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Transformation on Diffusion Processes and First Passage Time to The Moving Boundaries

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ABSTRACT

In this paper, we study the first passage time of a diffusion process to a moving boundary. Under some special conditions we apply a transformation to diffusion process and to the boundary function and then in each case obtain the first passage time distribution of the original process by the first passage time distribution of transformed process to transformed boundary. In addition, by applying these transformations to the Ornstein-Uhlenbeck and Wiener processes the first passage time distributions for the new boundaries are presented as examples.

Keywords: Diffusion process; Ornstein-Uhlenbeck process; First passage times; Moving boundaries; Lambert function; Hyperbolic function; Fixed boundaries

1. INTRODUCTION

The first passage time for moving boundary has a wide application area in the literature such as modeling neuron cells [12, 20], animal movements [16], mathematical finance [10], in obtaining estimators [7], and chemical physics Szabo et al.[22], Hänggi et al. [23]. Despite its importance in the literature, the exact forms of first passage time distributions are not known except some special cases whereas the exact forms known it is often encountered some challenges such as being enormous difficulties with the calculation.

Dominé [8] obtained the moments of first passage times for Brownian motions with a drift through an elastic boundary. These results are generalized by Wang and Yin [21] to time homogeneous diffusion processes and they also obtained a recurrence relation among the moments of first passage times.

Giorno et al. [13], studied the asymptotic properties of first passage times for some special boundaries, also including the periodic ones. In addition, they applied these results to a new diffusion process which is obtained by the spatial transformation of Ornstein-Uhlenbeck process. Durbin [9], presented a general theorem for the first passage time to a continuous boundary of the Gaussian processes who have positive definite covariance functions and have first order continuous partial derivatives, and have a finite limit for the rate of the variance of their increments to the elapsed time. Using this result, Salminen [19] obtained the distribution of first passage time of a linear and quadratic boundary functional for the Brownian process. In addition, in [19] it is also mentioned the last passage time probabilities to linear and square root boundaries for the Brownian motion. First passage times to quadratic boundaries for

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the Brownian process are also investigated by Martin-Löf [15].

In the investigation of first passage time distributions, Fortet [11] and Buonocore et al. [4] proposed the integral equations for the probabilities of the first passage times.

In his study Kolmogorov [14] showed that the new processes obtained by the transformations applied on the diffusion processes are again diffusion processes and obtained their drift and diffusion coefficients.

Using this result Cherkasov [6] and Ricciardi [17] examined the transformation of diffusion processes to Brownian process whereas Capocelli and Ricciardi [5] used the same transformations to convert the diffusion processes to a Feller process. Ricciardi and Sato [18] studied the distributions of first passage time by these transformations. However, they confined themselves only to the diffusion processes which can be transformed into Wiener processes. In this study, applications of similar transformations to a wider family of diffusion processes are discussed.

2. PRELIMINARY DISCUSSIONS

Let X_t be a diffusion process with a diffusion $a(t,x)$ and a drift $b(t,x)$. Let $g:\mathbb{R}\to\mathbb{R}$ be a continuous function and denote the first passage time of the process X_t to the boundary g with τ_g^X; that is,

$\tau_g^X = \inf\{t>0: X_t = g(t); X_0 = x_0\}$.

By the following transformation of the process X_t, let us define a new stochastic process Y_t,

$$s' = \varphi(s),\ \ x' = \Psi(x,t) \tag{1}$$

where the continuous functions $\varphi:\mathbb{R}\to\mathbb{R}$ and $\Psi:\mathbb{R}^2\to\mathbb{R}$ satisfy the following features:

1. Let the derivative $\varphi'(t)$ and the partial derivatives
$$\begin{aligned}\Psi_x(x,t) &:= \frac{\partial}{\partial x}\Psi(x,t)\\ \Psi_t(x,t) &:= \frac{\partial}{\partial t}\Psi(x,t)\\ \Psi_{xx}(x,t) &:= \frac{\partial^2}{\partial x^2}\Psi(x,t)\end{aligned}$$
be definite for all x and t, and
2. $\varphi'(t) > 0$ for all t,
3. $\Psi_x(x,t) \neq 0$ for all x and t,
4. $\Psi(\Psi^{-1}(x,t),t) = \Psi^{-1}(\Psi(x,t),t) = x$ for all x.

