}$, $W_{(1,2)}$ and $s_2W_{(2,1)} = \langle s_2s_1s_2, t_1, t_2 \rangle$. We note that the reflection subgroup $s_2W_{(2,1)}$ does not coincide with any subgroup of $W_3$ corresponding to any signed composition of 3.

**Remark 13.** For $A, B \in \mathcal{S}(n)$ such that $B \subseteq A$ and for any $x \in HB(W_n)$, by using the definition of $s_A$ one can see that there exists the relation $s_B^A(\text{res}_W^A(x)) = s_B(x)$.

We can now give the following proposition.

**Proposition 14.** Let be $A, B \in \mathcal{S}(n)$ and let $A_1, A_2, \ldots, A_r$ be representatives of the $W_A$-equivalent classes of subsets of $A$, which are $W_n$-equivalent to $B$. Then,

$$\text{res}_W^A e_B = \sum_{i=1}^r e_{A_i}.$$  

If $B$ is not $W_n$-equivalent to any subset of $A$ then $\text{res}_W^A e_B = 0$.

**Proof.** Since $\text{res}_W^A e_B$ is an element of $HB(W_A)$, then we have

$$\text{res}_W^A e_B = \sum_{A_i \subseteq A} s_{A_i}(\text{res}_W^A(e_B)) e_{A_i}.$$  

Then by using Remark 13 and the relation 2, we get

$$\text{res}_W^A e_B = \sum_{A_i \subseteq A} s_{A_i}(e_B) e_{A_i}.$$
\[
= \sum_{A_i \subseteq A, A_i \neq \emptyset, B} e_{A_i}
\]
\[
= \sum_{i=1}^{r} e_{A_i}.
\]

**Proposition 15.** Let \( A, B \in \mathcal{S}(n) \) and let \( B \subset A \). Then we have
\[
\text{Ind}_{W_n}^{W_n} e_A^B = \frac{|W(B)|}{|W_A \cap W(B)|} \cdot e_B.
\]

**Proof.** Firstly, we assume that \( A = B \) and \( \text{cox}_A \) is a Coxeter element of \( W_A \). Since the image of \( \text{cox}_A \) under permutation character of \( W_n \) on the cosets of \( W_A \) is \( |W(A)| \) then it follows from the fact that
\[
x^{-1} \text{cox}_A x \in W_A \Leftrightarrow x \in N_{W_n}(W_A).
\]
Thus we obtain
\[
\text{Ind}_{W_A}^{W_n} e_A^B(\text{cox}_A) = |D_A \cap N_{W_n}(W_A)|
\]
\[
= |W(A)|.
\]
As \( \text{Ind}_{W_A}^{W_n} e_A^B \) takes value zero except for the elements conjugate to \( \text{cox}_A \) and so we get
\[
\text{Ind}_{W_A}^{W_n} e_A^A = |W(A)| e_A.
\]
By transitivity of induced characters, we generally get
\[
\text{Ind}_{W_A}^{W_n} e_A^B = \text{Ind}_{W_A}^{W_n} \left( \frac{1}{|W_A \cap W(B)|} |W_A \cap W(B)| e_A^B \right)
\]
\[
= \text{Ind}_{W_A}^{W_n} \left( \frac{1}{|W_A \cap W(B)|} \right) \cdot \text{Ind}_{W_B}^{W_n} e_B^B
\]
\[
= \frac{|W(B)|}{|W_A \cap W(B)|} e_B.
\]

Furthermore, there is also the equality \( \text{Ind}_{W_A}^{W_n} e_A^B = |N_{W_n}(W_B) : N_{W_n}(W_B)| e_B \).

