



PRO- \mathcal{C} COMPLETIONS OF CROSSED SQUARES OF COMMUTATIVE ALGEBRAS

HATICE GÜLSÜN AKAY

ABSTRACT. In this paper we give the explicit construction of a pro- \mathcal{C} completion functor which is defined in the category of crossed squares of commutative algebras. Afterwards, we study some functorial properties of this pro- \mathcal{C} completion process.

1. INTRODUCTION

A profinite group [9] occurs in a wide range of problems related to number theory, commutative algebra, algebraic geometry and algebraic topology. Although the category of profinite groups forms a natural extension of the category of finite groups, it carries a richer structure. Because it has some categorical objects and constructions which do not exist in finite case; e.g. projective limits and free products. The existence of such constructions in extended category leads to the definition of profinite analogues of the usual constructions of combinatorial group theory, such as free groups and presentations of group by generators and relations.

The theory of crossed modules [10] plays an important role in combinatorial and cohomological group theory. Profinite crossed modules are introduced in [7]. They examined the pro- \mathcal{C} completion of crossed modules for a full class of finite groups \mathcal{C} . The crossed square version of this completion process is given in [3]. The definition of crossed modules over a commutative algebra is given in [8]; also see [5] for the general case. Crossed squares in the category of commutative algebras studied in [1, 4].

2. PRELIMINARIES

In this paper, k will be a fixed commutative ring with $1 \neq 0$ for abstract cases and k will be a fixed commutative profinite ring with $1 \neq 0$ for topological cases.

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All k -algebras will be commutative and associative. \mathcal{C} will denote a class of finite k -algebras which is closed under the formation of subalgebras, homomorphic images, finite products and which contains at least one non-trivial algebra. Pro- \mathcal{C} algebras are profinite algebras whose finite quotients are in \mathcal{C} . The class \mathcal{C} will be assumed to be full in the sense that \mathcal{C} must also be closed under extension of algebras.

Throughout this paper we denote an action of P on M by pm , where P and M are k -algebras. We recall the definition of a crossed module, and its pro- \mathcal{C} analogue.

A crossed module [8] is a k -algebra homomorphism $\partial: M \rightarrow P$ together with an action of P on M such that following two Peiffer relations hold (for all $m, m' \in M$ and $p \in P$):

$$\begin{aligned} CM1) \quad & \partial(pm) = p\partial(m), \\ CM2) \quad & \partial(m)m' = mm'. \end{aligned}$$

We denote such a crossed module by (M, P, ∂) .

If (M, P, ∂) and (M', P', ∂') are two crossed modules, a crossed module morphism $(\phi, \psi): (M, P, \partial) \rightarrow (M', P', \partial')$ is a tuple which consists of k -algebra homomorphisms, $\phi: M \rightarrow M'$, $\psi: P \rightarrow P'$ such that $\psi\partial = \partial'\phi$ and $\phi(pm) = \psi(p)\phi(m)$. Thus we get the category of crossed modules, denoted by **XMod**.

There are special classes of morphisms, those in which $P = P'$ and ψ is the identity morphism. For fixed P , a morphism $(\phi, id_P): (M, P, \partial) \rightarrow (M', P, \partial')$ will be called a morphism of crossed modules over P . Then we have a subcategory **XMod/P** of **XMod**.

Let (M, P, ∂) be a crossed module. (M_1, P_1, ∂_1) is a subcrossed module of (M, P, ∂) if:

- i) M_1 is a subalgebra of M and P_1 is a subring of P ,
- ii) $\partial_1 = \partial|_{M_1}$, the restriction of ∂ to M_1 ,
- iii) The action of P_1 on M_1 is induced by the action of P on M .

A subcrossed module (M_1, P_1, ∂_1) of (M, P, ∂) is a crossed ideal if:

- i) P_1 is an ideal of P and M_1 is an ideal of M ,
- ii) $pm_1 \in M_1$, for all $p \in P, m_1 \in M_1$,
- iii) $p_1m \in M_1$, for all $p_1 \in P_1, m \in M$.

A pro- \mathcal{C} crossed module (M, P, ∂) is a crossed module in which M and P are pro- \mathcal{C} k -algebras, ∂ is a continuous k -algebra homomorphism and the action of P on M is a continuous P -action [6].

