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Coincidence and Common Fixed Point Theorems in Fuzzy Metric Spaces Using a Meir-Keeler Type Contractive Condition

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Abstract

In this paper we prove coincidence and common fixed point theorems for two pairs of weakly compatible self maps in fuzzy metric space using a fuzzy analogue of the Meir-Keeler type contractive condition. Our results substantially extend, generalize, and improve a multitude of well known results of the form existing in the literature for metric as well as fuzzy metric spaces.

Key words: Weakly compatible, fuzzy metric space, common property *(E.A)*, $JCLR_{ST}$ property, common fixed point.

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1. INTRODUCTION

The introduction of the notion of a fuzzy set, is a new way to represent the vagueness in everyday life, by Zadeh [41], proved a turning point in the development of fuzzy mathematics. However, it appears that Kramosil and Michhlek's study of fuzzy metric spaces paves the way for very soothing machinery to develop fixed point theorems especially for contractive type and nonexpansive type maps [20]. Later on, Grabiec [14] followed Kramosil-Michklek [20] and obtained the fuzzy version of the Banach contraction principle. George and Veeramani [10, 11] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek [20] and defined a Hausdorff topology for metric spaces, which later proved to be metrizable. They also showed that every metric induces a fuzzy metric. For a good bibliography on fundamentals and the development of fuzzy mathematics, refer to A.P. Shostak [39]. Fuzzy mathematics has very fruitful applications in quantum physics [9], nonlinear dynamical systems [16], population dynamics [4] and computer programming [12]. The theory of a fuzzy set is of fundamental importance in computerized Medical diagnosis, effect of drugs, and diagnosis processes [3].

The existence of a common fixed point of maps satisfying contractive type conditions has been a very active area of research. Common fixed point theorems for four maps, say *A, B, S* and *T* either use

(*A*). a Banach type contractive condition:

$d(Ax, Ty) \leq k\ max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty)\}$, where $0 \leq k < 1$, or

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(*B*). a Meir-Keeler type contractive condition:

given $\varepsilon > 0$ there exists a $\delta > 0$ such that

ε ≤ *max*{*d*(*Sx, Ty*), *d*(*Ax, Sx*), *d*(*By, Ty*) }< $\varepsilon + \delta$ implies *d*(*Ax, Ty*) < ε, or

(*C*). slightly weaker Meir-Keeler type contractive condition: given $\varepsilon > 0$ there exists a $\delta > 0$ such that

ε < *max*{*d*(*Sx, Ty*), *d*(*Ax, Sx*), *d*(*By, Ty*) }< $\varepsilon + \delta$ implies *d*(*Ax, Ty*) ≤ ε, or

(*D*). a φ- contractive condition:

d(*Ax, By*) ≤ φ (*max* {*d*(*Sx, Ty*), *d*(*Ax, Sx*), *d*(*By, Ty*)}) where φ :$R_+ \to R_+$ is such that φ (*t*) < *t* for each *t* > 0.

Jachymski [18] has shown that contractive condition (*B*) implies (*C*) but the reverse implication is not true. Moreover φ- contractive conditions do not guarantee the existence of a fixed point unless some additional condition is assumed on φ. Also φ- contractive conditions, in general, do not imply Meir Keeler type contractive conditions. Hence fixed point theory, for the class of Meir Keeler type contractive conditions, is much wider than the class of Banach or the class of φ - contractive conditions.

In the literature many results have been proved using different contractive conditions in fuzzy metric spaces; see for instance [5],[17],[21],[22],[27],[38],[40] and references therein. The aim of this paper is to prove coincidence and common fixed point theorems for two pairs of maps in a fuzzy metric space using a fuzzy analogue of the Meir-Keeler type contractive condition (B). Our results extend, generalize, and improve multitude of well known results of the same form in a metric space, as well as fuzzy metric spaces, including Meir-Keeler [25], Boyd and Wong [6], J. Matkowski [24], Maiti and Pal [23], Park-Rhoades[33], Rao-Rao[35], Jungck [19], Pant [31], Pathak *et al.* [34], Cho *et al* [8].

