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Connections Between Legendre with Hermite and Laguerre Matrix Polynomials

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ABSTRACT

The aim of this paper is to develop a connection between Legendre and Hermite matrix polynomials recently introduced in [8, 25] is derived. We also obtain various new generalized forms of the Legendre matrix polynomials by using the integral representation method. An expansion of Legendre matrix polynomials in a series of Laguerre matrix polynomials is established.

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1. INTRODUCTION

It is well known that many of the ordinary special functions of mathematical physics [5, 15] can be derived from the theory of group representations. More recently, Laguerre, Hermite and Gegenbauer matrix polynomials recent appeared in connection with the study of differential equations in papers [1-4, 8, 11, 13, 16- 27]. The problem of the development of matrix functions in a series of Legendre's matrix polynomials requires, which is not available in the literature, some new results about the matrix operational calculus. From this motivation, we prove some new properties for the Legendre matrix polynomials. The outline of this paper is as follows: Section 2 is to establish a connection between Legendre and Hermite matrix polynomials recently introduced in [8, 25]. In particular, the two-index Legendre matrix polynomials of two variables are presented. We get an expansion of the Legendre matrix polynomials in a series of Laguerre matrix polynomials. Finally, we define and study of the generalized Legendre matrix polynomials by means of the hypergeometric matrix function.

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1.1. Preliminaries

In this section, we will give some known facts, definitions, notation, lemma, theorem and properties to be used throughout the development in the following section.

Throughout the paper, we assume that A is a matrix in $C^{N\times N}$, its spectrum $\sigma(A)$ will denotes the set of all the eigenvalues of A. Furthermore the identity matrix of $C^{N\times N}$ will be denoted by I.

Fact 1.1 *(Dunford and Schwartz [6]) If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane and if A and B are matrices in $C^{N\times N}$ such that $\sigma(A)\subset\Omega$, $\sigma(B)\subset\Omega$ and $AB = BA$, then*

$$f(A)g(B) = g(B)f(A).$$

If D_0 is the complex plane cut along the negative real axis and $\log(z)$ denotes the principle logarithm of z, then $\sqrt{z}$ represents $z^{\frac{1}{2}} = \sqrt{z} = \exp(\frac{1}{2}\log(z))$. If A is a matrix in $C^{N\times N}$ with $\sigma(A)\subset D_0$, then $A^{\frac{1}{2}} = \sqrt{A} = \exp(\frac{1}{2}\log(A))$ denotes the image by $\sqrt{z}$ of the matrix functional calculus acting on the matrix A.

Definition 1.1 *(Jódar and Cortés [7]) If A is a positive stable matrix in $C^{N\times N}$, then the Gamma matrix function $\Gamma(A)$ has been defined as follows*

$$\Gamma(A) = \int_0^\infty e^{-t}t^{A-I}dt; \quad t^{A-I} = \exp((A-I)\ln t). \tag{1.1}$$

Definition 1.2 *(Jódar and Defez [10]) If A is a matrix in $C^{N\times N}$ such that*

$$A+nI \quad \text{is an invertible matrix for all integers } n\geq 0, \tag{1.2}$$

we have

$$(A)_n = A(A+I)...(A+(n-1)I) = \Gamma(A+nI)\Gamma^{-1}(A); \quad n\geq 1; \quad (A)_0 = I. \tag{1.3}$$

where $(A)_n$ is the Pochhammer symbol (the shifted factorial).

Notation 1.1 *Relying on [18], from the relation (1.3), it is easy to obtain*

$$(A)_{n-k} = (-1)^k (A)_n[(I-A-nI)_k]^{-1}; \quad 0\leq k\leq n, \tag{1.4}$$

and

$$\frac{(-1)^k}{(n-k)!}I = \frac{(-n)_k}{n!}I = \frac{(-nI)_k}{n!}; \quad 0\leq k\leq n. \tag{1.5}$$

Lemma 1.1 *(Defez and Jódar [5]) If $A(k,n)$ and $B(k,n)$ are matrices in $C^{N\times N}$ for $n\geq 0$, $k\geq 0$, the following relations are satisfied*

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]}A(k,n-2k),\\ \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}B(k,n-k).\end{aligned} \tag{1.6}$$

Similarly, we can write

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+2k),$$
$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}B(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B(k,n+k), \tag{1.7}$$

where $[.]$ is the greatest integer symbol.