A morphism of pro- \mathcal{C} crossed modules

$$(\phi, \psi): (M, P, \partial) \rightarrow (M', P', \partial')$$

is a morphism of the underlying crossed modules in which both ϕ and ψ are continuous morphisms of pro- \mathcal{C} k -algebras. Thus we get the categories **Pro- \mathcal{C} .XMod**

and similarly $\mathbf{Pro-C.XMod}/P$ for a fixed codomain P ; we therefore obtain the forgetful functor:

$$\mathcal{U}_{\mathbf{XMod}} : \mathbf{Pro-C.XMod} \rightarrow \mathbf{XMod}.$$

Recall from [8] that a cat^1 -algebra is a triple (E, s, t) , where E is an k -algebra and s, t are endomorphisms of E satisfying the following conditions:

- i) $st = t$ and $ts = s$
- ii) $[Kers, Kert] = 0$.

It is well known that there is an equivalence of between the categories \mathbf{XMod} and $\mathbf{Cat}^1(\mathbf{Alg})$.

A pro- \mathcal{C} cat^1 -algebra is a cat^1 -algebra (E, s, t) in which E is a pro- \mathcal{C} algebra and s and t are continuous endomorphisms of E . A morphism of pro- \mathcal{C} cat^1 -algebra is a morphism

$$\phi : (E, s, t) \rightarrow (E', s', t')$$

of the underlying cat^1 -algebras such that $\phi: E \rightarrow E'$ is a continuous morphism of pro- \mathcal{C} algebras. Thus we get the category of pro- \mathcal{C} cat^1 -algebras, denoted by $\mathbf{Pro-C.Cat}^1(\mathbf{Alg})$. There is a forgetful functor:

$$\mathcal{U}_{\mathbf{CAlg}} : \mathbf{Pro-C.Cat}^1(\mathbf{Alg}) \rightarrow \mathbf{Cat}^1(\mathbf{Alg})$$

It is proven in [6] that, there exists an equivalence of categories $\mathbf{Pro-C.XMod}$ and $\mathbf{Pro-C.Cat}^1(\mathbf{Alg})$ compatible with the forgetful functors, in the sense of the equivalence between \mathbf{XMod} and $\mathbf{Cat}^1(\mathbf{Alg})$.

3. CROSSED SQUARES AND THEIR PRO- \mathcal{C} ANALOGUE

3.1. **Crossed squares.** The following definition is due to [1].

A crossed square of commutative algebras is a commutative diagram:

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ \lambda' \downarrow & & \downarrow \mu \\ N & \xrightarrow{\mu'} & P \end{array}$$

together with actions of P on L, M and N (there are thus actions of N on L and M via μ' , and of M on L and N via μ) and a function $h : M \times N \rightarrow L$ such that:

- 1) The maps $\lambda, \lambda', \mu, \mu'$ and the composite $\mu\lambda = \mu'\lambda'$ are crossed modules,
- 2) The maps λ, λ' preserve the action of P ,
- 3) $kh(m, n) = h(km, n) = h(m, kn)$,
- 4) $h(m + m', n) = h(m, n) + h(m', n)$,
- 5) $h(m, n + n') = h(m, n) + h(m, n')$,
- 6) $ph(m, n) = h(pm, n) = h(m, pn)$,
- 7) $\lambda h(m, n) = m\mu'(n)$,
- 8) $\lambda' h(m, n) = \mu(m)n$,

- 9) $h(\lambda(l), n) = l\mu'(n)$,
- 10) $h(m, \lambda'(l)) = \mu(m)l$

for all $n, n' \in N$, $m, m' \in M$, $p \in P$, $l \in L$, $k \in k$. We denote such a crossed square by (L, M, N, P) .

Let μ and μ' are normal subalgebra inclusions and $L = M \cap N$, with h is given by the multiplication in P , i.e., $h(m, n) = mn$. Then, we have the crossed square:

$$\begin{array}{ccc} M \cap N & \xrightarrow{\lambda} & M \\ \lambda' \downarrow & & \downarrow \mu \\ N & \xrightarrow{\mu'} & P \end{array}$$