2. PRELIMINARIES

Definition 2.1 [41]: Let *X* be any set. A fuzzy set in *X* is a function with domain *X* and values in [0, 1]. The concept of triangular norms (*t*-norms) was originally introduced by Menger [26] in study of statistical metric spaces.

Definition 2.2 [37] A binary operation * : [0,1]×[0,1] ⟶ [0,1] is continuous in the *t*-norm if * satisfies the following conditions:

(i) * is commutative and associative;
(ii) * is continuous;
(iii) *a* * 1 = *a* for all $a \in [0,1]$;
(iv) *a* * *b* ≤ *c* * *d* whenever *a* ≤ *c* and *b* ≤ *d* for all $a,b,c,d \in [0,1]$.

Examples of *t*-norms are: *a***b* = min{*a,b*}, *a***b* = *ab* and *a***b* = max{*a*+*b*-1,0} for all $a,b \in [0,1]$.

Definition 2.3 [10] A 3-tuple $(X,M,*)$ is a fuzzy metric space if *X* is an arbitrary set, * is a continuous *t*-norm and *M* is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions for all $x,y,z \in X$ and *s, t* > 0

(i) *M*(*x, y, t*) > 0;
(ii) *M*(*x, y, t*) = 1 for all *t > 0* if and only if *x = y*;
(iii) *M*(*x, y, t*) = M(*y, x, t*);
(iv) *M*(*x, y, t*) * M(*y, z, s*) ≤ M(*x, z, t + s*);
(v) *M*(*x, y*,.): [0, ∞) → [0, 1] is continuous.

The function *M*(*x, y, t*) denotes the degree of nearness between *x* and *y* with respect to *t* .
We identify *x = y* with M(x, y, t) = 1 for all t >0 and M(x, y, t) = 0 with *d*(*x, y*) = □.
In what follows $(X,M,*)$ is a fuzzy metric space with the following property:
(vi) $\lim_{t\to\infty} M(x,y,t) = 1$ for all $x,y \in X$ and t > 0.

Remark 2.4.[10]. In a fuzzy metric space $(X,M,*)$, *M*(*x, y,* *) is non-decreasing for all $x,y \in X$.

Definition 2.5.[10]. A sequence {x_n} in a fuzzy metric space $(X,M,*)$ is

(i) convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n,x,t) = 1$ for all *t* > 0,
(ii) a Cauchy sequence if $\lim_{n\to\infty} M(x_{n+p},x_n,t) = 1$ for all *t* > 0 and *p* > 0,
(ii) complete if every Cauchy sequence in *X* is convergent to a some point in *X*.

Example 2.6 [10]. Let *X* = {*1/n*: $n \in N$ } ∪ {0} and let * be the continuous *t*-norm defined by *a* * *b* = *ab* (or *a* * *b* = min{*a, b*}) respectively, for all $a,b \in [0,1]$. For each *t* > 0 and $x,y \in X$, let

$$M(x,y,t) = \begin{cases} \dfrac{t}{t+|x-y|}, & t > 0, \\ 0 & t=0 \end{cases}$$

Clearly $(X,M,*)$ is a complete fuzzy metric space.

Definition 2.7. A pair of self maps *(f, g)* of a fuzzy metric space $(X,M,*)$ is

(i) compatible **[19]** if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

(ii) non-compatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$ but either $\lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$ or is nonexistent.

(iii) weakly compatible **[13]** if f and g commute at coincidence points; that is, $fgx = gfx$ whenever $fx = gx$.

(iv) satisfy property (*E.A)* **[2]** if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = z = \lim_{n\to\infty} gx_n$ for some $z \in X$.

Note that the class maps of satisfying property (*E.A.)* contains the class of compatible as well as non-compatible maps.