Let us recall some important properties of the Hermite matrix polynomials.

Definition 1.3 *(Jódar and Company [8]) Let* A *be a positive stable matrix in* $C^{N\times N}$ *such that*

$$Re(z)>0, \quad \text{for all eigenvalues } z\in\sigma(A). \tag{1.8}$$

Then the n^{th} Hermite matrix polynomials are defined by

$$H_n(x,A)=n!\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k}{k!(n-2k)!}(x\sqrt{2A})^{n-2k}, n\geq 0. \tag{1.9}$$

Definition 1.4 *(Metwally et al. [14]) If* A *is a positive stable matrix in* $C^{N\times N}$ *, actually it satisfies the condition (1.8), then the two-index Hermite matrix polynomials of two variables are defined by the series*

$$H_{n,m}(x,y,A)=n!\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k y^k}{k!(n-mk)!}(x\sqrt{mA})^{n-mk}; \quad n\geq 0 \tag{1.10}$$

and the generating matrix function

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_{n,m}(x,y,A)=\exp(xt\sqrt{mA}-yt^m I). \tag{1.11}$$

Theorem 1.1 *(Metwally et al. [14]) The two-index Hermite matrix polynomials of two variables satisfy the following addition, multiplication theorems*

$$\alpha^n H_{n,m}(x,y,A)=H_{n,m}(\alpha x,\alpha^m y,A) \tag{1.12}$$

and

$$H_{n,m}(\alpha x+\beta x_1,y,A)=n!\sum_{k=0}^{n}\frac{H_{k,m}\left(\beta x_1,\frac{1}{2}y,A\right)H_{n-k,m}\left(\alpha x,\frac{1}{2}y,A\right)}{k!(n-k)!} \tag{1.13}$$

where α and β are constants, respectively.

Nota that for $m=2$, $H_{n,2}(x,y,A)=H_n(x,y,A)$ where $H_n(x,y,A)$ is Hermite matrix polynomials of two variables.

Definition 1.5 *(Jódar and Sastre [12]) Let* A *be a matrix in* $C^{N\times N}$ *such that*

$$-k \notin \sigma(A) \text{ for every integer } k > 0, \tag{1.14}$$

and λ is a complex number with $Re(\lambda) > 0$. Then the Laguerre matrix polynomials are defined by

$$L_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} \frac{(-1)^k (A+I)_n \left[(A+I)_k\right]^{-1} (\lambda x)^k}{k!(n-k)!}. \tag{1.15}$$

There are numerous interesting relations connecting the Hermite matrix polynomials and the Legendre matrix polynomials [26]. We quote two simple integral representation relations. One of relation is

$$P_n(x,A) = \frac{2}{n!\sqrt{\pi}} \int_0^\infty e^{-t^2} t^n H_n(xt,A)dt. \tag{1.16}$$

A real integral relation giving $H_n(x,A)$ in terms of $P_n(x,A)$ is

$$H_n(x,A) = 2^{n+1} \exp(x^2) \int_x^\infty e^{-t^2} t^{n+1} P_n\left(\frac{x}{t}, A\right) dt. \tag{1.17}$$

2. CONNECTIONS BETWEEN LEGENDRE, HERMITE AND LAGUERRE MATRIX POLYNOMIALS

Let A be a positive stable matrix in $C^{N\times N}$. Then the Legendre matrix polynomials is defined by (Upadhyaya and Shehata [25])

$$P_n(x,A) = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (2n-2k)! (x\sqrt{2A})^{n-2k}}{2^{2n-2k} k!(n-k)!(n-2k)!} \tag{2.1}$$

or by using the Hermite matrix polynomials of integral representation as follows

$$P_n(x,A) = \frac{2}{n!\sqrt{\pi}} \int_0^\infty t^n e^{-t^2} H_n(xt,A)dt. \tag{2.2}$$

