A General Fixed Point Theorem In Complete $G$ - Metric Spaces For Weakly Compatible Pairs

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Received: 19.11.2012 Accepted: 05.06.2013

ABSTRACT

In this paper a general fixed point theorem in complete $G$ - metric space for weakly compatible pairs of mappings is proved, which generalize the results by Theorems 3.2 and 3.3 [18] and obtained another particular results.

Key words: complete $G$ - metric space, fixed point, weakly compatible mappings, implicit relation.
1. INTRODUCTION

Let \((X,d)\) be a metric space and \(S, T : (X,d) \to (X,d)\) be two mappings. In 1994, Pant [13] introduced the notion of pointwise \(R\) - weakly commuting mappings. It is proved in [14] that the notion of pointwise \(R\) - weakly commutativity is equivalent to commutativity in coincidence points. Jungeck [4] defined \(S\) and \(T\) to be weakly compatible if \(Sz = Ts\) implies \(STz = TSz\). Thus, \(S\) and \(T\) are weakly compatible if and only if \(S\) and \(T\) are pointwise \(R\) - weakly commuting.

In [2], [3] Dhage introduced a new class of generalized metric spaces, named \(D\) - metric space. Mustafa and Sims [6], [7] proved that most of the claims concerning the fundamental topological structures on \(D\) - metric spaces are correct and introduced appropriate notion of generalized metric space, named \(G\) - metric space. In fact, Mustafa, Sims and other authors studied many fixed point results for self mappings in \(G\) - metric spaces under certain conditions [5] – [12], [17].

Quite recently, Srivastava et al. [18] proved two fixed point theorems for weakly compatible mappings in complete \(G\) - metric spaces.

In [15] and [16], Popa initiated the study of fixed points for mappings satisfying implicit relations.

The purpose of this paper is to prove a general fixed point theorem in \(G\) - metric spaces for weakly compatible pairs of mappings satisfying an implicit relation which generalize the results from Theorems 3.2 and 3.2 [18].

2. PRELIMINARIES

Definition 2.1 [7] Let \(X\) be a nonempty set and \(G : X^3 \to \mathbb{R}_+\) be a function satisfying the following properties:

\((G_1)\): \(G(x,y,z) = 0\) if \(x = y = z\),

\((G_2)\): \(0 < G(x,x,y)\) for all \(x, y \in X\) with \(x \neq y\),

\((G_3)\): \(G(x,y,z) \leq G(x,y,z)\) for all \(x, y, z \in X\) with \(z \neq y\),

\((G_4)\): \(G(x,y,z) = G(y,z,x) = G(z,x,y) = \ldots\) (symmetry in all three variables),

\((G_5)\): \(G(x,y,z) \leq G(x,a,a) + G(a,y,z)\) for all \(x, y, z, a \in X\) (rectangle inequality).

Then the function \(G\) is called a \(G\) - metric on \(X\) and the pair \((X,G)\) is called a \(G\) - metric space.

Note that \(G(x,y,z) = 0\), then \(x = y = z\).

Definition 2.2 [7] Let \((X,G)\) be a \(G\) - metric space. A sequence \((x_n)\) in \(X\) is said to be

\(a)\) \(G\) - convergent if for \(\varepsilon > 0\), there exists \(n \in X\) and \(k \in N\) such that for all \(m, n \geq k\), \(G(x_n,x_m,x_k) < \varepsilon\),

\(b)\) \(G\) - Cauchy if for each \(\varepsilon > 0\), there exists \(k \in N\) such that for all \(n, m, p \geq k\), \(G(x_n,x_m,x_p) < \varepsilon\), that is \(G(x_n,x_m,x_p) \to 0\) as \(n, m, p \to \infty\).

\(c)\) A \(G\) - metric space is said to be \(G\) - complete if every \(G\) - Cauchy sequence is \(G\) - convergent.

Lemma 2.1 [7] Let \((X,G)\) be a \(G\) - metric space. Then, the following properties are equivalent:

1) \((x_n)\) is \(G\) - convergent to \(x\);

2) \(G(x_n,x,x) \to 0\) as \(n \to \infty\);

3) \(G(x_n,x) \to 0\) as \(n \to \infty\);

4) \(G(x_n,x_m) \to 0\) as \(m, n \to \infty\).