A morphism of crossed square

$$\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4) : (L_1, M_1, N_1, P_1) \rightarrow (L_2, M_2, N_2, P_2),$$

consists of homomorphisms:

$$\begin{array}{ll} \Phi_1 : L_1 \rightarrow L_2, & \Phi_2 : M_1 \rightarrow M_2, \\ \Phi_3 : N_1 \rightarrow N_2, & \Phi_4 : P_1 \rightarrow P_2, \end{array}$$

such that the diagram commutes:

$$\begin{array}{ccccc} & & L_1 & \xrightarrow{\lambda_1} & M_1 \\ & \searrow \Phi_1 & \downarrow \lambda'_1 & & \downarrow \mu_1 \\ & & N_1 & \xrightarrow{\mu'_1} & P_1 \\ & \swarrow \Phi_2 & & & \swarrow \Phi_3 \\ L_2 & \xrightarrow{\lambda_2} & M_2 & & P_2 \\ \lambda'_2 \downarrow & \swarrow \Phi_4 & \downarrow \mu_2 & & \downarrow \mu_2 \\ N_2 & \xrightarrow{\mu'_2} & P_2 & & \end{array}$$

and

$$\Phi_1 h(m_1, n_1) = h(\Phi_2(m_1), \Phi_3(n_1)),$$

for all $m_1 \in M_1, n_1 \in N_1$, and the homomorphisms Φ_1, Φ_2, Φ_3 are Φ_4 -equivariant.

Thus we get the category of crossed squares, denoted by \mathbf{Crs}^2 .

There are special classes of morphisms, those in which $P_1 = P_2$ and Φ_4 is the identity morphism. For a fixed P , such a morphism

$$\Phi = (\Phi_1, \Phi_2, \Phi_3, id) : (L_1, M_1, N_1, P) \rightarrow (L_2, M_2, N_2, P),$$

will be called a morphism of crossed squares over P , yields a subcategory \mathbf{Crs}^2/P .

A crossed square

$$\begin{array}{ccc} L_1 & \xrightarrow{\lambda_1} & M_1 \\ \lambda'_1 \downarrow & & \downarrow \mu_1 \\ N_1 & \xrightarrow{\mu'_1} & P_1 \end{array}$$

is a subcrossed square of

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ \lambda' \downarrow & & \downarrow \mu \\ N & \xrightarrow{\mu'} & P \end{array}$$

if,

- i) L_1 is a subalgebra of L , M_1 is a subring of M , N_1 is a subring of N , P_1 is a subring of P ,
- ii) λ_1 is restriction of λ to L_1 , μ_1 is restriction of μ to M_1 , λ'_1 is restriction of λ' to L_1 , μ'_1 is restriction of μ' to N_1 ,
- iii) Actions of P_1 on L_1, M_1 and N_1 are induced by the actions of P on L, M and N , respectively.
- iv) $h_1 : M_1 \times N_1 \rightarrow L_1$ is the restriction of $h : M \times N \rightarrow L$ to $M_1 \times N_1$.

A subcrossed square

$$\begin{array}{ccc} L_1 & \xrightarrow{\lambda_1} & M_1 \\ \lambda'_1 \downarrow & & \downarrow \mu_1 \\ N_1 & \xrightarrow{\mu'_1} & P_1 \end{array}$$

is an ideal of

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ \lambda' \downarrow & & \downarrow \mu \\ N & \xrightarrow{\mu'} & P \end{array}$$

if,

- i) P_1 is an ideal of P , M_1 is an ideal of M , N_1 is an ideal of N ,
- ii) For all $p \in P$, $l_1 \in L_1$, $m_1 \in M_1$ and $n_1 \in N_1$,

$$\begin{aligned} pl_1 &\in L_1, \\ pm_1 &\in M_1, \\ pn_1 &\in N_1. \end{aligned}$$

iii) For all $p_1 \in P_1, l \in L, m \in M$ and $n \in N$,

$$\begin{aligned} p_1 l &\in L_1, \\ p_1 m &\in M_1, \\ p_1 n &\in N_1. \end{aligned}$$

Let

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ \lambda' \downarrow & & \downarrow \mu \\ N & \xrightarrow{\mu'} & P \end{array}$$

be a crossed square and

$$\begin{array}{ccc} L_1 & \xrightarrow{\lambda_1} & M_1 \\ \lambda'_1 \downarrow & & \downarrow \mu_1 \\ N_1 & \xrightarrow{\mu'_1} & P_1 \end{array}$$

be an ideal of (L, M, N, P) . Let $\bar{\lambda}$ is induced by $\lambda, \bar{\mu}$ is induced by $\mu, \bar{\lambda}'$ is induced by $\lambda', \bar{\mu}'$ is induced by μ' . Then there are actions of P/P_1 on $L/L_1, M/M_1$ and N/N_1 given by

$$\begin{aligned} (p + P_1)(l + L_1) &= (pl) + L_1, \\ (p + P_1)(m + M_1) &= pm + M_1, \\ (p + P_1)(n + N_1) &= pn + N_1 \end{aligned}$$