By using the relation

$$H_n(x,y,A) = y^{\frac{1}{2}n} H_n\left(\frac{x}{\sqrt{y}}, A\right), \tag{2.3}$$

we can give a new connection Legendre matrix polynomials and Hermite matrix polynomials of two variables

$$P_n(x,A) = \frac{2}{n!\sqrt{\pi}} \int_0^\infty e^{-t^2} t^{2n} H_n\left(x, \frac{1}{t^2}, A\right) dt. \tag{2.4}$$

or

$$P_n(x,A) = \frac{2}{n!\sqrt{\pi}} \int_0^\infty e^{-t^2} H_n(xt^2, t^2, A)dt. \tag{2.5}$$

Thus, the following result has been established.

Theorem 2.1 *Let* A *be a positive stable matrix in* $C^{N\times N}$. *Then the integral expressions (2.4) and (2.5) hold true.*

Multiplying (2.5) by u^n and then summing up over n, we find

$$\sum_{n=0}^{\infty} P_n(x,A)u^n = \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-t^2} \sum_{n=0}^{\infty} H_n(xt^2,t^2,A)\frac{u^n}{n!}dt. \tag{2.6}$$

Using the generating matrix function of Hermite matrix polynomials of two variables, we get the following desired generating matrix function of the Legendre matrix polynomials:

$$\sum_{n=0}^{\infty} P_n(x,A)u^n = \left(I - xu\sqrt{2A} + u^2 I\right)^{-\frac{1}{2}}; \left\| xu\sqrt{2A} - u^2 I \right\| < 1.$$

Thus the result has been established.

Theorem 2.2 *If A is a positive stable matrix in $C^{N\times N}$, then the Legendre matrix polynomials have the following generating matrix function:*

$$\sum_{n=0}^{\infty} P_n(x,A)u^n = \left(I - xu\sqrt{2A} + u^2 I\right)^{-\frac{1}{2}}; \left\| xu\sqrt{2A} - u^2 I \right\| < 1. \tag{2.7}$$

Let us now introduce the two-index Legendre matrix polynomials of two variables of two matrices through the integral representation

$$P_{m,n}(x,y,A,B) = \frac{2}{m!n!\sqrt{\pi}}\int_0^{\infty} e^{-t^2} t^{2m+2n}\, H_m\left(x,\frac{1}{t^2},A\right) H_n\left(y,\frac{1}{t^2},B\right)dt. \tag{2.8}$$

where A and B are two positive stable matrices in $C^{N\times N}$. From (1.10) and (2.8), we obtain

$$P_{m,n}(x,y,A,B) = \sum_{k=0}^{[\frac{1}{2}m]}\sum_{l=0}^{[\frac{1}{2}n]} \frac{(-1)^{k+l}(2m+2n-2k-2l)!(x\sqrt{2A})^{m-2k}(y\sqrt{2B})^{n-2l}}{2^{2n+2m-2k-2l}\,k!l!\,(m-2k)!(n-2l)!(m+n-k-l)!}. \tag{2.9}$$

From (1.12), the integral representation (2.12) becomes

$$P_{m,n}(x,y,A,B) = \frac{2}{m!n!\sqrt{\pi}}\int_0^{\infty} e^{-t^2}\, H_m(xt^2,t^2,A)\, H_n(yt^2,t^2,B)dt. \tag{2.10}$$

It is worthy to mention that, on taking $m=0$ or $n=0$, (2.8), (2.9) and (2.10) of the two-index Legendre matrix polynomials of two variables reduce to (2.1), (2.4) and (2.5) of the Legendre matrix polynomials, respectively.

