Some Fixed Point Results for Multi Valued Mappings in Ordered G-Metric Spaces

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ABSTRACT

Using the setting of G− metric spaces, some new fixed point theorems for multivalued monotone mappings in ordered G− metric space X are proved, where the partial ordered \( \leq \) in X is obtained by a pair of functions \((\psi, \varphi)\).

Key Words: Common fixed point, generalized weak contractive condition, lower semicontinuous functions, G− metric space.

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1. INTRODUCTION AND PRELIMINARIES

Many authors studied many fixed and common fixed points in metric and order metric spaces. Dhage introduced the concept of D-metric spaces and studied several fixed point results (see [1]-[4]). Mustafa and Sims [5] showed that the structure of D metric spaces didn’t generate a metric space. They introduced a new concept of generalized metric spaces, called G-metric spaces. Since then many authors introduced many fixed and common fixed point results using the concept of G-metric spaces (see [5]-[25]).

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In 1976, Caristi [26] defined an order relation in a metric space by using a functional

as follows: Let \((X, d)\) be a metric space, \(\varphi : X \to \mathbb{R}\) be a functional. Define the relation \(\leq\) on \(X\) by

\[ x \leq y \text{ iff } d(x, y) \leq \varphi(x) - \varphi(y). \]

Then \(\leq\) is a partial order relation on \(X\) introduced by \(\varphi\)

and \((X, \leq)\) is called an ordered metric space introduced by \(\varphi\). After that many authors discussed the existence of a fixed point and a common fixed point using Caristi type mapping (see [26]-[31]). Consistent with Mustafa and Sims [6], the following definitions and results will be needed in the sequel.

**Definition 1.1.** Let \(X\) be a nonempty set. Suppose that a mapping \(G : X \times X \times X \to \mathbb{R}^+\) satisfies

\begin{enumerate}[(G1)]
    \item \(G(x, y, z) = 0\) if \(x = y = z\);
    \item \(0 < G(x, y, z)\) for all \(x, y, z \in X\) with \(x \neq y\)
\end{enumerate}

(G3) \(G(x, x, y) \leq G(x, y, z)\) for all \(x, y, z \in X\) with \(y \neq z\)

(G4) \(G(x, y, z) = G(x, z, y) = G(y, x, z) = \ldots\)

(symmetry in all three variables); and

(G5) \(G(x, y, z) \leq G(x, a, a) + G(a, y, z)\) for all \(x, y, z, a \in X\).

Then \(G\) is called a \(G\)-metric on \(X\) and \((X, G)\) is called a \(G\)-metric space.

**Definition 1.2.** A sequence \(\{x_n\}\) in a \(G\)-metric space \(X\) is:

(i) a \(G\)-Cauchy sequence if for any \(\varepsilon > 0\), there is a natural number \(n_0 \in \mathbb{N}\) such that for all \(n, m, l \geq n_0\), \(G(x_n, x_m, x_l) < \varepsilon\),

(ii) a \(G\)-convergent sequence if for any \(\varepsilon > 0\), there is an \(x \in X\) and an \(n_0 \in \mathbb{N}\) such that for all \(n, m \geq n_0\), \(G(x_n, x_m, x) < \varepsilon\).

A \(G\)-metric space on \(X\) is said to be \(G\)-complete if every \(G\)-Cauchy sequence in \(X\) is \(G\)-convergent in \(X\). It is known that \(\{x_n\}\) \(G\)-converges to \(x \in X\) if and only if

\[ G(x_n, x_m, x) \to 0 \text{ as } n, m \to +\infty. \]

**Proposition 1.3.** [6] Let \(X\) be a \(G\)-metric space. Then the following are equivalent:

1. The sequence \(\{x_n\}\) is \(G\)-convergent to \(x\).
2. \(G(x_n, x_m, x) \to 0\) as \(n \to +\infty\).
3. \(G(x_n, x, x) \to 0\) as \(n \to +\infty\).
4. \(G(x_n, x_m, x) \to 0\) as \(n, m \to +\infty\).

**Proposition 1.4.** [6] Let \(X\) be a \(G\)-metric space. Then the following are equivalent:

1. The sequence \(\{x_n\}\) is \(G\)-Cauchy.
2. For every \(\varepsilon > 0\), there exists \(n_0 \in \mathbb{N}\), such that for all \(n, m \geq n_0\), \(G(x_n, x_m, x) < \varepsilon\); that is \(G(x_n, x_m, x) \to 0\) as \(n, m \to +\infty\).