Definition 2.8[1] The Pairs (A, S) and (B, T) on a fuzzy metric space $(X, M, *)$ satisfy common property (*E.A)* if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z$ for some $z \in X$.

Example 2.9 Define self maps A, B, S and T on X = [1, 15) by

$$Ax = \begin{cases} 1 & x \in \{1\} \cup (3,15) \\ 14 & x \in (1,3] \end{cases}, \quad Bx = \begin{cases} 1 & x \in \{1\} \cup (3,15) \\ 5 & x \in (1,3] \end{cases},$$

$$Sx = \begin{cases} 3 & x = 1 \\ 6 & x \in (1,3] \\ \dfrac{x+1}{4} & x \in (3,15) \end{cases} \text{ and } Tx = \begin{cases} 2 & x = 1 \\ 11 & x \in (1,3] \\ x-2 & x \in (3,15) \end{cases}.$$

Take $\left\{x_n = 3 + \frac{1}{n}\right\}$ and $\left\{y_n = 3 + \frac{1}{n}\right\}$, clearly

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1 \in X.$$

⇒A, B, S and T satisfy common property (*E.A.*).

Definition 2.10[7]. Pairs (A, S) and (B, T) on a fuzzy metric space $(X, M, *)$ satisfy the $JCLR_{ST}$ property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = Sz = Tz$$

for some $z \in X$.

Example 2.11. Let $(X, M, *)$ be a fuzzy metric space with X = [-1, 1] and for all $x, y \in X$

$$M(x,y,t) = \begin{cases} e^{\frac{-|x-y|}{t}}, & t > 0, \\ 0, & t=0 \end{cases}$$

Define self maps A, B, S and T on X by $Ax = x/3$, $Bx = -x/3$, $Sx = x$, $Tx = -x$ for all $x \in X$. Then with sequences $\{x_n = 1/n\}$ and $\{y_n = -1/n\}$ in X, one can easily verify that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = S0 = T0$$

. This shows that the pairs (A, S) and (B, T) satisfy the $JCLR_{ST}$ property.

3. MAIN RESULT

Theorem 3.1. Let A, B, S and T be four self maps in a fuzzy metric space $(X, M, *)$ such that

(3.1) $AX \subseteq TX$ and $B(X) \subseteq S(X)$

(3.2) given an $\varepsilon > 0$, there exists a $\delta \in (0, \varepsilon)$ such that

$$\varepsilon - \delta < m(x,y,t) < \varepsilon \Rightarrow M(Ax, By, t) \geq \varepsilon$$

where

$$m(x,y,t) = \min \begin{Bmatrix} M(Sx,Ty,t), M(Ax,Sx,t), M(By,Ty,t), \\ M(Ax,By,t), M(By,Sx,2t) \end{Bmatrix}$$

(3.3) one of AX, BX, SX or TX is a complete subspace of X.

Then

(I) A and S have a coincidence point,

(II) B and T have a coincidence point.

Moreover, if the pairs (A, S) and (B, T) are weakly compatible, then the self maps A, B, S and T have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X by

$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$. This can be done by virtue of (3.1).

We claim that $\{y_n\}$ is a Cauchy sequence.

Let $M_n = M(y_n, y_{n+1}, t)$ where $t > 0$. Two cases arise.

Suppose that $M_n = 1$ for some $n = 2k$-1. Then $M(y_{2k-1}, y_{2k}, t) = 1 \Rightarrow y_{2k-1} = y_{2k}$;

i.e., $Tx_{2k-1} = Ax_{2k-2} = Sx_{2k} = Bx_{2k-1}$. Hence the maps *B* and *T* have a coincidence point.

Further, if $M_n = 1$ for some $n = 2k$.

Then $M(y_{2k}, y_{2k+1}, t) = 1 \Rightarrow y_{2k} = y_{2k+1}$;

i.e., $Tx_{2k+1} = Ax_{2k} = Sx_{2k} = Bx_{2k-1}$. Hence the maps *A* and *S* have a coincidence point.