By (2.4), we have

$$P_n(x+z,A) = \frac{2}{n!\sqrt{\pi}}\int_0^{\infty} e^{-t^2} t^{2n}\, H_n\left((x+z),\frac{1}{t^2},A\right)dt. \tag{2.11}$$

Applying (1.13) in (2.11), we obtain

$$P_n(x+z,A) = \frac{2}{\sqrt{\pi}}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\int_0^{\infty} e^{-t^2} t^{2n}\, H_{n-k}\left(x,\frac{1}{2t^2},A\right) H_k\left(z,\frac{1}{2t^2},A\right)dt. \tag{2.12}$$

From (1.12), one gets

$$P_n(x+z,A)=\frac{2^{1-\frac{n}{2}}}{\sqrt{\pi}}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\int_0^{\infty}e^{-t^2}t^{2n}\,H_{n-k}\left(\sqrt{2}x,\frac{1}{t^2},A\right)H_k\left(\sqrt{2}z,\frac{1}{t^2},A\right)dt. \quad (2.13)$$

These results are summarized below.

Theorem 2.3 *If* A *is a positive stable matrix in* $C^{N\times N}$ *, then the Legendre matrix polynomials satisfy the addition formula as follow:*

$$P_n(x+z,A)=2^{-\frac{n}{2}}\sum_{k=0}^{n}P_{n-k,k}\left(\sqrt{2}x,\sqrt{2}z,A,A\right). \quad (2.14)$$

More generally, we can also the generalized Legendre-type matrix polynomials by using their integral representation.

Let A and B be positive stable matrices in $C^{N\times N}$, and $AB=BA$. We say generalized Legendre-type matrix polynomials, the new matrix polynomials defined by the following relation:

$$P_n(x,A,B)=\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k B^{k-n-\frac{1}{2}}(2n-2k)!(x\sqrt{2A})^{n-2k}}{2^{2n-2k}k!\,(n-2k)!(n-k)!} \quad (2.15)$$

or

$$P_n(x,A,B)=\frac{2}{n!\sqrt{\pi}}\int_0^{\infty}e^{-Bt^2}t^n\,H_n(xt,A)dt. \quad (2.16)$$

In a similar way, we define the generalized Legendre-type matrix polynomials of two variables by using Hermite matrix polynomials as follows:

$$P_n(x,y,A,B)=\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k B^{k-n-\frac{1}{2}}(2n-2k)!\,y^k(x\sqrt{2A})^{n-2k}}{2^{2n-2k}k!\,(n-2k)!(n-k)!} \quad (2.17)$$

or

$$P_n(x,y,A,B)=\frac{2}{n!\sqrt{\pi}}\int_0^{\infty}e^{-Bt^2}t^n\,H_n(xt,y,A)dt. \quad (2.18)$$

According to (1.12), the integral representation (2.18) becomes

$$P_n(x,y,A,B)=\frac{2}{n!\sqrt{\pi}}\int_0^{\infty}e^{-Bt^2}\,H_n(xt^2,yt^2,A)dt. \quad (2.19)$$

Also, we give the relation

$$P_n(x,y,A,B)=y^{\frac{n}{2}}P_n\left(\frac{x}{\sqrt{y}},A,B\right). \quad (2.20)$$

The generalized Legendre-type matrix polynomials of two variables involving Hermite matrix polynomials can be define in the following form

$$P_n^{\mu}(x,y,A,B)=\frac{2}{n!\sqrt{\pi}\Gamma(\mu)}\int_0^{\infty}e^{-Bt^2}t^{n+\mu-1}\,H_n(xt,y,A)dt. \quad (2.21)$$

From (1.10) and (2.21), we find that the generalized Legendre-type matrix polynomials are defined by the following series

$$P_n^{\mu}(x,y,A,B) = \frac{1}{\Gamma(\mu)} \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k B^{k-n-\frac{\mu}{2}} \Gamma(2n-2k+\mu-1) y^k (x\sqrt{2A})^{n-2k}}{2^{2n-2k+\mu} k!(n-2k)!\Gamma\left(n-k+\frac{1}{2}(\mu+1)\right)}, \qquad (2.22)$$

where $Re(\mu) > 0$.

Here, we give an expansion of the Legendre matrix polynomials in a series of Laguerre matrix polynomials relevant to the present investigation is summarized in the following theorem.