Moreover let denote the diffusion and drift coefficient of this new process with $\bar{a}(t,x)$ and $\bar{b}(t,x)$. Then, we get the following relations:

$$\bar{a}(t,x) = \frac{a(t,x)\varphi'(t)}{\Psi_x^2(x,t)} \tag{2}$$

$$\bar{b}(t,x) = \left[b(t,x)\varphi'(t) - \frac{a(t,x)\varphi'(t)}{\Psi_x^2(x,t)}\Psi_{xx}(x,t) - \Psi_t(x,t)\right]/\Psi_x(x,t) \tag{3}$$

The relations (2) and (3) are obtained by Kolmogorov [14]. Cherkasov [6], Capocelli and Ricciardi [5], and Ricciardi [17] also studied on the transformation of the original process into a Wiener or Feller process by these transformations.

If the first passage time of the process Y_t to the boundary $h(t) \equiv \Psi^{-1}(g(\varphi(t)),t)$ is denoted by

$$\tau_h^Y = \inf\{t>0: Y_t = h(t); Y_0 = h(0)\}$$

and its distribution is known, then the distribution properties for τ_g^X can be derived from that of τ_h^Y, since $\tau_g^X \overset{d}{=} \tau_h^Y$.

Above mentioned method is proposed by Ricciardi and Sato [18], but they investigated only the first passage time of diffusion processes which can be transformed into a Wiener process. Unlike Ricciardi and Sato [18], in place of the transformations into a Wiener process, in this study it is considered the transformations on the processes whose first passage time distributions are known.

According to some specific forms of $a(t,x)$ and $b(t,x)$, the following propositions show how the functions φ and Ψ can be selected. These propostions and corollaries are from Aksop [2].

Proposition 1. (Aksop [2]) The transformations Ψ and φ which provide the independence of (2) from t, and defined as above also satisfy the following equation

$$2\frac{\partial}{\partial t}\ln\Psi_x(x,t) = \frac{\partial}{\partial t}\ln a(t,x) + \frac{d}{dt}\ln\varphi'(t) \tag{4}$$

Corollary 1. (Aksop [2]) If $a(t,x)$ is independent of t, then the transformations Ψ and φ which are defined above and make (2) independent from t satisfy the following equations:

$$\begin{aligned}\Psi(x,t) &= c_1 x\sqrt{\varphi'(t)} + c_2\\ \Psi_t(x,t) &= c_1\frac{x\varphi''(t)}{2\sqrt{\varphi'(t)}}\\ \Psi_x(x,t) &= c_1\sqrt{\varphi'(t)}\\ \Psi_{xx}(x,t) &= 0\end{aligned}$$

where $c_1 > 0$ and $c_2 \in \mathbb{R}$. In this case, the diffusion and drift coefficients of the new process are given by

$$\begin{aligned}\bar{a}(t,x) &= \frac{1}{c_1^2}a(t,x)\\ \bar{b}(t,x) &= \frac{1}{c_1}b(t,x)\sqrt{\varphi'(t)} - x\frac{\varphi''(t)}{\varphi'(t)}\end{aligned}$$

Example 1. (*Brownian motion and square root boundary*) Let B_t be a standard Brownian motion and for a function g which will be defined later, let us look at the first passage time $\tau = \inf\{t>0: B_t = g(t); B_0 = 0\}$. So we have the diffusion coefficient $a(t,x) = 1/2$ and drift coefficient $b(t,x)=0$ for the process. Hence Corollary 1 gives

$$\bar{a}(x,t) = \frac{1}{2c_1^2}, \qquad \bar{b}(x,t) = -x\frac{\varphi''(t)}{2\varphi'(t)}.$$

Since we want $\bar{b}(x,t)$ to be independent of t, then the part $\varphi''(t)/(2\varphi'(t))$ must be a constant. Therefore we have

$$\varphi(t)=\frac{c_4}{2c_3}e^{2c_3t}+c_5 \tag{6}$$

where $c_3\neq 0$, $c_4>0$ and $c_5\in\mathbb{R}$. Using Corollary 1 once again we obtain

$$\Psi(x,t)=c_1xe^{c_3t}+c_2, \quad \Psi^{-1}(x,t)=\frac{e^{-c_3t}}{c_1}(x-c_2). \tag{7}$$