and N/N_1 on L/L_1 and M/M_1 via $\bar{\mu}'$, i.e.,

$$\begin{aligned} (n + N_1)(l + L_1) &= \mu'(n)l + L_1, \\ (n + N_1)(m + M_1) &= \mu'(n)m + M_1 \end{aligned}$$

and then M/M_1 acts on L/L_1 and N/N_1 via $\bar{\mu}$, i.e.,

$$\begin{aligned} (m + M_1)(l + L_1) &= \mu(m)l + L_1, \\ (m + M_1)(n + N_1) &= \mu(m)n + N_1. \end{aligned}$$

for all $p \in P, l \in L, m \in M$ and $n \in N$. The conditions for (L_1, M_1, N_1, P_1) to be ideal in (L, M, N, P) ensure that the actions are well defined. Let $h_1 : M_1 \times N_1 \rightarrow L_1$ is defined by $h(m + M_1, n + N_1) = h(m, n) + L_1$. It is clear that:

$$\begin{array}{ccc} L/L_1 & \xrightarrow{\bar{\lambda}} & M/M_1 \\ \bar{\lambda}' \downarrow & & \downarrow \bar{\mu} \\ N/N_1 & \xrightarrow{\bar{\mu}'} & P/P_1 \end{array}$$

is a crossed square. It is called the quotient crossed square of (L, M, N, P) by (L_1, M_1, N_1, P_1) and denoted by $(L, M, N, P)/(L_1, M_1, N_1, P_1)$.

3.2. **Pro- \mathcal{C} crossed squares.** A pro- \mathcal{C} crossed square of algebras

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ \lambda' \downarrow & & \downarrow \mu \\ N & \xrightarrow{\mu'} & P \end{array}$$

is a crossed square in which L, M, N and P are pro- \mathcal{C} algebras, $\lambda, \lambda', \mu, \mu'$ are continuous homomorphisms, all the actions are continuous and the h -map is continuous.

A morphism

$$\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4) : (L_1, M_1, N_1, P_1) \rightarrow (L_2, M_2, N_2, P_2)$$

of pro- \mathcal{C} crossed squares is a morphism of the underlying crossed squares in which all the maps Φ_1, Φ_2, Φ_3 and Φ_4 are continuous morphisms of pro- \mathcal{C} algebras. Thus we get the categories **Pro- \mathcal{C} .Crs²** and similarly **Pro- \mathcal{C} .Crs²/ P** for a fixed codomain P ; and we also get the forgetful functor:

$$\mathcal{U}_{\text{Crs}^2} : \mathbf{Pro-}\mathcal{C}.\mathbf{Crs}^2 \rightarrow \mathbf{Crs}^2.$$

Recall the corresponding situation for k -algebras; the forgetful functor

$$\mathcal{U}_{\mathbf{Alg}} : \mathbf{Pro-}\mathcal{C}.\mathbf{Alg} \rightarrow \mathbf{Alg}$$

has a left adjoint, known as the pro- \mathcal{C} completion functor, which we will denote by a “ $\widehat{}$ ”.

This is defined as follows:

Let P be a k -algebra and let $\Omega(P)$ be the directed set of finite index ideals W of P with $P/W \in \mathcal{C}$, then

$$\widehat{P} = \varprojlim_{W \in \Omega(P)} P/W.$$

We will sometimes write $W_{\text{fin}}P$ as indicating that $W \in \Omega(P)$. This notation is useful in as it is more suggestive of the actual concept involved, but can also become somewhat cumbersome so we will use both notations.

We wish to see if the crossed square forgetful functor

$$\mathcal{U}_{\text{Crs}^2} : \mathbf{Pro-}\mathcal{C}.\mathbf{Crs}^2 \rightarrow \mathbf{Crs}^2$$

also has a left adjoint. The obvious approach using some idea of finite index ideals is technically messy so we use an equivalence formulation involving Loday’s notion of cat^2 -algebras.

4. CAT²-ALGEBRAS AND THEIR PRO-C ANALOGUE

4.1. **Cat²-algebras.** A cat²-algebra [1] is a 5-tuple (E, s_1, t_1, s_2, t_2) where (E, s_i, t_i) , $i = 1, 2$ are cat¹-algebras and

$$s_i s_j = s_j s_i, \quad t_i t_j = t_j t_i, \quad s_i t_j = t_j s_i$$

for $i, j = 1, 2, i \neq j$.