Lemma 2.2 [7] If \((X,G)\) is a \(G\) - metric space and \((x_n) \in X\), then the following properties are equivalent:

1) \((x_n)\) is \(G\) - Cauchy;

2) For every \(\varepsilon > 0\), there exists \(k \in N\) such that \(G(x_n,x_m,x_k) < \varepsilon\) for all \(n, m \geq k\).

Lemma 2.3 [7] 1 Let \((X,G)\) be a \(G\) - metric space, then the function \(G(x,y,z)\) is jointly continuous in all three of its variables.

Lemma 2.4 [7] Let \((X,G)\) be a \(G\) - metric space. Then \(G(x,y,x) \leq 2G(y,x,x)\) for all \(x, y \in X\).

Quite recently, the following theorems are proved in [18].

Theorem 2.1 Let \((X,G)\) be a complete \(G\) - metric space and let \(S, T : X \to X\) be two mappings which satisfy the following conditions:

\(i)\) \(T(X) \subseteq S(X)\),

\(ii)\) \(T(X)\) or \(S(X)\) is \(G\) - complete, and

\(iii)\) \(G(Tx,Ty,Tz) \leq \alpha G(Sx,Sy,Sz) + \beta G(Tx,Sx,Sz) + \gamma G(Ty,Sy,Sz) + \delta G(Tz,Sz,Sz) + \eta G(Tx,Sy,Sy)\),

for all \(x, y, z \in X\), where \(\alpha, \beta, \gamma, \delta, \eta \geq 0\) and \(\alpha + 2\beta + 2\gamma + 2\delta + 2\eta < 1\).
Then $S$ and $T$ have an unique point of coincidence in $X$. Moreover, if $S$ and $T$ are weakly compatible, then $S$ and $T$ have an unique common fixed point.

(iii) $G(Tx, Ty, Tz) \leq \alpha \max\{G(Sx, Sy, Sz), G(Tx, Sx, Sz), G(Ty, Sy, Sz), G(Tz, Sz, Sz), (Tz, Sy, Sx)\}.$

for all $x, y, z \in X$, where $\alpha \in \left(0, \frac{1}{2}\right).$

Then $S$ and $T$ have an unique point of coincidence in $X$. Moreover, if $S$ and $T$ are weakly compatible, then $S$ and $T$ have an unique common fixed point in $X$.

3. IMPLICIT RELATIONS

Definition 3.1 [2] Let $T_3$ be the set of all continuous functions $F(t_1, \ldots, t_5) : \mathbb{R}^5_+ \to \mathbb{R}$ satisfying the following conditions:

$F_1$: $F$ is nonincreasing in variables $t_3$ and $t_4$.

$F_2$: There exists $h \in [0, 1)$ such that for all $u, v \geq 0$, $F(u, v, 2u, 0) \leq 0$ implies $u \leq hv$.

$F_3$: $F(t, t, 0, 0, t) > 0, \forall t > 0$.

Example 3.1 $F(t_1, \ldots, t_5) = t_1 - at_2 - bt_3 - (c + d)t_4 - et_5$, where $a, b, c, d, e \geq 0$ and $a + 2b + 2c + 2d + e < 1$.

$F_1$: Obviously.

$F_2$: Let $u, v \geq 0$ be and $F(u, v, 2v, 2u, 0) = u - av - 2bv - 2(c + d)u \leq 0$. Then, $u \leq hv$ where $0 \leq h = \frac{a + 2b}{1 - (c + d)}$.

$F_3$: $F(t, t, 0, 0, t) = t(1 - (a + e)) > 0, \forall t > 0$.

Example 3.2 $F(t_1, \ldots, t_5) = t_1 - k \max\{t_2, t_3, t_4, t_5\}$, where $k \in \left[0, \frac{1}{2}\right]$.

$F_1$: Obviously.

$F_2$: Let $u, v \geq 0$ be and $F(u, v, 2v, 2u, 0) = u - k \max\{t_2, 2v, 2u\} \leq 0$. If $u > v$, then $u(1 - 2k) \leq 0$, a contradiction. Hence $u \leq v$ and $u \leq hv$, where $0 \leq h = 2k < 1$.

$F_3$: $F(t, t, 0, 0, t) = t(1 - k) > 0, \forall t > 0$.

Theorem 2.2 Let $(X, G)$ be a complete $G$-metric space and let $S, T : X \to X$ be two mappings which satisfy the following conditions:

(i) $T(X) \subset S(X)$,

(ii) $T(X)$ or $S(X)$ is $G$-complete, and

Example 3.3 $F(t_1, \ldots, t_5) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5^2$, where $a, b, c, d \geq 0$, $a + 2b + 2c < 1$ and $a + d < 1$.