Let s be a constant and choose the function g such that $\Psi^{-1}(g(\varphi(t)),t)=s$. Then we get,

$$g(t)=\Psi(s,\varphi^{-1}(t))$$

Since $\varphi^{-1}(t)=\left[\ln\left(\frac{2c_3}{c_4}(t-c_5)\right)\right]/(2c_3)$ this gives

$$g(t)=\Psi(s,\varphi^{-1}(t))=c_1s\sqrt{\frac{2c_3}{c_4}(t-c_5)}+c_2. \tag{8}$$

Therefore, the first passage time of a diffusion process obtained by applying the transformations in (1) to standard Brownian motion has the same distribution with the first passage time of Brownian motion through a square root boundary in (8). Moreover, the process defined by $Y_t=\Psi^{-1}(B_{\varphi(t)},t)$ satisfies the following stochastic differential equation

$$dY_t=-c_3Y_tdt+\frac{1}{c_1}dB_t$$

which is an Ornstein-Uhlenbeck process. If we particularly choose $c_1=1,\ c_2=0,\ c_3=1,\ c_4=2,$ $c_5=-1$ and $s=1$, then we obtain the first passage time of Brownian motion through the boundary $\sqrt{t+1}$ and the transformation $Y_t=e^{-t}B_{e^{2t}-1}$. This transformation is the Doob's transformation which is also used in Breiman [3].

Proposition 2. (Aksop [2]) The transformation Ψ which provides the independence of (2) from x, and defined as above also satisfy the following equation

$$\Psi(x,t)=c_1\int\sqrt{a(t,x)}dx+c_2$$

where c_1 and c_2 are constants.

3. ORNSTEIN-UHLENBECK PROCESS AND LINEAR BOUNDARY

Let X_t be an Ornstein-Uhlenbeck process defined by the stochastic differential equation

$$dX_t=-X_tdt+dB_t, \quad X_0=0.$$

So we have diffusion and drift coefficients $a(x,t)=1/2$ and $b(x,t)=-x$ and

$$\bar{a}(t,x)=\frac{\varphi'(t)}{2\Psi_x^2(x,t)}, \tag{9}$$

$$\bar{b}(t,x)=\left[-x\varphi'(t)-\frac{\varphi'(t)}{2\Psi_x^2(x,t)}\Psi_{xx}(x,t)-\Psi_t(x,t)\right]/\Psi_x(x,t) \tag{10}$$

Let $k\neq 0$ and $l\in\mathbb{R}$ be constants and h be a function which will be defined later, and take

$$\bar{a}(t,x) = kh(t), \qquad \bar{b}(t,x) = lh(t).$$

In this case, for any function $\phi(t)\neq 0$, we should have $\Psi_x(x,t)=\phi(t)$. Thus we get

$$\Psi(x,t) = x\phi(t)+\zeta(t)+c_1, \quad \Psi_t(x,t) = x\phi'(t)+\zeta'(t), \quad \Psi_{xx}(x,t) = 0,$$

where $c_1\in\mathbb{R}$ and $\zeta:\mathbb{R}\to\mathbb{R}$ is a continuous function. Substituting these equations in (10) we obtain

$$-x\left(\frac{\varphi'(t)+\phi'(t)}{\phi(t)}\right)-\frac{\zeta(t)}{\phi(t)}=\frac{l\varphi'(t)}{2k\phi^2(t)}.$$

Since the right-hand side of this equation is independent of x, then the equation $\varphi'(t)=-\phi'(t)$ must be satisfied. Therefore it is true that

$$h(t)=-\frac{\phi'(t)}{2\phi^2(t)}$$

On the other hand, for any $c_2\in\mathbb{R}$ we have

$$\zeta(t)=\frac{l}{2k}\ln\phi(t)+c_2-c_1.$$

Therefore

$$\begin{aligned}\Psi(x,t) &= x\phi(t)+\frac{l}{2k}\ln\phi(t)+c_2,\\ \Psi^{-1}(x,t) &= \left(x-\frac{l}{2k}\ln\phi(t)-c_2\right)/\phi(t),\\ \varphi(t) &= \phi(t)+c_3, \quad c_3\in\mathbb{R}\end{aligned}$$