Now suppose that $M_n \neq 1$ for all *n*.

If, for some $x, y \in X$, $m(x, y, t) = 1$ then $Ax = Sx$ and $By = Ty$. This proves (I) and (II).

If $m(x, y, t) < 1$ for all $x, y \in X$, then, by (3.2), we have

$$\varepsilon - \delta < m(x, y, t) < \varepsilon \Rightarrow M(Ax, By, t) \geq \varepsilon$$
$$\Rightarrow M(Ax, By, t) > m(x, y, t) \text{ ....(1)}$$

Hence

$$\begin{aligned}
M_{2n-1} = M(y_{2n-1}, y_{2n}, t) = M(Ax_{2n-2}, Bx_{2n-1}, t) \\
&> m(x_{2n-2}, x_{2n-1}, t) \\
&= \min\left\{\begin{matrix} M(Sx_{2n-2}, Tx_{2n-1}, t), M(Ax_{2n-2}, Sx_{2n-2}, t), M(Bx_{2n-1}, Tx_{2n-1}, t), \\ M(Ax_{2n-2}, Bx_{2n-1}, t), M(Bx_{2n-1}, Sx_{2n-2}, 2t) \end{matrix}\right\} \\
&= \min\left\{\begin{matrix} M(y_{2n-2}, y_{2n-1}, t), M(y_{2n-1}, y_{2n-2}, t), M(y_{2n}, y_{2n-1}, t), \\ M(y_{2n-1}, y_{2n}, t, M(y_{2n}, y_{2n-2}, 2t)) \end{matrix}\right\} \\
&\geq \min\left\{\begin{matrix} M(y_{2n-2}, y_{2n-1}, t), M(y_{2n}, y_{2n-1}, t), \\ M(y_{2n}, y_{2n-1}, t), M(y_{2n-1}, y_{2n-2}, t) \end{matrix}\right\} \\
&= \min\{M_{2n-2}, M_{2n-1}\} = M_{2n-2} \text{....(2)}
\end{aligned}$$

Therefore, $M_{2n-1} > M_{2n-2}$. Similarly we can show that $M_{2n} > M_{2n-1}$.Thus $M_n > M_{n-1}$ for all n; i.e., $\{M_n\}$ is a strictly increasing sequence of positive real numbers in [0, 1]. Hence it converges to $p \in [0,1]$, which is the *l.u.b*. of $\{M_n\}$.

Next we claim that $p = 1$. If not, there exists a $\delta > 0$ and a natural number *m* such that, for each $n \geq m$,

$p - \delta < M(y_n, y_{n+1}, t) = M_n \leq p$ .....(3)

In particular, as $m(x_{2n}, x_{2n-1}, t) = \min\{M_{2n}, M_{2n-1}\} = M_{2n-1}$, we get $p - \delta < M_{2n-1} < p$. Therefore, by using (3.2),

$M(Ax_{2n}, Bx_{2n-1}, t) = M(y_{2n+1}, y_{2n}, t) = M_{2n} > p$,

a contradiction. Thus $p = 1$; i.e., $\lim_{n\to\infty} M_n = \lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1.$

We now show that $\{y_n\}$ is a Cauchy sequence in *X*. If not, then there exists an $\varepsilon > 0$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$M(y_{n_i}, y_{n_{i+1}}, t) = M_{n_i} \leq p$.

Since, $\lim_{n\to\infty} M_n = \lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1$,

We claim that there exist integers m_i, n_i satisfying $n_i < m_i < n_{i+1}$ such that for each $t > 0$, $M(y_{n_i}, y_{m_i}, t) < \frac{(1+p)}{2}$ …..(4)

Suppose not. Then $M(y_{n_i}, y_{n_{i+1}}, t) \geq M\left(y_{n_i}, y_{n_{i+1}-1}, \frac{t}{2}\right) * M\left(y_{n_{i+1}-1}, y_{n_{i+1}}, \frac{t}{2}\right)$.