**Definition 1.5.** A \(G\)-metric on \(X\) is said to be symmetric if \(G(x, y, y) = G(x, y, y)\) for all \(x, y \in X\).

**Proposition 1.6.** Every \(G\)-metric on \(X\) will define a metric \(d_G\) on \(X\) by

\[ d_G(x, y) = G(x, y, y) + G(y, x, x) \text{ for all } x, y \in X. \]

For a symmetric \(G\)-metric space, one obtains

\[ d_G(x, y) = 2G(x, y, y) \text{ for all } x, y \in X. \]

However, if \(G\) is not symmetric, then the following inequality holds:

\[ \frac{3}{2} G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y) \text{ for all } x, y \in X. \]

**Definition 1.7.** The two classes of following mappings are defined as

\[ \Phi = \{\varphi / \varphi : [0, +\infty) \to [0, +\infty) \text{ is lower semi continuous, } \varphi(t) > 0 \text{ for all } t > 0, \varphi(0) = 0\}. \]
\[ \Psi = \{ \psi / \psi : [0, +\infty) \to [0, +\infty) \text{ is continuous and nondecreasing with } \psi(t) = 0 \text{ if and only if } t = 0 \} \]  

Using the setting of \( G \)-metric spaces, some new fixed point theorems for multivalued monotone mappings in ordered \( G \)-metric space \( X \) are proved, where the partial ordered \( \leq \) in \( X \) is obtained by a pair of functions \( (\psi, \phi) \).

### 2. MAIN RESULTS

Throughout this paper, we let \( \psi : [0, +\infty) \to [0, +\infty) \) be a function with following properties:

1. \( \psi \) is nondecreasing continuous.

2. \( \psi^{-1}(\{0\}) = \{0\} \).

3. \( \psi(a + b) \leq \psi(a) + \psi(b) \) for all \( a, b \in [0, +\infty) \).

Let \( (X, G) \) be a \( G \)-metric space, define a relation \( \leq \) by using functional \( \phi : X \to R \) and \( \psi \) as follows:

\[ x \leq y \text{ iff } \psi(G(x, y, y)) \leq \phi(x) - \phi(y) \]

for all \( x, y \in X \). Then it is an easy matter to prove the following lemma:

**Lemma 2.1** \( \leq \) is partial order and \( (X, \leq) \) is a partial ordered set.

**Proof:** \( \leq \) is reflexive because \( \psi(G(x, x, x)) = \phi(x) - \phi(x) \) for all \( x \in X \).

\( \leq \) is antisymmetric because if \( x, y \in X \) with \( x \leq y \) and \( y \leq x \), then

\[ \psi(G(x, y, y)) \leq \phi(x) - \phi(y) \]

and

\[ \psi(G(y, x, x)) \leq \phi(y) - \phi(x) \]

Thus

\[ \psi(G(x, y, y)) + \psi(G(y, x, x)) = 0. \]

Hence \( \psi(G(x, y, y)) = \psi(G(y, x, x)) = 0 \). Therefore \( G(x, y, x) = 0 \) and hence \( x = y \).

\( \leq \) is transitive because if \( x, y, z \in X \) with \( x \leq y \) and \( y \leq z \), then

\[ \psi(G(x, y, y)) \leq \phi(x) - \phi(y) \]

and

\[ \psi(G(y, z, z)) \leq \phi(y) - \phi(z) \]

Thus

\[ \psi(G(x, y, y)) + \psi(G(y, z, z)) \leq \phi(x) - \phi(z). \]

Using (G5) of the definition \( G \)-metric space and property (3) of the function \( \psi \), we get

\[ \psi(G(x, z, z) \leq \psi(G(x, y, y) + G(y, z, z)) \]

\[ \leq \psi(G(x, y, y)) + \psi(G(y, z, z)) \]

\[ \leq \phi(x) - \phi(z) \]

Thus, we have \( x \leq z \).

From now on, we let \( (X, G, \leq) \) be an ordered \( G \)-metric space introduced by \( (\psi, \phi) \).

Let \( (X, G, \leq) \) be an ordered \( G \)-metric space introduced by \( (\psi, \phi) \). For \( x, y \in X \) we define the ordered interval in \( X \) as:

\[ [x, y] = \{ z \in X : x \leq z \leq y \}, \]

\[ [x, +\infty) = \{ z \in X : x \leq z \}, \]

\[ (-\infty, x] = \{ z \in X : z \leq x \}. \]

Let \( F : X \to 2^X \) be a multivalued mapping, we say that \( F \) is upper semi-continuous if whenever \( x_n \in X \) and \( y_n \in F(x_n) \) with \( x_n \to x_0 \in X \) and \( y_n \to y_0 \in X \), then \( y_0 \in F(x_0) \).