Hence the following are true for any constant $s\in\mathbb{R}$

$$\begin{aligned}g(t) &= \Psi(s,\varphi^{-1}(t))\\ &= \Psi(s,\phi^{-1}(t-c_3))\\ &= s\phi(\phi^{-1}(t-c_3))+\frac{l}{2k}\ln\phi(\phi^{-1}(t-c_3))+c_2\\ &= s(t-c_3)+\frac{l}{2k}\ln(t-c_3)+c_2\\ &= st+\frac{l}{2k}\ln(t-c_3)+c_4, \quad c_4=c_2-sc_3.\end{aligned}$$

In this case in which the function $\phi(t)$ can be properly selected, the distribution of the first passage time through the boundary s of the diffusion process Y_t defined by

$$dY_t=-l\frac{\phi'(t)}{2\phi^2(t)}dt-k\sqrt{\frac{\phi'(t)}{\phi^2(t)}}dB_t, \quad Y_0=\Psi^{-1}\left(X_{\varphi(0)},\varphi(0)\right)$$

will be the same that of the first passage time through the boundary $st+\frac{l}{2k}\ln(t-c_3)+c_4$ of the Ornstein-Uhlenbeck process.

If we particularly take $l = 0$, then we can obtain the distributional information of the first passage time

$$\tau = \inf\{t > 0: X_t = st + c_4; X_0 = 0\}$$

from that of the first passage time through the boundary s of the process

$$dY_t = kh(t)dB_t.$$

Particularly, choose $h(t) = 1/k$. Thus, for any constant $c_5 < 0$,

$$\phi(t) = \frac{k^2}{-t + c_5}$$

transforms the process into the standard Brownian motion.

4. ORNSTEIN-UHLENBECK PROCESS AND LAMBERT BOUNDARY

Consider the equalities (9) and (10). Unlike the previous section, for this time the diffusion and drift coefficients of the transformed process will be determined independent of t. By the Corollary 1 we have the following diffusion and drift coefficients, respectively

$$\begin{aligned} \bar{a}(x) &= \frac{1}{2c_1^2}, \\ \bar{b}(x) &= -x\left(\frac{1}{c_1}\sqrt{\varphi'(t)} + \frac{\varphi''(t)}{2\varphi'(t)}\right), \end{aligned}$$

and the following equalities for the transformations φ and Ψ

$$\begin{aligned} \Psi_x(x,t) &= c_1\sqrt{\varphi'(t)} \\ \Psi(x,t) &= c_1 x\sqrt{\varphi'(t)} + c_2 \\ \Psi_{xx}(x,t) &= 0 \\ \Psi_t(x,t) &= \frac{c_1 x\varphi''(t)}{2\sqrt{\varphi'(t)}} \end{aligned} \qquad (11)$$

Since $\bar{b}$ depends only on x, for a constant $c_3 > 0$, the following

$$\frac{1}{c_1}\sqrt{\varphi'(t)} + \frac{\varphi''(t)}{2\varphi'(t)} = c_3 \qquad (12)$$

should be satisfied. Consequently by the equation

$$dY_t = -c_3 Y_t dt + \frac{1}{c_1}dB_t, \qquad (13)$$

it is concluded that the process $Y_t = \Psi^{-1}(X_{\varphi(t)}, t)$ is again an Ornstein-Uhlenbeck process. The solution of (12) is given as follows:

$$\varphi(t) = \frac{c_1^2 c_3^2 c_5}{c_4 e^{c_3 t} + c_3 c_5} + c_1^2 c_3 \ln(c_4 e^{c_3 t} + c_3 c_5) + c_6, \quad c_6 \in \mathbb{R}$$

Substituting this result in (11) gives

$$\Psi(x,t) = \frac{c_1^2 c_4 x e^{c_3 t}}{\frac{c_4}{c_3}e^{c_3 t} + c_5} + c_2. \qquad (14)$$