Taking $n \to \infty$, we have $\lim_{n\to\infty} M(y_{n_i}, y_{n_{i+1}}, t) \geq \frac{(1+p)}{2} * 1 = \frac{(1+p)}{2} > p,$
a contradiction.

Hence we choose the smallest positive integer m_i satisfying $n_i < m_i < n_{i+1}$ such that,
for each $t > 0$, $M(y_{n_i}, y_{m_i}, t) < \frac{(1+p)}{2}$.
Therefore, $M(y_{n_i}, y_{m_i-1}, t) \geq \frac{(1+p)}{2}$.
This gives $M(y_{n_i-1}, y_{m_i-1}, t) \geq M\left(y_{n_i-1}, y_{n_i}, \frac{t}{2}\right) * M\left(y_{n_i}, y_{m_i-1}, \frac{t}{2}\right)$.
Taking $n_i \to \infty$, we have

$$\lim_{n_i\to\infty} M(y_{n_i}, y_{m_i}, t) \geq \lim_{n_i\to\infty} M(y_{n_i-1}, y_{m_i-1}, t) \geq 1 * \frac{(1+p)}{2} = \frac{(1+p)}{2},$$

which is a contradiction to (4). Therefore our supposition is wrong. Hence $\{y_n\}$ is a Cauchy sequence in X.
Now suppose that SX is a complete subspace of X. Then the subsequence $\{y_{2n}\}$, being contained in SX, must converge to a point $z \in SX$, so there exists a $v \in S^{-1}z$, and hence $Sv = z$. Note that the subsequence $\{y_{2n-1}\}$ also converges to z. As $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence, the sequence $\{y_n\}$ also converges to z.
First we claim that $Av = z$. Suppose not. Then, on setting $x = v$ and $y = x_{2n-1}$ in (1), one gets, for $t > 0$,

$$M(Av, Bx_{2n-1}, t) > m(v, x_{2n-1}, t) = \min\begin{Bmatrix} M(Sv, Tx_{2n-1}, t), M(Sv, Av, t), M(Bx_{2n-1}, Tx_{2n-1}, t) \\ M(Av, Bx_{2n-1}, t), M(Bx_{2n-1}, Sv, 2t) \end{Bmatrix}.$$

Taking the limit as $n \to \infty$, we have

$$\begin{aligned} M(Av, z, t) &> \min\begin{Bmatrix} M(z, z, t), M(z, Av, t), M(z, z, t), \\ M(Av, z, t), M(z, z, 2t) \end{Bmatrix} \\ &= M(z, Av, t), \end{aligned}$$

a contradiction. Therefore, $Av = z = Sv$. Hence maps A and S have a point of coincidence.
As $AX \subseteq TX$, $Av = z \Rightarrow z \in TX$. Let $w \in T^{-1}z$, then $Tw = z$.
Next we claim that $Bw = z$. Suppose not. Again by using (1), we get

$$M(Ay_{2n}, Bw, t) > m(y_{2n}, w, t)$$
$$= \min\begin{Bmatrix} M(Sy_{2n}, Tw, t), M(Sy_{2n}, Ay_{2n}, t), M(Bw, Tw, t) \\ M(Ay_{2n}, Bw, t), M(Bw, Sy_{2n}, 2t) \end{Bmatrix}.$$

Taking the limit as $n \to \infty$, we have

$$M(z, Bw, t) > \min\{M(z,z,t), M(z,z,t), M(Bw,z,t), M(z,Bw,t), M(Bw,z,2t)\}$$
$$\geq \min\{1, M(Bw,z,t), M(Bw,z,t), M(z,z,t)\}$$
$$= M(Bw, z, t),$$

a contradiction. Therefore, $Bw = z = Tw$. Thus the pair (B, T) has a point of coincidence. Hence we have shown that $z = Sv = Av = Bw = Tw$. The same result is obtained if we assume TX to be complete. Indeed, if AX is complete, then $z \in AX \subseteq TX$ and, if BX is complete, then $z \in BX \subseteq SX$ and hence the same result is obtained. As the pairs (A, S) and (B, T) are weakly compatible, then

$Az = ASv = SAv = Sz$ and $Bz = BTw = TBw = Tz$.