$F_1$: Obviously.

$F_2$: Let $u, v \geq 0$ be and $F(u, v, 2v, 2u, 0) = u^2 - u(2v + 2b + 2c)u \leq 0$. If $u > 0$, then $u - uv - 2bv - 2cu \leq 0$ which implies $u \leq hv$, where $0 \leq h = \frac{a + 2b}{1 - 2c} < 1$. If $u = 0$ then $u \leq hv$.

$F_3$: $F(t, t, 0, 0, t) = t^2(1 - (a + d)) > 0, \forall t > 0$.

Example 3.4 $F(t_1, \ldots, t_5) = t_1 - a \frac{t_2 + t_3}{2} - b \frac{t_4 + t_5}{2}$, where $a, b \geq 0$ and $3a + 2b < 2$.

$F_1$: Obviously.

$F_2$: Let $u, v \geq 0$ be and $F(u, v, 2v, 2u, 0) = u - a \frac{3v}{2} - bu \leq 0$. Hence $u \leq hv$, where $0 \leq h = \frac{3a}{2 - 2b} < 1$.

$F_3$: $F(t, t, 0, 0, t) = \left(1 - \frac{a + b}{2}\right) > 0, \forall t > 0$.

Example 3.5 $F(t_1, \ldots, t_5) = t_1^2 - at_2^2 - b \frac{t_3^2 + t_4^2}{2} \frac{1}{t_5^2}$, where $a + 8b < 1$.

$F_1$: Obviously.

$F_2$: Let $u, v \geq 0$ be and $F(u, v, 2v, 2u, 0) = u^2 - av^2 - (4a^2 + 4v^2)b \leq 0$ which implies $u \leq hv$, where $0 \leq h = \frac{a + 4b}{1 - 4b}$.

$F_3$: $F(t, t, 0, 0, t) = t^2(1 - a) > 0, \forall t > 0$.

Example 3.6 $F(t_1, \ldots, t_5) = t_1 - at_2 - bt_3 - c \min\{t_4, t_5\}$, where $a, b, c \geq 0$ and $a + 2b < 1$. 


(F_1): Obviously.

(F_2): Let \( u, v \geq 0 \) and 
\[ F(u, v, 2v, 2u, 0) = u - av - 2bv \leq 0 \] 
which implies \( u \leq hv \), where \( 0 \leq h = a + 2b < 1 \).

(F_3): \( F(t, t, 0, 0, t) = t(1 - a) > 0, \forall t > 0 \).

Example 3.7 \( F(t_1, ..., t_5) = t_1 - c \max\{t_2, t_3, \sqrt{t_4/t_5}\} \),
where \( c \in \left(0, \frac{1}{2}\right)\).

(F_1): Obviously.

\[ F(u, v, 2v, 2u, 0) = u - k \max\{v, 2v, v + 4u, 2u\} = u - k \max\{2v, v + 2u\} \leq 0 \]
which implies \( u \leq 2k(u + v) \). Hence \( u \leq hv \), where \( 0 \leq h = \frac{2k}{1 - 2k} < 1 \).

(F_3): \( F(t, t, 0, 0, t) = t(1 - k) > 0, \forall t > 0 \).

4. MAIN RESULTS

Definition 4.1 Let \( S \) and \( T \) two self mappings of a nonempty set \( X \). If \( w = T(x) = S(x) \) for some \( x \in X \),

\[ F(G(T_x, T_y, T_y), G(S_x, S_y, S_y), G(T_x, S_x, S_x), G(T_y, S_y, S_y), G(T_y, S_x, S_y)) \leq 0 \quad (4.1) \]

for all \( x, y \in X \) and \( F \) satisfying property \((F_3)\). Then \( T \) and \( S \) have at most a point of coincidence.

Theorem 4.2 Let \( X, G \) be a \( G \)-metric space and \( T, S \) self mappings of \( X \) such that

\[ F(G(T_q, T_p, T_p), G(S_q, S_p, S_p), G(T_q, S_q, S_q), G(T_p, S_p, S_p), G(T_q, S_p, S_p)) \leq 0, \]

(iii) \( T \) and \( S \) satisfy the inequality \((4.1)\) for all \( x, y \in X \) and \( F \in F_q \).