**Theorem 16.** Let \( A, B \in \mathcal{S}(n) \) be such that \( \lambda(B) \subset \lambda(A) \). If \( B_1, B_2, \ldots, B_r \) are the representatives of the \( W_A \)-equivalence classes of subsets of \( A \) which are \( W_n \)-equivalent to \( B \), then for \( \text{cox}_B \in W_n \),
\[
\text{Ind}_{W_A}^{W_n} 1_A(\text{cox}_B) = \sum_{i=1}^{r} \frac{|W(B)|}{|W_A \cap W(B_i)|}.
\]
Proof. Let $A, B \in \mathcal{SC}(n)$. If $A \equiv B$ then it is easy to prove that $|W(A)| = |W(B)|$. We write $1_A = \sum_E e_E^2$, where $E \in \mathcal{SC}(n)$ runs over $W_A$-conjugacy classes of subsets of $A$. From Proposition [15] we have

$$\text{Ind}^{W_n}_{W_A} 1_A = \sum_E \text{Ind}^{W_n}_{W_A} e_E^2 = \text{Ind}^{W_n}_{W_A} 1_A = \sum_E \frac{|W(E)|}{|W_A \cap W(E)|} \cdot e_E.$$

Since each $B_i$ is $W_n$-equivalent to $B$, then $e_{E}(\text{cox}_B) = 1$ if and only if $E \equiv W_A B_i$. Thus we obtain that

$$\text{Ind}^{W_n}_{W_A} 1_A(\text{cox}_B) = \sum_{i=1}^r \frac{|W(B)|}{|W_A \cap W(B_i)|}.$$

Hence the theorem is proved. $\square$

Theorem [17] and Proposition [18] give us a useful computation to determine the coefficient of the sign character $\varepsilon_n$ in the expression of the orthogonal primitive idempotent $e_\lambda$, $\lambda \in \mathcal{DP}(n)$ in terms of irreducible characters of $W_n$.

**Theorem 17.** $u_{n(\emptyset), (\emptyset; 1, \ldots, 1)} = \frac{(-1)^n}{2^n}$.

**Proof.** Let $\chi_{\text{reg}} : W_n \to \mathbb{Z}$ be the regular character of $W_n$. For $A = (-1, \ldots, -1)$ it is satisfied $\text{Ind}^{W_n}_{W_A} 1_A = \chi_{\text{reg}}$. The character $\varepsilon_n$ is contained in $\chi_{\text{reg}}$ with the property that its coefficient is just 1, thus we have

$$\langle \text{Ind}^{W_n}_{W_A} 1_A, \varepsilon_n \rangle = 1.$$

Now let $A \neq (-1, \ldots, -1)$. By using Frobenius Reciprocity and the formula $\text{res}^{W_n}_{W_A} \varepsilon_n = \varepsilon_A$, it is obtained that $\langle \text{Ind}^{W_n}_{W_A} 1_A, \varepsilon_n \rangle = 0$. If $w$ is conjugate to $\text{cox}_{W_n}$ under $W_n$, then we have $e_{\langle n; \emptyset \rangle}(w) = 1$ and $e_n(w) = \varepsilon_n(\text{cox}_{W_n}) = (-1)^{l(w)} = (-1)^n$. Let $\text{ccl}_{W_n}(<\text{cox}_{W_n}>)$ denote the conjugacy class of $\text{cox}_{W_n}$ in $W_n$. By considering the formula $|\text{ccl}_{W_n}(\text{cox}_{W_n})| = \frac{|W_n|!}{n}$ in [14], we obtain

$$\langle e_{\langle n; \emptyset \rangle}, \varepsilon_n \rangle = \frac{(-1)^n}{2^n},$$

On the other hand, $\langle e_{\langle n; \emptyset \rangle}, \varepsilon_n \rangle = \sum_{\mu \in \mathcal{DP}(n)} u_{\langle n; \emptyset \rangle, \mu} \langle \varphi_{\mu}, \varepsilon_n \rangle = u_{\langle n; \emptyset \rangle, (\emptyset; 1, \ldots, 1)}$ and so the proof is completed. $\square$

**Proposition 18.** For $\lambda \in \mathcal{DP}(n)$ and $\lambda \neq (n; \emptyset)$, then we have

$$u_{\lambda, (\emptyset; 1, \ldots, 1)} = (-1)^{|S_{\lambda}|} \cdot \frac{|K_{\lambda}|}{|W_n|}.$$