If (E, s_1, t_1, s_2, t_2) and $(E', s'_1, t'_1, s'_2, t'_2)$ are cat²-algebras a cat²-algebra morphism:

$$\phi : (E, s_1, t_1, s_2, t_2) \rightarrow (E', s'_1, t'_1, s'_2, t'_2)$$

is an algebra homomorphism $\phi : E \rightarrow E'$ such that:

$$\begin{aligned} s'_1 \phi &= \phi s_1, \\ t'_1 \phi &= \phi t_1, \\ s'_2 \phi &= \phi s_2, \\ t'_2 \phi &= \phi t_2. \end{aligned}$$

Thus we get the category of cat²-algebras, denoted by **Cat²(Alg)**.

Proposition 1. *There is an equivalence of categories between the category of cat²-algebras and that of crossed squares.*

Proof. The cat¹-algebra (E, s_1, t_1) will give us a crossed module

$$\partial : C \rightarrow B$$

with $C = \text{Kers}$, $B = \text{Im}s$ and $\partial = t \mid C$, but as the two cat¹-algebra structures are independent, (E, s_2, t_2) restricts to give cat¹-algebra structures on C and B and makes ∂ a morphism of cat¹-algebras. Thus we get a morphism of crossed modules

$$\begin{array}{ccc} \text{Kers}_1 \cap \text{Kers}_2 & \longrightarrow & \text{Im}s_1 \cap \text{Kers}_2 \\ \downarrow & & \downarrow \\ \text{Kers}_1 \cap \text{Im}s_2 & \longrightarrow & \text{Im}s_1 \cap \text{Im}s_2 \end{array}$$

where each morphism is a crossed module for the natural action, i.e. multiplication in E . It remains to produce an h -map, but it is given by the multiplication within E since if $x \in \text{Kers}_2 \cap \text{Im}s_1$ and $y \in \text{Im}s_2 \cap \text{Kers}_1$ then $xy \in \text{Kers}_1 \cap \text{Kers}_2$. It is easy to check the crossed squares axioms.

Conversely, if

$$\begin{array}{ccc} L & \longrightarrow & M \\ \downarrow & & \downarrow \\ N & \longrightarrow & R \end{array}$$

is a crossed square, then we can think of it as a morphism of crossed modules

$$\begin{array}{c} L \rightarrow N \\ \downarrow \\ M \rightarrow R. \end{array}$$

Using the equivalence between crossed modules and cat^1 -algebras this gives a morphism

$$\partial : (L \rtimes N, s, t) \longrightarrow (M \rtimes R, s', t')$$

of cat^1 -algebras. There is an action of $M \rtimes R$ on $L \rtimes N$ given by

$$(m, r)(l, n) = (rl + \partial(m)l + h(m, n), rn + mn)$$

for all $(m, r) \in M \rtimes R$ and $(l, n) \in L \rtimes N$. Using this action, we thus form its associated cat^1 -algebra with algebra $(L \rtimes N) \rtimes (M \rtimes R)$ and induced endomorphisms s_1, t_1, s_2, t_2 . \square

4.2. Pro- \mathcal{C} cat^2 -algebras.

Definition 2. A pro- \mathcal{C} cat^2 -algebra is a cat^2 -algebra (E, s_1, t_1, s_2, t_2) in which E is a pro- \mathcal{C} algebra and s_1, s_2, t_1 and t_2 are continuous endomorphisms of E .

A morphism of pro- \mathcal{C} cat^2 -algebra

$$\phi : (E, s_1, t_1, s_2, t_2) \rightarrow (E', s'_1, t'_1, s'_2, t'_2)$$

is a morphism of the underlying cat^2 -algebras such that ϕ is a continuous morphism of pro- \mathcal{C} algebras. Thus we get the category of pro- \mathcal{C} cat^2 -algebras, denoted by **Pro- \mathcal{C} . $\text{Cat}^2(\text{Alg})$** .