Then \( T \) and \( S \) have an unique point of coincidence. Moreover, if \( T \) and \( S \) are weakly compatible, then \( T \) and \( S \) have an unique common fixed point.

Proof. Suppose that \( u = Tp = Sp \) and \( v = Tq = Sq \) are two distinct points of coincidence. Then, by \((4.1)\) we have successively:

\[ F(G(T_x, T_q, T_q), G(S_x, S_q, S_q), G(T_x, S_x, S_x), G(T_q, S_q, S_q), G(T_x, S_x, S_q)) \leq 0, \]

a contradiction of \((F_2)\) if \( G(S_q, S_p, S_p) > 0 \). Hence \( G(S_q, S_p, S_p) = 0 \), so \( SQ = SP \) which implies \( u = v \).

Theorem 4.2 Let \( (X, G) \) be a \( G \)-metric space and let \( T, S : (X, G) \to (X, G) \) be two mappings such that

2. (i) \( T(X) \subset S(X) \),

(ii) \( T(X) \) or \( S(X) \) is \( G \)-complete.

By Lemma 2.4

\[ G(S_{x_n+1}, S_{x_n}, S_{x_n}) \leq 2G(S_{x_n}, S_{x_n+1}, S_{x_n+1}) \]
\[ G(Sx_n, Sx_{n-1}, Sx_{n-1}) \leq 2G(Sx_{n-1}, Sx_n, Sx_n). \]

By \((F_1)\) we obtain:
\[ F(G(Sx_n, Sx_{n+1}, Sx_{n+1}), G(Sx_{n-1}, Sx_n, Sx_n), 2G(Sx_{n-1}, Sx_n, Sx_n), 2G(Sx_n, Sx_{n+1}, Sx_{n+1}), 0) \leq 0 \]

which implies by \((F_2)\) that
\[ G(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq hG(Sx_{n-1}, Sx_n, Sx_n). \]

By repeated application of the above inequality, we have
\[ G(Sx_n, Sx_m, Sx_m) \leq G(Sx_n, Sx_{n+1}, Sx_{n+1}) + G(Sx_{n+1}, Sx_{n+2}, Sx_{n+2}) + \ldots + G(Sx_m, Sx_m, Sx_m) \]
\[ \leq (h^n + h^{n+1} + \ldots + h^{m-1})G(Sx_0, Sx_1, Sx_1) \]
\[ \leq \frac{h^n}{1-h}G(Sx_0, Sx_1, Sx_1). \]

Taking limit as \(n, m \to \infty\), we get
\[ \lim_{n, m \to \infty} G(Sx_n, Sx_m, Sx_m) = 0. \] Hence \(\{Sx_n\}\) is a G-Cauchy sequence. Now, since \(S(X)\) is G-complete, there exists a point \(q \in S(X)\) such that \(Sx_n \to q\) as \(n \to \infty\). Consequently, we can find a point \(p \in X\) such that \(Sp = q\).

If \(T(X)\) is G-complete, there exists \(q \in T(X)\) such that \(Sx_n \to q\). As \(T(X) \subset S(X)\), we have \(q \in Sx\). Then, there exists \(p \in X\) such that \(Sp = q\).

We prove that \(p\) is a coincidence point for \(T\) and \(S\). By \((4.1)\) we have successively:

\[ F(G(Tx_n, Tp, Tp), G(Sx_n, Sp, Sp), G(Tx_n, Sx_{n-1}, Sx_{n-1}), G(Tp, Sp, Sp), G(TX_n-1, Sp, Sp)) \leq 0, \]
\[ F(G(Sx_n, Tp, Tp), G(Sx_n-1, Sp, Sp), G(Sx_n, Sx_{n-1}, Sx_{n-1}), G(Tp, Sp, Sp), G(Sx_n, Sp, Sp)) \leq 0. \]

Letting \(n\) tend to infinity, we obtain
\[ F(G(Sp, Tp, Tp), 0, 0, G(Tp, Sp, Sp), 0) \leq 0. \]

By Lemma 2.4, \(G(Tp, Sp, Sp) \leq 2G(Sp, Tp, Tp)\). By \((F_1)\) we obtain
\[ F(G(Sp, Tp, Tp), 0, 0, 2G(Sp, Tp, Tp), 0) \leq 0. \]

By \((F_2)\), \(G(Sp, Tp, Tp) = 0\) which implies \(w = Tp = Sp\) and \(p\) is a coincidence point of \(T\) and \(S\). By Theorem 4.1, \(w\) is the unique point of coincidence of \(T\) and \(S\). Moreover, if \(T\) and \(S\) are weakly compatible, by Lemma 4.1 \(w\) is the unique common fixed point of \(T\) and \(S\).