On the other hand, since we want the following equality to be satisfied

$$\Psi^{-1}(g(\varphi(t)), t) = s,$$

where s is a constant, then

$$g(t) = s\frac{c_1^2 c_4 \exp\{c_3\varphi^{-1}(t)\}}{\frac{c_4}{c_3}\exp\{c_3\varphi^{-1}(t)\} + c_5} + c_2$$

will be implied. Let $L_\omega(\cdot)$ be the Lambert ω function. Then we have (necessary calculations are done with MATLAB ©)

$$\varphi^{-1}(t) = \frac{1}{c_3}\ln\left[-\frac{c_3 c_5\left(1 + L_\omega\left(-c_3 c_5 \exp\left\{-\frac{x - c_6}{c_1^2 c_3}\right\}\right)\right)}{c_4 L_\omega\left(-c_3 c_5 \exp\left\{-\frac{x - c_6}{c_1^2 c_3}\right\}\right)}\right]$$

and

$$g(t) = sc_1^2 c_3\left[1 + L_\omega\left(-c_3 c_5 \exp\left\{-\frac{t - c_6}{c_1^2 c_3}\right\}\right)\right] + c_2.$$

Particularly, letting $c_1 = c_3 = c_5 = s = 1$ and $c_2 = c_6 = 0$ we have

$$g(t) = 1 + L_\omega(-e^{-t}).$$

Thus, the distribution of first passage time of the Ornstein-Uhlenbeck process defined by

$$dX_t = -X_t dt + dB_t, \quad X_0 = 0$$

through the boundary $g(t) = 1 + L_\omega(-e^{-t})$ is the same of the Ornstein-Uhlenbeck process $Y_t = \Psi^{-1}(X_{\varphi(t)}, t)$ through the level s (see (13)), where

$$\begin{aligned} \Psi(x,t) &= \frac{e^t}{e^t + 1}x, \\ \varphi(t) &= (e^t + 1)^{-1} + \ln(e^t + 1). \end{aligned}$$

5. ORNSTEIN-UHLENBECK PROCESS AND THE HYPERBOLIC BOUNDARY

For $h(t) = \alpha t$, $\alpha \in \mathbb{R}$ Salminen [19] showed that the probability density function of $\tau_h = \inf\{t > 0: B_t = h(t)\}$ is

$$P_x\{\tau_h \in dt\} = \exp\left\{\alpha x - \frac{1}{2}\alpha^2 t\right\}\frac{|x|}{\sqrt{2\pi t^3}}\exp\left\{-\frac{x^2}{2t}\right\}dt.$$

Using this information, for the constants $\mu, \sigma > 0$ and the Ornstein-Uhlenbeck process Y_t defined by

$$dY_t = -\mu Y_t dt + \sigma dB_t, \quad Y_0 = 0,$$

the probability density function of first passage time through the boundary

$$g(t) = -\alpha\beta\sigma e^{\mu t} + \beta\sigma e^{-\mu t}, \quad \beta \in \mathbb{R}$$

can be determined as follows:

Since it is known that $Y_t = \Psi^{-1}(B_{\varphi(t)}, t)$ constitutes an Ornstein-Uhlenbeck process for the transformations φ and Ψ defined by (6) and (7), and

$$\Psi^{-1}(h(\varphi(t)), t) = \alpha \frac{c_4}{2c_1c_3} e^{c_3 t} + \frac{c_5 - c_2}{c_1} e^{-c_3 t},$$

the particular choices of $c_1 = 1/\sigma$, $c_2 = 0$, $c_3 = \mu$, $c_4 = -2\mu\beta$, $c_5 = \beta$ result in $\tau_g \overset{d}{=} \tau_h$ for $\tau_g = \inf\{t > 0: Y_t = g(t)\}$. The first passage times of an Ornstein-Uhlenbeck process through a hyperbolic boundary is investigated by Buonocore et al. [4] under integral transformations. A special case of this result is also given in Aksop [2].

Furthermore, for $h(t) = b + at - \frac{t^2}{2}$, Martin-Löf [15] showed that the probability density function of $\tau_h = \inf\{t > 0: B_t = h(t)\}$ is

$$\begin{aligned} &P_b\{\tau_h \in \mathrm{d}t\} \\ &= \exp\left\{-\frac{(t-a)^3 + a^3}{6} \right. \\ &\left. - ab\right\} \int_{-\infty}^{\infty} e^{tu} \frac{B(u)A(u-b) - A(u)B(u-b)}{\pi(A^2(u) + B^2(u))} du \end{aligned}$$

where $A(u) = Ai(2^{1/3}u)$, $B(u) = Bi(2^{1/3}u)$ and Ai, Bi are Airy functions (see, [1]).