Next we claim that $Az = z$. If not, then by (1), we have

$$M(Az, z, t) = M(Az, Bw, t)$$
$$> m(z, w, t)$$
$$= \min\begin{Bmatrix} M(Sz, Tw, t), M(Sz, Az, t), M(Bw, Tw, t), \\ M(Az, Bw, t), M(Bw, Sz, 2t) \end{Bmatrix}$$
$$= \min\{M(Az,z,t), M(Az,Az,t), M(z,z,t), M(Az,z,t), M(z,Az,2t)\}$$
$$\geq \min\{M(Az,z,t), 1, 1, M(Az,z,t), M(z,Az,t), M(Az,Az,t)\}$$
$$= M(Az, z, t)$$

a contradiction. Therefore, $Az = z$. Similarly one can easily show that $Bz = z$. Thus z is the unique common fixed point of A, B, S and T. The uniqueness of fixed point is an easy consequence of inequality (3.2). Hence the result.

We now give an example to illustrate the above theorem.

Example 3.2. Let $X = [2, 20]$ and for each $t > 0$ and $x, y \in X$, and define $(X, M, *)$ by

$$M(x,y,t) = \begin{cases} \dfrac{t}{t + |x - y|}, & t > 0, \\ 0 & t = 0 \end{cases}.$$

Define self maps A, B, S and T on X by

$Ax = 2$ if $x = 2$ or > 5, $Ax = x + 1$ if $2 < x \leq 5$,
$Bx = 2$ if $x = 2$ or > 5, $Bx = x + 2$ if $2 < x \leq 5$,
$S2 = 2$, $Sx = 8$ if $2 < x \leq 5$, $Sx = \dfrac{x+1}{3}$ if $x > 5$,
$Tx = 2$ if $x = 2$ or > 5, $Tx = x + 1$ if $2 < x \leq 5$.

Then self maps A, B, S and T satisfy all of the conditions of the above theorem and have a unique common fixed point at $x = 2$. Moreover, maps neither satisfy the φ-contractive condition nor the Banach type contractive condition. Also one may verify that the self maps A, B, S and T are discontinuous at the common fixed point $x = 2$ and only $S(X)$ is a complete subspace of X. Now we shall improve the above theorem using the common property (*E.A.*), as it relaxes the containment of the range of one map into the range of another, which is utilized to construct the sequence of joint iterates in common fixed point considerations. As a consequence, a multitude of recent common fixed point theorems of the form existing in the literature are sharpened and enriched. Also we are replacing completeness of subspaces by a more natural condition of closedness of subspace. Recall that a subspace SX *is* closed if, for a sequence $\{x_n\}$ in SX and a point $x \in X$,
$\lim_{n\to\infty} d(x_n, x) = 0 \Rightarrow x \in SX$.

Theorem 3.3: Let *A, B, S* and *T* be four self maps in fuzzy metric space $(X, M, *)$ satisfying condition (3.2) such that

(3.4) the pairs (*A, S*) and (*B, T*) satisfy common property (*E.A.*),

(3.5) *SX* and *TX* are closed subspaces of *X*.

Then

(I) *A* and *S* have a coincidence point,

(II) *B* and *T* have a coincidence point.

Moreover, if the pairs (*A, S)* and (*B, T*) are weakly compatible then the maps *A, B, S* and *T* have a unique common fixed point in *X*.

Proof: This theorem easily proved along the same lines as Theorem 3.1

We now give an example to illustrate the above theorem.