Theorem 2.4 *Let* A *be a positive stable matrix in* $C^{N\times N}$. *An expansion of Legendre matrix polynomials in a series of Laguerre matrix polynomials is given by*

$$\begin{aligned} P_n(x,A) = & \frac{(\sqrt{2A})^n \left(\frac{1}{2}\right)_n}{n!} (A+I)_n \sum_{s=0}^{n} {}_2F_3\Bigg(-\frac{1}{2}(n-s)I, -\frac{1}{2}(n-s-1)I \\ & ;(\frac{1}{2}-n)I, -\frac{1}{2}(A+nI), -\frac{1}{2}(A+(n-1)I); -\left(\frac{\lambda}{2\sqrt{2}}(\sqrt{A})^{-1}\right)^2\Bigg) \qquad (2.23) \\ & (-n)_s [(A+I)_s]^{-1} \lambda^{-n} L_s^{(A,\lambda)}(x) \end{aligned}$$

where $L_n^{(A,\lambda)}(x)$ stands for Laguerre matrix polynomials [12].

Proof. In [12], we recall the relation

$$x^n I = n!\, \lambda^{-n} \sum_{k=0}^{n} \frac{(-1)^k (A+I)_n [(A+I)_k]^{-1}}{(n-k)!} L_k^{(A,\lambda)}(x). \qquad (2.24)$$

From (2.1) and (2.24), we get

$$\begin{aligned} \sum_{n=0}^{\infty} P_n(x,A) t^n & = \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} (x\sqrt{2A})^{n-2k}}{k!(n-2k)!} t^n \\ & = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)_{n+k} (x\sqrt{2A})^n}{k!n!} t^{n+2k} \\ & = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{s=0}^{n} \frac{(-1)^{k+s} (\sqrt{2A})^n \lambda^{-n} \left(\frac{1}{2}\right)_{n+k} (A+I)_n [(A+I)_s]^{-1}}{k!(n-s)!} L_s^{(A,\lambda)}(x) t^{n+2k}. \end{aligned}$$

Using (1.7), we obtain

$$\sum_{n=0}^{\infty} P_n(x,A)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^{k+s}(\sqrt{2A})^{n+s}\left(\frac{1}{2}\right)_{n+s+k}(A+I)_{n+s}[(A+I)_s]^{-1}}{k!n!} \lambda^{-n-s} L_s^{(A,\lambda)}(x)t^{n+s+2k}.$$

From (1.6), we have

$$\sum_{n=0}^{\infty} P_n(x,A)t^n = \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^{k+s}(\sqrt{2A})^{n+s-2k}\left(\frac{1}{2}\right)_{n+s-k}(A+I)_{n+s-2k}[(A+I)_s]^{-1}}{k!(n-2k)!} \lambda^{-n-s+2k} L_s^{(A,\lambda)}(x)t^{n+s}.$$

Substituting the well-known identities

$$\frac{1}{(n-2k)!} = \frac{(-n)_{2k}}{n!}; 0 \le k \le \frac{1}{2}n,$$

$$\left(\frac{1}{2}\right)_{n+s-k} = (-1)^k \left(\frac{1}{2}\right)_{n+s}\left[\left(\frac{1}{2}-n-s\right)_k\right]^{-1},$$

$$(A+I)_{n+s-2k} = 2^{-2k}(A+I)_{n+s}\left[\left(-\frac{1}{2}\left(A+(n+s-1)I\right)\right)_k\right]^{-1}\left[\left(-\frac{1}{2}\left(A+(n+s)I\right)\right)_k\right]^{-1}.$$