**Proof.** Since the sign character is constant on the conjugacy classes, then we have

$$\langle e_{\lambda}, \varepsilon_n \rangle = \frac{1}{|W_n|} \sum_{w \in K_{\lambda}} (-1)^{l(w)} (\text{rank} W_{\lambda} = |S_{\lambda}|)$$

$$= (-1)^{|S_{\lambda}|} \cdot \frac{|K_{\lambda}|}{|W_n|}.$$
Note that \( \langle \varphi_\mu, \varepsilon_n \rangle \) has value 1 for \( \mu = (0; 1, \cdots, 1) \) and zero for the others. Henceforth, we obtain \( \langle \lambda, \varepsilon_n \rangle = \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda \mu} \langle \varphi_\mu, \varepsilon_n \rangle = u_{\lambda,(\emptyset;1,\cdots,1)} \). Eventually, we have \( u_{\lambda,(\emptyset;1,\cdots,1)} = (-1)^{|S_\lambda|} \cdot \frac{|K_\lambda|}{|W|} \).

Notice that calculation of the inner product \( \langle e_\lambda, 1_{W_n} \rangle \) leads to the following corollary.

**Corollary 19.** Let \( \lambda \in \mathcal{DP}(n) \). Then
\[
|W_n| \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda \mu} = |K_\lambda|,
\]

By means of Corollary 19 and the matrix \((u_{\lambda \mu})_{\lambda, \mu \in \mathcal{DP}(n)}\), one can readily determine the sizes of all the conjugacy classes of \( W_n \).

**Theorem 20.** Let \( A \in \mathcal{SC}(n) \) and \( \lambda \in \mathcal{DP}(n) \). Then
\[
\sum_{\mu \in \mathcal{DP}(n)} u_{\lambda \mu} a_{\hat{\mu} A}(-1, \cdots, -1) = (-1)^{|S_\lambda|} \frac{|K_\lambda \cap W_A|}{|W_A|},
\]

where \( a_{\hat{\mu} A}(-1, \cdots, -1) = \{x \in D_{\hat{\mu} A} : x^{-1} \hat{\mu} \cap A = (-1, \cdots, -1)\} \).

**Proof.** The term \( d_{(-1, \cdots, -1)} \) in the multiplication \( d_A d_A \) lies in the summand \( \sum_{x \in D_{\hat{\mu} A}} d_{f_{\hat{\mu} A}(x)} \) from the structure of \( \text{Ker} \Phi_n \) and part (i) of Proposition 1. If we write the coefficient of \( d_{(-1, \cdots, -1)} \) in this summand as \( a_{\hat{\mu} A}(-1, \cdots, -1) \), and so we get
\[
a_{\hat{\mu} A}(-1, \cdots, -1) = \{x \in D_{\hat{\mu} A} : f_{\hat{\mu} A}(x) = (-1, \cdots, -1)\}.
\]

By using part (i) of Proposition 1 along with the fact \( f_{\hat{\mu} A}(x) \equiv_A x^{-1} \hat{\mu} \cap A \), it is seen that there is the equivalence \( x^{-1} \hat{\mu} \cap A \equiv A (-1, \cdots, -1) \). Since no element in \( \mathcal{SC}(n) \) is congruent to \((1, \cdots, 1)\) except for \((-1, \cdots, -1)\), it then follows that \( x^{-1} \hat{\mu} \cap A = (-1, \cdots, -1) \). Hence we have deduced the equality \( a_{\hat{\mu} A}(-1, \cdots, -1) = \{x \in D_{\hat{\mu} A} : x^{-1} \hat{\mu} \cap A = (-1, \cdots, -1)\} \) holds. Therefore, by Frobenius Reciprocity and Mackey Theorem, we have
\[
\langle e_\lambda, \text{ind}_{W_A}^W e_A \rangle = \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda \mu} \sum_{x \in D_{\hat{\mu} A}} \langle \text{ind}_{W_{x^{-1} \hat{\mu} \cap A}}^W 1_{x^{-1} \hat{\mu} \cap A} e_A \rangle
\]
\[
= \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda \mu} \sum_{x \in D_{\hat{\mu} A}} 1_{x^{-1} \hat{\mu} \cap A}
\]
\[
= \sum_{\mu \in \mathcal{DP}(n)} u_{\lambda \mu} a_{\hat{\mu} A}(-1, \cdots, -1).
\]

Also, \( \varepsilon_n(w) \) is the same value for every \( w \in K_{\lambda} \) and so \( \varepsilon_n(c_{\lambda}) = \varepsilon_n(\text{co}_{\lambda}) = (-1)^{|S_\lambda|} \).