Example 3.4. Let $X = [2, 20]$ and for each $t > 0$ and $x, y \in X$, define $(X, M, *)$ by

$$M(x,y,t) = \begin{cases} \dfrac{t}{t+|x-y|}, & t > 0, \\ 0 & t=0 \end{cases}.$$

Define self maps *A, B, S* and *T* on *X* by

$Ax = 2$ if $x = 2$ or > 5, $Ax = x + 1$ if $2 < x \le 5$,

$Bx = 2$ if $x = 2$ or > 5, $Bx = x + 2$ if $2 < x \le 5$,

$Sx = 2$ if $x = 2$ or $x > 5$, $Sx = 8$ if $2 < x \le 5$,

$Tx = 2$ if $x = 2$ or > 5, $Tx = 9$ if $2 < x \le 5$,

Take $\left\{x_n = 5 + \frac{1}{n}\right\}$ and $\left\{y_n = 5 + \frac{1}{n}\right\}$.

Then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 2 \in X$. Thus the pairs (*A, S*) and (*B, T*) satisfy common property (*E.A.*). One can easily verify that self maps *A, B, S* and *T* satisfy all the conditions of the above theorem and have a unique common fixed point $x = 2$. Here *SX* and *TX* are closed subspaces of *X* where as neither *AX* nor *BX* is closed subspace of *X*. Also none of the four self maps is complete.

Moreover maps satisfy neither a φ-contractive condition nor a Banach type contractive condition. Also one may note that neither $BX \not\subset SX$ nor $AX \not\subset TX$ and, at the common fixed point x = 2, the self maps *A, B, S* and *T* are discontinuous. Moreover it is observed that common property (*E.A*) require completeness / closedness of subspaces for the existence of common fixed point, so attempt has been made to drop closedness of subspaces from theorem 3.3 using $JCLR_{ST}$ property.

Theorem 3.5: Let *A, B, S* and *T* be four self maps in fuzzy metric space $(X, M, *)$ satisfying condition (3.2) such that

(3.6) (*A, S*) and (*B, T*) satisfy the $JCLR_{ST}$ property.

Then the pairs (*A, S*) and (*B, T*) have a coincidence point. Further, if the pairs (*A, S*) and (*B, T*) are weakly compatible then *A, B, S* and *T* have a unique common fixed point in *X*.

Proof: As the pairs (*A, S*) and (*B, T*) satisfy the $JCLR_{ST}$ property, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in *X* such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = Su = Tu$$

for some $u \in X$.

We assert that $Au = Su$. From (3.2), taking $x = u, y = y_n$,

$$M(Au, By_n, t) > m(u, y_n, t) = \min\begin{Bmatrix} M(Su, Ty_n, t), M(Su, Au, t), M(By_n, Ty_n, t), \\ M(Au, By_n, t), M(By_n, Su, 2t) \end{Bmatrix},$$

and, taking the limit as $n \to \infty$, we have

$$\begin{aligned} M(Au, Su, t) &> \min\begin{Bmatrix} M(Sz, Su, t), M(Su, Au, t), M(Su, Su, t), \\ M(Au, Su, t), M(Su, Su, 2t) \end{Bmatrix} \\ &= M(Su, Au, t), \end{aligned}$$

a contradiction. Therefore, $Au = Su$, which shows that *u* is a coincidence point of the pair (*A, S*).