If we define $g(t) = \Psi^{-1}(h(\varphi(t)), t)$ similar to the operations as done above, we obtain the probability density function of first passage time of the Ornstein-Uhlenbeck process $Y_t = \Psi^{-1}(B_{\varphi(t)}, t)$ through the boundary

$$g(t) = \sigma e^{-\gamma t} + \varsigma e^{\gamma t} - \nu e^{3\gamma t},$$

where

$$\begin{aligned} \sigma &= \left(b + c_4 - \frac{c_4^2}{2}\right) \exp\left\{-\frac{c_1 + c_3}{2}\right\}, \\ \gamma &= c_2, \\ \varsigma &= \frac{a + c_4}{2c_2} \exp\left\{-\frac{c_1}{2}\right\}, \\ \upsilon &= \frac{1}{8c_2^2} \exp\left\{-\frac{c_1 - 3c_3}{2}\right\}. \end{aligned}$$

6. ORNSTEIN-UHLENBECK PROCESS AND THE MIXED BOUNDARY

From Section 5, for the Ornstein-Uhlenbeck process defined by

$$dX_t = -X_t dt + dB_t, \quad X_0 = 0$$

we know the probability density function of first passage time through the boundary $g(t) = \alpha e^t + \beta e^{-t}$, $\alpha, \beta \in \mathbb{R}$. In addition, by means of the following transformations given in Section 4

$$\begin{aligned} \varphi(t) &= c_1^2 c_3 \ln(e^{c_3 t} + c_3 c_5) + \frac{c_1^2 c - 9c_5}{e^{c_3 t} + c_3 c_5} + c_6, \\ \Psi(x, t) &= x \frac{c_1^2 c_3 e^{c_3 t}}{e^{c_3 t} + c_3 c_5} + c_2, \end{aligned}$$

we showed that the process Y_t defined as $Y_t = \Psi^{-1}(X_{\varphi(t)}, t)$ is again an Ornstein-Uhlenbeck process since it can be written as

$$dY_t = -c_3 Y_t dt + \frac{1}{c_1} dB_t.$$

With this in mind, let us define a new boundary as follows:

$$\begin{aligned} h(t) &= \Psi^{-1}(g(\varphi(t)), t) \\ &= \frac{c_4 e^{c_3 t} + c_3 c_5}{c_1^2 c_3 c_4 e^{c_3 t}} \left[\alpha \exp\left\{\frac{c_1^2 c_3^2 c_5}{\zeta(t)} + c_1^2 c_3 \ln(\zeta(t)) + c_6\right\} \right. \\ &\left. + \beta \exp\left\{-\frac{c_1^2 c_3^2 c_5}{\zeta(t)} - c_1^2 c_3 \ln(\zeta(t)) - c_6\right\} - c_2\right], \end{aligned}$$

where $\zeta(t) = c_4 \exp\{c_3 t\} + c_3 c_5$.

Then it is true that $\tau_g^X \overset{d}{=} \tau_h^Y$, where $\tau_g^X = \inf\{t > 0: X_t = g(t)\}$ and $\tau_h^Y = \inf\{t > 0: Y_t = h(t)\}$.

7. CONCLUSION

In this study, for the examination of the first passage time through a moving boundary some transformations are applied to the diffusion processes and the distribution of first passage time of the original process obtained by that of first passage time of the transformed process. Unlike the study of Ricciardi and Sato [18], not only the processes that can be transformed into the Wiener process considered, but all of the transformations to diffusion processes whose first passage time distributions are known are discussed. By these transformations known results in the literature are more easily obtained, and the distributions of first passage time of the Ornstein-Uhlenbeck process through a linear boundary are also presented.

REFERENCES

[1] Abramowitz, M., Stegun, I. “AHandbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables”, ***National Bureau of Standards.*** . (1972).

[2] Aksop, C. “Opsiyon Tercihlerinin Stokastis Analiz İle İncelenmesi”, Doktora Tezi, ***Gazi Üniversitesi, Fen Bilimleri Enstitüsü***, (2012)

[3] Breiman, L. “First Exit Time from a Square root Boundary”. ***Proceedings 5th Berkley Symposium Mathematical Statistics and Probability***, 2: 9-16(1966).