Our first result is:

**Theorem 2.1** Let \( (X, G, \leq) \) be an ordered complete \( G \)-metric space introduced by \( (\psi, \phi) \), where \( \phi : X \to R \) be a function bounded below. Let \( F : X \to 2^X \) be a multivalued mapping and

\[ M = \{ x \in X : F(x) \cap [x, +\infty) \neq \phi \} \]

Suppose that:

i. \( F \) is upper semi-continuous;

ii. for each \( x \in M \), \( F(x) \cap M \cap [x, +\infty) \neq \phi \);  

iii. \( M \neq \phi \).

Then there exists a sequence \( \{x_n\} \) with

\[ x_{n+1} \leq x_n \in F(x_{n+1}), \quad \forall \ n \in N, \]
and $F$ has a fixed point $x^*$ such that $x_n \rightarrow x^*$. Moreover if $\varphi$ is lower semi-continuous, then $x_n \leq x^*$ for all $n$.

**Proof:** Since $M \neq \emptyset$, we choose $x_0 \in M \subseteq X$.

By (ii), we have

$$F(x_0) \cap M \cap [x_0, +\infty) \neq \emptyset.$$  

Thus we choose

$$x_1 \in F(x_0) \cap M \cap [x_0, +\infty).$$

Therefore $x_0 \leq x_1$. Again by (ii), we have

$$F(x_1) \cap M \cap [x_1, +\infty) \neq \emptyset.$$  

Thus, we choose

$$x_2 \in F(x_1) \cap M \cap [x_1, +\infty).$$

Hence $x_1 \leq x_2$. Continuing in the same process, we construct a sequence $(x_n)$ in $X$ such that

$$x_{n-1} \leq x_n \in F(x_{n-1}), \quad \forall \ n \in N.$$  

Since $(X, G, \leq)$ is an ordered $G$-metric space introduced by $(\psi, \varphi)$, we get that

$$\psi(G(x_{n-1}, x_n)) \leq \varphi(x_{n-1}) - \varphi(x_n).$$  

Since $\psi$ is a nonnegative function, we get that

$$\varphi(x_{n-1}) - \varphi(x_n) \geq 0 \quad \forall \ n \in N.$$  

Thus

$$\varphi(x_{n-1}) \geq \varphi(x_n) \quad \forall \ n \in N.$$  

Since $\varphi$ is a function which is bounded below, we have ($\varphi(x_n)$) is a decreasing sequence which is bounded below. By completeness property of $\mathbb{R}$, we have

$$\lim_{n \rightarrow +\infty} \varphi(x_n) = \inf \{x_n : n \in N\}.$$  

For $m > n$, we have $x_n \leq x_m$. Thus, we get

$$\psi(G(x_n, x_m)) \leq \varphi(x_n) - \varphi(x_m).$$

Let $n, m \rightarrow +\infty$, then

$$\lim_{n, m \rightarrow +\infty} \psi(G(x_n, x_m)) \leq \lim_{n \rightarrow +\infty} \varphi(x_n) - \lim_{m \rightarrow +\infty} \varphi(x_m).$$

Thus

**Proof:** Let

$$\psi(G(x_n, x_m)) = 0.$$  

Using the continuity of $\psi$ and the fact that $\psi^{-1}(\{0\}) = \{0\}$, we get that

$$\lim_{n, m \rightarrow +\infty} G(x_n, x_m) = 0.$$  

Hence $(x_n)$ is a Cauchy sequence in $X$. Since $X$ is $G$-complete, then there is $x^* \in X$ such that $(x_n)$ is $G$-convergent to $x^*$. Since $x_{n-1} \in X, x_n \in F(x_{n-1})$, $x_{n-1} \rightarrow x^*$ and $x_n \rightarrow x^*$ by definition of upper semi-continuous of $F$, we have $x^* \in F(x^*)$. Now, suppose that $\varphi$ is lower semi-continuous, then for each $n \in N$, we have

$$\psi(G(x_n, x^*, x^*)) \leq \limsup_{m \rightarrow +\infty} \varphi(x_n) - \varphi(x_m)$$

$$= \varphi(x_n) - \liminf_{m \rightarrow +\infty} \varphi(x_m) \leq \varphi(x_n) - \varphi(x^*).$$

Therefore $x_n \leq x^*$ for all $n \in N$. 