If \(S(X)\) is complete, the proof it follows by \(T(X) \subset S(X)\).

**Corollary 4.1** Let \(T\) and \(S\) be self mappings of a G-metric space satisfying the following conditions:

\[ (i) \quad T(X) \subset S(X), \]
\[ (ii) \quad S(X) \text{ or } T(X) \text{ is G-complete}, \]
\[ (iii) \quad \text{One of the following inequalities hold for all } x, y \in X \] (1)

\[ G(Tx, Ty) \leq aG(Sx, Sy) + bG(Tx, Sx, Sx) + (c + d)G(Ty, Sy, Sy) + eG(Tx, Sy, Sy), \] (3)

where \(a, b, c, d, e \geq 0\) and \(a + 2b + 2c + 2d + e < 1\). (2)

\[ G(Tx, Ty) \leq k \max\{G(Sx, Sy, Sy), G(Tx, Sx, Sx), G(Ty, Sy, Sy), G(Tx, Sy, Sy)\}, \]
where \( k \in \left(0, \frac{1}{2}\right)\). (3)

\[ G^2(Tx, Ty, Ty) \leq G(Tx, Ty, Ty)G(Sx, Sy, Sy) + bG(Tx, Sx, Sx) + cG(Ty, Sy, Sy) + dG^2(Ty, Sy, Sy), \]

where \( a, b, c, d \geq 0, a + 2b + 2c < 1 \) and \( a + d < 1 \). (4)

\[ G(Tx, Ty, Ty) \leq a \frac{G(Sx, Sy, Sy) + G(Tx, Sx, Sx)}{2} + b \frac{G(Ty, Sy, Sy) + G(Tx, Sx, Sx)}{2}, \]

where \( a, b \geq 0 \) and \( 3a + 2b < 2 \). (5)

\[ G^2(Tx, Ty, Ty) \leq aG^2(Sx, Sy, Sy) + G(Tx, Sx, Sx) + bG^2(Ty, Sy, Sy) + \frac{G^2(Tx, Sx, Sx)}{1 + G^2(Tx, Sy, Sy)}, \]

where \( a, b \geq 0 \) and \( a + 8b < 1 \). (6)

\[ G(Tx, Ty, Ty) \leq aG(Sx, Sy, Sy) + bG(Tx, Sx, Sx) + c \min\{G(Ty, Sy, Sy), G(Tx, Sx, Sx)\}, \]

where \( a, b, c \geq 0 \) and \( a + 2b < 1 \). (7)

\[ G(Tx, Ty, Ty) \leq c \max\{G(Sx, Sy, Sy), G(Tx, Sx, Sx), G(Ty, Sy, Sy), (1 + G(Tx, Sy, Sy))^2\}, \]

where \( c \in \left(0, \frac{1}{2}\right)\). (8)

\[ G(Tx, Ty, Ty) \leq k \max\{G(Sx, Sy, Sy), G(Tx, Sx, Sx), \frac{1}{2}\{G(Tx, Sx, Sx) + 2G(Ty, Sy, Sy)\}, \]

where \( k \in \left(0, \frac{1}{4}\right)\).

If \( S \) and \( T \) are weakly compatible, then \( S \) and \( T \) have an unique common fixed point.

**Proof.** The proof follows by Theorem 4.2 and Examples 3.1 – 3.8.

**Remark 4.1** Because in Theorem 2.1 and \( a + 2b + 2c + 2d + e < 1 \), for \( y = z \) we obtain

\[ G(Tx, Ty, Ty) \leq aG(Sx, Sy, Sy) + bG(Tx, Sx, Sx) + (c + d)G(Ty, Sy, Sy) + eG(Tx, Sy, Sy) \]

and \( a + 2b + 2c + 2d + e < 1 \), Theorem 2.1 follows from Corollary 4.1 (iii) (1).

**Remark 4.2** Because in Theorem 2.2 for \( y = z \) we obtain

\[ G(Tx, Ty, Ty) \leq k \max\{G(Sx, Sy, Sy), G(Tx, Sx, Sx), G(Ty, Sy, Sy), G(Tx, Sy, Sy)\}, \]

and Theorem 2.2 follows from Corollary 4.1 (iii) (2).
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