[4] Buonocore, A., Nobile, A. G., Ricciardi, L. M. “A New Integral Equation for the Evaluation of First-Passage-Time Probability Densities”. ***Advances in Applied Probability***, 19: 784-800(1987).

[5] Capocelli, R. M., Ricciardi, L. M. “On the Transformation of Diffusion Processes into the Feller Process”, ***Mathematical Biosciences***, 29: 219-234(1976)..

[6] Cherkasov, I. D. On the Transformation of the Diffusion Process to a Wiener Process. *Theory of* ***Probability and Applications***, 2: 373-377(1957).

[7] Ditlevsen, S., Ditlevsen, O. Parameter Estimation from Observations of First-Passage Times of the Ornstein-Uhlenbeck Process and the Feller Process. ***Probabilistic Engineering Mechanics***, 23: 170-179(2008).

[8] Dominé, M. Moments of the First-Passage Time of a Wiener Process with Drift Between Two Elastic Barriers. ***Journal of Applied Probability***, 33: 1007-1013(1995).

[9] Durbin, J. The First-Passage Density of a Continuous Gaussian Process to a General Boundary. ***Journal of Applied Probability***, 22: 99-122(1985).

[10] Fang, Y., Wu, R. Optimal Dividends in the Brownian Motion Risk model with Interest. ***Journal of Computational and Applied Mathematics***, 229: 145-151(2009).

[11] Fortet, R. Les Functiones Aléatories du Type de Markov Associées a Certaines Equationes Linéaries aux Dérivées Partielles du Type Parabolique. ***Journal de Mathématiques Pures et Appliqées***, 22: 177-243(1943).

[12] Giorno, V., Lansky, P. Nobile, A., Ricciardi, L. "Diffusion Approximation and First-Passage-Time Problem for a Model Neuron", ***Biological Cybernetics***, 58: 387-404(1988).

[13] Giorno, V., Nobile, A. G., Ricciardi, L. M. On the Asymptotic Behavior of First-Passage-Time Densities for One-Dimensional Diffusion Process and Varying Boundaries. ***Advances in Applied Probability***, 22: 883-914(1990).

[14] Kolmogorov, A. N. (1992) On Analytical Methods in Probability Theory. In *Selected Works of A. N. Kolmogorov*. Ed. A. N. Shiryayev. Vol II. Kluwer Academic. English translation from Kolmorogov, A. N. (1931) Über die Analytischen Methoden in der Warscheinlichkeitsrechnung, Math. Ann. 104: 415-458.

[15] Martin-Löf, A. "The Final Size of a Nearly Critical Epidemic, and the First Passage Time of a Wiener Process to a Parabolic Barrier", ***Journal of Applied Probability***, 35: 671-682(1998).

[16] McKenzie, H. W., Lewis, M. A., Merrill, E. H. "First Passage Time Analysis of Animal Movement and Insights into the Functional Response", ***Bulletin of Mathematical Biology***, 71: 107-129(2009).

[17] Ricciardi, L. M. On the Transformation of Diffusion Processes into the Wiener Process. ***Journal of Mathematical Analysis and Applications***, 54: 185-199(1976).

[18] Ricciardi, L. M., Sato, S. A Note on the Evaluation of First-Passage-Time Probability Densities. ***Journal of Applied Probability***, 20: 197-201(1983).

[19] Salminen, P. "On the First Hitting Time and the Last Exit Time for a Brownian Motion to/from a Moving Boundary". ***Advances in Applied Probability***, 20: 411-426 (1988).

[20] Smith, C. E. A "Lauguerre Series Approximation to the Probability Density of the First Passage Time of the Ornstein-Uhlenbeck", ***Noise in Physical Systems and 1/f Fluctuations***, 389-392(1991).

[21] Wang, H., Yin, C. "Moments of the First Passage Time of One-Dimensional Diffusion with Two-Sided Barriers", ***Statistics and Probability Letters***, 78: 3373-3380(2008).

[22] Szabo, A., Schulten, K., Schulten, Z. "First Passage Time Approach to Diffusion Controlled Reactions", ***Journal of Chemical Physics***, 72 (8): 4350 – 4357(1980)

[23] Hänggi, P., Talkner, P., Borkovec, M. "Reaction-rate Theory: Fifty Years After Kramers", ***Reviews of Modern Physics***, 62: 2(1990)