**Corollary 2.1** Let $(X, G, \leq)$ be an ordered complete $G$-metric space introduced by $(\psi, \varphi)$, where $\varphi : X \rightarrow \mathbb{R}$ be a function bounded below. Let $F : X \rightarrow 2^X$ be a multivalued mapping. Suppose that:

i. $F$ is upper semi-continuous;

ii. $F$ satisfies the monotonic condition: For each $x, y \in X$ with $x \leq y$ and any $u \in F(x)$, there exists $v \in F(y)$ such that $u \leq v$;

iii. There exists $x_0 \in X$ such that $F(x_0) \cap [x_0, +\infty) \neq \emptyset$.

Then there exists a sequence $(x_n)$ in $X$ with $x_{n-1} \leq x_n \in F(x_{n-1})$, $\forall \ n \in N$, and $F$ has a fixed point $x^*$ such that $x_n \rightarrow x^*$. Moreover if $\varphi$ is lower semi-continuous, then $x_n \leq x^*$ for all $n$.

$$M = \{x \in X : F(x) \cap [x, +\infty) \neq \emptyset\}.$$
By (iii) we conclude that $M \neq \phi$. For $x \in M$, take $y \in F(x)$ and $x \leq y$. Since $F$ satisfies the monotonic condition, there exist $z \in F(y)$ such that $y \leq z$. Thus $y \in M$, and $F(x) \cap M \cap [x, +\infty) \neq \phi$. Thus we get the result from Theorem 2.1. 

**Corollary 2.2** Let $(X, G, \leq)$ be an ordered complete G-metric space introduced by $(\psi, \varphi)$, where $\varphi : X \rightarrow R$ be a function bounded below. Let $f : X \rightarrow X$ be a map.

Suppose that:

i. $f$ is continuous.

ii. $f$ is monotone increasing.

iii. There exists $x_0 \in X$ such that $x_0 \leq f(x_0)$.

Then there exists a sequence $(x_n)$ in $X$ with $x_{n-1} \leq x_n \in f(x_{n-1})$, $\forall n \in N$, and $f$ has a fixed point $x^*$ such that $x_n \rightarrow x^*$. Moreover if $\varphi$ is lower semi-continuous, then $x_n \leq x^*$ for all $n$.

**Proof**: Define $F : X \rightarrow 2^X$ by $F(x) = \{f(x)\}$ for all $x \in X$. Then $F$ and $X$ satisfy all the hypotheses of Theorem 2.1. Thus the result follows from Theorem 2.1. 

**Theorem 2.2** Let $(X, G, \leq)$ be an ordered complete G-metric space introduced by $(\psi, \varphi)$, where $\varphi : X \rightarrow R$ be a function bounded above. Let $F : X \rightarrow 2^X$ be a multivalued mapping and $M = \{x \in X : F(x) \cap (-\infty, x] \neq \phi\}$. Suppose that:

i. $F$ is upper semi-continuous;

ii. for each $x \in M$, $F(x) \cap M \cap (-\infty, x] \neq \phi$;

iii. $M \neq \phi$.

Then there exists a sequence $(x_n)$ with $x_{n-1} \geq x_n \in F(x_{n-1})$, $\forall n \in N$, and $F$ has a fixed point $x^*$ such that $x_n \rightarrow x^*$. Moreover if $\varphi$ is lower semi-continuous, then $x_n \geq x^*$ for all $n$.

**Proof**: Since $M \neq \phi$, we choose $x_1 \in F(x_0) \cap M \cap (-\infty, x_0]$. Therefore $x_0 \geq x_1$. Again by (ii), we choose $x_2 \in F(x_1) \cap M \cap (-\infty, x_1]$. Hence $x_1 \geq x_2$. Continuing in the same process, we construct a sequence $(x_n)$ in $X$ such that $x_{n-1} \geq x_n \in F(x_{n-1})$, $\forall n \in N$.

Since $(X, G, \leq)$ is an ordered G-metric space introduced by $(\psi, \varphi)$, we get that $\psi(G(x_n, x_{n-1}), x_n)) \leq \varphi(x_n) - \varphi(x_{n-1})$. Since $\psi$ is a nonnegative function, we get that $\varphi(x_n) - \varphi(x_{n-1}) \geq 0$ $\forall n \in N$.

Thus $\varphi(x_n) \geq \varphi(x_{n-1})$ $\forall n \in N$.