Therefore, we have

$$\begin{aligned}\sum_{n=0}^{\infty} P_n(x,A)t^n = & \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^{k+s}(\sqrt{2A})^{n+s-2k}[(A+I)_s]^{-1}}{k!(n-2k)!} \\ & (-1)^k\left(\frac{1}{2}\right)_{n+s}[\left(\frac{1}{2}-n-s\right)_k]^{-1} 2^{-2k}(A+I)_{n+s} \\ & [\left(-\frac{1}{2}\left(A+(n+s-1)I\right)\right)_k]^{-1}[(-\frac{1}{2}(A+(n+s)I))_k]^{-1}\,\lambda^{-n-s+2k} L_s^{(A)}(x)t^{n+s} \\ = & \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} {}_2F_3\Bigg(-\frac{1}{2}nI, -\frac{1}{2}(n-1)I \\ & ;(\frac{1}{2}-n-s)I, -\frac{1}{2}(A+(n+s)I), -\frac{1}{2}(A+(n+s-1)I); -\left(\frac{\lambda}{2\sqrt{2}}(\sqrt{A})^{-1}\right)^2\Bigg) \\ & \frac{(-1)^s(\sqrt{2A})^{n+s}\left(\frac{1}{2}\right)_{n+s}}{n!}(A+I)_{n+s}[(A+I)_s]^{-1}\,\lambda^{-n-s} L_s^{(A,\lambda)}(x)t^{n+s}.\end{aligned}$$

From (1.6), we get

$$\sum_{n=0}^{\infty} P_n(x,A)t^n = \sum_{n=0}^{\infty}\sum_{s=0}^{n} {}_2F_3\Bigg(-\frac{1}{2}(n-s)I, -\frac{1}{2}(n-s-1)I$$
$$;(\frac{1}{2}-n)I, -\frac{1}{2}(A+nI), -\frac{1}{2}(A+(n-1)I); -\left(\frac{\lambda}{2\sqrt{2}}(\sqrt{A})^{-1}\right)^2\Bigg)$$
$$\frac{(-1)^s(\sqrt{2A})^n\left(\frac{1}{2}\right)_n}{(n-s)!}(A+I)_n[(A+I)_s]^{-1}\lambda^{-n}L_s^{(A,\lambda)}(x)t^n.$$

On comparing the coefficients of t^n, we obtain (2.23). Thus the proof is completed. Finally, it is now interesting to extend the above results to new generalized forms of generalized Legendre matrix polynomials of two variables can be defined in the form:

$$P_n^B(x,y,A) = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k(2n-2k)!}{2^{2n-2k}k!(n-k)!(n-2k)!} \mathrm{B}_{n,k}(x\sqrt{2A})^{n-2k} \qquad (2.25)$$

where

$$\mathrm{B}_{n,k} = \mathrm{B}_{n,k}(y;B) = y^k \; {}_2F_1(-kI, B; -nI; y) = \sum_{i=0}^{k} \frac{k!(n-i)!}{i!n!(k-i)!}(B)_i y^{k+i}, \qquad (2.26)$$

where A and B are commutative matrices in $C^{N\times N}$ such that A satisfies the condition (1.8) and B satisfies the condition (1.2).

If B is the zero matrix, then the Legendre matrix polynomials of two variables reduce to

$$P_n^0(x,y,A) = P_n(x,y,A) \qquad (2.27)$$

From (2.26), we can write in the following integral representation

$$\mathrm{B}_{n,k} = \frac{y^k}{n!}\Gamma^{-1}(B)\int_0^{\infty}\int_0^{\infty} e^{-(t+u)}t^n u^{B-I}\left(1+\frac{yu}{t}\right)^k dtdu. \qquad (2.28)$$

Theorem 2.5 *Let A and B be commutative matrices in $C^{N\times N}$ such that A satisfy the condition (1.8) and B satisfy the condition (1.2). Then the generalized Legendre matrix polynomials of two variables have the following integral representation:*

$$P_n^B(x,y,A) = \frac{1}{n!}\Gamma^{-1}(B)\int_0^{\infty}\int_0^{\infty} e^{-(t+u)}t^n u^{B-I} P_n\left(x, y(1+\frac{yu}{t}), A\right)dtdu. \qquad (2.29)$$

Proof. Using (2.25) and (2.28), we obtain (2.29). Thus the result is completed.

In this paper, several new families of Legendre matrix polynomials are introduced using the integral representation. The possibility of combining these results to study new families of Legendre matrix polynomials is a problem for further work.

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CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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