Now we assert that $Bu = Tu$. From (3.2), taking $x = u, y = u$, we get

$$M(Au,Bu,t) > m(u,u,t) = \min\left\{\begin{matrix} M(Su,Tu,t), M(Su,Au,t), M(Bu,Tu,t), \\ M(Au,Bu,t), M(Bu,Su,2t) \end{matrix}\right\}$$
$$= \min\{M(Su,Su,t), M(Tu,Tu,t), M(Bu,Tu,t), M(Tu,Bu,t), M(Bu,Tu,2t)\}$$
$$= M(Tu,Bu,t),$$

a contradiction. Hence $Bu = Tu$, which shows that u is a coincidence point of the maps B and T.
Thus $Tu = Bu = Au = Su$. Now we assume that $z = Tu = Bu = Au = Su$. Since the pairs (A, S) and (B, T) are weakly compatible, $Az = ASu = SAu = Sz$, $Bz = BTu = TBu = Tz$.
If $Az \neq z$, then, by using inequality (3.2) and taking $x = z$, $y = u$, we have

$$M(Az,Bu,t) > m(z,u,t) = \min\left\{\begin{matrix} M(Sz,Tu,t), M(Sz,Az,t), M(Bu,Tu,t), \\ M(Az,Bu,t), M(Bu,Sz,2t) \end{matrix}\right\}$$
$$= \min\{M(Az,z,t), M(Az,Az,t), M(z,z,t), M(Az,z,t), M(z,Az,2t)\}$$
$$= M(Az,z,t),$$

a contradiction. Therefore, $Az = z = Sz$.
Similarly, one can prove that $Bz = Tz = z$. Hence $Az = Bz = Sz = Tz$, and z is common fixed point of A, B, S and T. The uniqueness of the fixed point is an easy consequence of inequality (3.2). Hence the result.
We now give an example to illustrate the above theorem.

Example3.6. Let $X = [2, 20]$ and for each $t > 0$ and $x, y \in X$, define $(X, M, *)$ by

$$M(x,y,t) = \begin{cases} \dfrac{t}{t+|x-y|}, & t > 0, \\ 0 & t=0 \end{cases}.$$

Define self maps A, B, S and T on X by
$Ax = 2$ if $x = 2$ or > 5, $Ax = x + 1$ if $2 < x \le 5$,
$Bx = 2$ if $x = 2$ or > 5, $Bx = x + 2$ if $2 < x \le 5$,
$Sx = 2$ if $x = 2$ or $x > 5$, $Sx = x + 1$ if $2 < x \le 5$,
$Tx = 2$ if $x = 2$ or > 5, $Tx = x + 9$ if $2 < x \le 5$,
Take $\left\{x_n = 5 + \frac{1}{n}\right\}$ and $\left\{y_n = 5 + \frac{1}{n}\right\}$.

Then $lim_{n\to\infty}Ax_n = lim_{n\to\infty}Sx_n = lim_{n\to\infty}By_n = lim_{n\to\infty}Ty_n = 2 = S2 = T2$. Thus the pairs ($A$, S) and (B, T) satisfy the $JCLR_{ST}$ property. Also self maps A, B, S and T satisfy all of the conditions of the above theorem and have a unique common fixed point at $x = 2$. Notice that none of AX, BX, SX and TX is a closed or complete subspace of X. Also at the common fixed point $x = 2$ all of the self maps A, B, S and T are discontinuous. Moreover, maps satisfy neither a φ-contractive condition nor a Banach type contractive condition. Also neither $B(X) \not\subset S(X)$ nor $A(X) \not\subset T(X)$.

Remark 3.7: 1. Note that none of the self maps A, B, S and T is continuous at their common fixed point in all of the examples. Thus we have not only generalized the fixed point theorems of the form, but have also provided an answer to the problem of Rhoades [36] on the existence of a contractive condition, which is strong enough to generate a fixed point but does not force the map to be continuous at the common fixed point.
2. It is well known that the Meir-Keeler type contractive condition does not ensure the existence of a fixed point unless some additional condition is imposed on δ or a φ-contractive type condition is also used. However, in the present paper, we have neither imposed any additional condition on δ nor used a φ-contractive condition together with in any theorem. So we have improved the known results existing in literature.
3. In theorem 3.5 we have neither used completeness /closedness of subspaces nor containment of range of maps which is known to be essential to prove a common fixed point theorem. We have illustrated our argument with the help of examples.

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