Since $\varphi$ be a function which is bounded above, we have $(\varphi(x_n))$ is an increasing sequence which is bounded above. By completeness property of $R$, we have

$$\lim_{n \rightarrow +\infty} \varphi(x_n) = \sup\{x_n : n \in N\}.$$ 

For $m > n$, we have $x_n \geq x_m$. Thus, we get

$$\psi(G(x_m, x_n, x_n)) \leq \varphi(x_m) - \varphi(x_n).$$

Let $n, m \rightarrow +\infty$, then

$$\lim_{n, m \rightarrow +\infty} \psi(G(x_m, x_n, x_n)) \leq \lim_{m \rightarrow +\infty} \varphi(x_m) - \lim_{n \rightarrow +\infty} \varphi(x_n).$$

Thus

$$\lim_{n, m \rightarrow +\infty} \psi(G(x_m, x_n, x_n)) = 0.$$ 

Using the continuity of $\psi$ and the fact that $\psi^{-1}(\{0\}) = \{0\}$, we get that

$$\lim_{n, m \rightarrow +\infty} G(x_m, x_n, x_n) = 0.$$
Hence \((x_n)\) is a Cauchy sequence in \(X\). Since \(X\) is \(G\)-complete, then there is \(x^* \in X\) such that 
\[
(x_n) \rightarrow x^*. \quad \text{Since} \quad x_{n+1} \in X, x_n \in F(x_{n-1}), \quad x_{n+1} \rightarrow x^* \quad \text{and} \quad x_n \rightarrow x^*,
\]
by definition of upper semi-continuous of \(F\), we have \(x^* \in F(x^*)\). Now, suppose that \(\phi\) is lower semi-continuous, then for each \(n \in \mathbb{N}\), we have
\[
\psi(G(x^*, x_n, x_n)) = \lim_{n \rightarrow +\infty} \psi(G(x_m, x_n, x_n)) \leq \limsup_{n \rightarrow +\infty} \phi(x_m) - \phi(x_n) \leq \phi(x^*) - \phi(x_n).
\]
Therefore \(x_n \geq x^*\) for all \(n \in \mathbb{N}\).

**Corollary 2.3** Let \((X, G, \leq)\) be an ordered complete \(G\)-metric space introduced by \((\psi, \phi)\),
where \(\phi : X \rightarrow R\) be a function bounded above. Let \(F : X \rightarrow 2^X\) be a multivalued mapping
Suppose that:

i. \(F\) is upper semi-continuous;

ii. \(F\) satisfies the monotonic condition: For each \(x, y \in X\) with \(x \geq y\) and any \(u \in F(x)\),
there exists \(v \in F(y)\) such that \(u \geq v\).

iii. There exists \(x_0 \in X\) such that \(F(x_0) \cap (-\infty, x_0] \neq \phi\).

Then there exists a sequence \((x_n)\) in \(X\) with 
\[
x_{n+1} \geq x_n \in F(x_{n-1}), \quad \forall \ n \in \mathbb{N},
\]
and \(F\) has a fixed point \(x^*\) such that \(x_n \rightarrow x^*\).

Moreover if \(\phi\) is lower semi-continuous,
then \(x_n \geq x^*\) for all \(n\).

**Proof :** Let 
\[
M = \{x \in X : F(x) \cap (-\infty, x] \neq \phi\}.
\]
By (iii) we conclude that \(M \neq \phi\). For \(x \in M\), take \(y \in F(x)\) and \(x \geq y\). Since \(F\) satisfies the monotonic condition, there exist \(z \in F(y)\) such that \(y \geq z\). Thus \(y \in M\), and
\[
F(x) \cap M \cap (-\infty, x] \neq \phi. \quad \text{Thus we get the result from Theorem 2.2.}
\]

**Corollary 2.4** Let \((X, G, \leq)\) be an ordered complete \(G\)-metric space introduced by \((\psi, \phi)\),
where \(\phi : X \rightarrow R\) be a function bounded above. Let 
\(f : X \rightarrow X\) be a map.
Suppose that:

i. \(f\) is continuous.

ii. \(f\) is monotone increasing.

iii. There exists \(x_0 \in X\) such that \(x_0 \geq f(x_0)\).

Then there exists a sequence \((x_n)\) in \(X\) with 
\[
x_{n+1} \geq x_n \in f(x_{n-1}), \quad \forall \ n \in \mathbb{N},
\]
and \(f\) has a fixed point \(x^*\) such that \(x_n \rightarrow x^*\).

Moreover if \(\phi\) is lower semi-continuous,
then \(x_n \geq x^*\) for all \(n\).

**Proof :** Define \(F : X \rightarrow 2^X\) by \(F(x) = \{f(x)\}\) for all \(x \in X\). Then \(F\) and \(X\) satisfy all the hypotheses of Theorem 2.2. Thus the result follows from Theorem 2.2.

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