Therefore, by Lemma 2, we have
\[
\langle e_\lambda, \text{ind}_{W_A}^W e_A \rangle = \frac{1}{|W_A|} \sum_{w \in K_{\lambda} \cap W_A} (-1)^{|A(w^{-1})|}
\]
\[
\frac{1}{|W_A|} \sum_{w \in K_\lambda \cap W_A} (-1)^{\ell(w)} = \frac{1}{|W_A|} (1)^{|S_\lambda|} |K_\lambda \cap W_A|
\]

Putting these two results together, we see that theorem is proved. \(\square\)

5. Example

We consider the Coxeter group \(W_3\). For all \(\lambda, \mu \in \mathcal{D}(3)\), by means of the character table of \(\mathcal{M}(W_3)\) in [3], we can write the values \(\varphi_\lambda(\cos_\mu)\) as in the following table:

<table>
<thead>
<tr>
<th>(\nu(3;0))</th>
<th>(\nu(0;3))</th>
<th>(\nu(2;1;0))</th>
<th>(\nu(1;2))</th>
<th>(\nu(0;2;1))</th>
<th>(\nu(1;1;0))</th>
<th>(\nu(0;1;1;0))</th>
<th>(\nu(0;0;1;0))</th>
<th>(\nu(0;0;0;1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>8</td>
<td>8</td>
<td>24</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8</td>
<td>24</td>
<td>24</td>
<td>48</td>
</tr>
</tbody>
</table>

The matrices \((u_{\lambda,\mu})_{\lambda,\mu \in \mathcal{D}(n)}\) is

\[
\begin{pmatrix}
1 & -1/2 & -1 & 0 & 0 & 1/2 & 1/3 & 0 & 0 & -1/6 \\
0 & 1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 1/6 \\
0 & 0 & 1 & -1/2 & -1/2 & 1/4 & -1/2 & 1/4 & 1/4 & -1/8 \\
0 & 0 & 0 & 1/2 & 0 & -1/4 & 0 & -1/4 & 0 & 1/8 \\
0 & 0 & 0 & 0 & 1/2 & -1/4 & 0 & 0 & -1/4 & 1/8 \\
0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 0 & 0 & -1/8 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/6 & -1/4 & 1/8 & -1/48 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & -1/4 & 1/16 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & -1/16 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/48
\end{pmatrix}
\]

For \(\lambda = (3;0), (2;1;\emptyset), (1,1,1;\emptyset) \in \mathcal{D}(3)\), the size of \(K_\lambda\) is calculated by means of Corollary [19] and matrix \((u_{\lambda,\mu})_{\lambda,\mu \in \mathcal{D}(n)}\) the above. Since \(|K_{(3;0)}| = 8, |K_{(2;1;\emptyset)}| = 6\) and \(|K_{(1,1,1;\emptyset)}| = 1\), then we have found that the number of elements of type \(S_3\) is \(|C(S_3)| = 15\).

Acknowledgment. We would like to thank the referee for useful comments and corrections.

References


Current address: Department of Mathematics, Faculty of Science, Erciyes University, 38039, Kayseri, Turkey.
E-mail address: hasanarslan@erciyes.edu.tr
ORCID Address: http://orcid.org/0000-0002-0430-8737

Current address: Department of Mathematics, Faculty of Science, Erciyes University, 38039, Kayseri, Turkey.
E-mail address: can@erciyes.edu.tr
ORCID Address: http://orcid.org/0000-0001-8485-6815