There is a forgetful functor:

$$\mathcal{U}_{\mathcal{C}^2\text{Alg}} : \text{Pro-}\mathcal{C}.\text{Cat}^2(\text{Alg}) \rightarrow \text{Cat}^2(\text{Alg}).$$

Theorem 3. There exists an equivalence of categories **Pro- \mathcal{C} . Crs^2** and **Pro- \mathcal{C} . $\text{Cat}^2(\text{Alg})$** compatible with the forgetful functors, in the sense of the equivalence between **Crs^2** and **$\text{Cat}^2(\text{Alg})$** , i.e. the following diagram commutes:

$$\begin{array}{ccc} \text{Pro-}\mathcal{C}.\text{Crs}^2 & \xrightarrow{\cong} & \text{Pro-}\mathcal{C}.\text{Cat}^2(\text{Alg}) \\ \mathcal{U}_{\text{Crs}^2} \downarrow & & \downarrow \mathcal{U}_{\mathcal{C}^2\text{Alg}} \\ \text{Crs}^2 & \xrightarrow{\cong} & \text{Cat}^2(\text{Alg}) \end{array}$$

Proof. In fact, if

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ \chi' \downarrow & & \downarrow \mu \\ N & \xrightarrow{\mu'} & P \end{array}$$

is a pro- \mathcal{C} crossed square, then $E = (L \times M) \times (N \times P)$ is a pro- \mathcal{C} algebra and the endomorphisms s_1, s_2, t_1 and t_2 , given before, are continuous, so resulting (E, s_1, t_1, s_2, t_2) is a pro- \mathcal{C} cat^2 -algebra. Similarly if (E, s_1, t_1, s_2, t_2) is a pro- \mathcal{C} cat^2 -algebra then

$$\begin{array}{ccc} \text{Kers}_1 \cap \text{Kers}_2 & \xrightarrow{\lambda} & \text{Im}s_1 \cap \text{Kers}_2 \\ \chi' \downarrow & & \downarrow \mu \\ \text{Kers}_1 \cap \text{Im}s_2 & \xrightarrow{\mu'} & \text{Im}s_1 \cap \text{Im}s_2 \end{array}$$

is a pro- \mathcal{C} crossed square. □

This lemma will enable us to prove the existence of a left adjoint for

$$\mathcal{U}_{\text{Crs}^2} : \mathbf{Pro-C.Crs}^2 \longrightarrow \mathbf{Crs}^2$$

by constructing one for

$$\mathcal{U}_{\mathbf{C}^2 \mathbf{Alg}} : \mathbf{Pro-C.Cat}^2(\mathbf{Alg}) \longrightarrow \mathbf{Cat}^2(\mathbf{Alg}).$$

This latter construction will need projective limit within $\mathbf{Pro-C.Cat}^2(\mathbf{Alg})$ and so we will briefly look at their construction as it sheds more light on the pro- \mathcal{C} completion functor that will result from their use.

5. THE COMPLETION PROCESS

Given a projective system $F : I \rightarrow \mathbf{Cat}^2(\mathbf{Alg})$, one notes that F is a projective system of algebras together with four endomorphisms of projective systems, $s_1, s_2, t_1, t_2 : F \rightarrow F$ satisfying

$$s_i s_j = s_j s_i, \quad t_i t_j = t_j t_i, \quad s_i t_j = t_j s_i$$

for $i, j = 1, 2, i \neq j$ and $[\text{Kers}_1, \text{Kert}_1] = 0, [\text{Kers}_2, \text{Kert}_2] = 0$. We form $\varprojlim F$ by taking the limit of this underlying system of pro- \mathcal{C} algebras together with the induced endomorphism $\varprojlim s$ and $\varprojlim t$. Writing the result as $(\overline{F}, \overline{s}_1, \overline{t}_1, \overline{s}_2, \overline{t}_2)$, we only need to check the conditions $[\text{Ker}\overline{s}_1, \text{Ker}\overline{t}_1] = 0$ and $[\text{Ker}\overline{s}_2, \text{Ker}\overline{t}_2] = 0$. However \overline{F} can be realized as a subalgebra of the product $\prod_{i \in I} F(i)$ and

$$\overline{s}_1((x_i)) = (s_1(i)x_i, \overline{t}_1(x_i)) = (t_1(i)x_i),$$

similarly for $\overline{s}_2, \overline{t}_1$ and \overline{t}_2 . So the commutator subalgebras $[\text{Ker}\overline{s}_1, \text{Ker}\overline{t}_1]$ and $[\text{Ker}\overline{s}_2, \text{Ker}\overline{t}_2]$ are trivial for each i in I .

Proposition 4. *A pro- \mathcal{C} completion functor from the category $\mathbf{Cat}^2(\mathbf{Alg})$ to the category $\mathbf{Pro-}\mathcal{C}.\mathbf{Cat}^2(\mathbf{Alg})$ exists, (i.e. the forgetful functor $\mathcal{U}_{\mathcal{C}^2\mathbf{Alg}}$ has a left adjoint).*

Proof. An exact sequence

$$0 \longrightarrow (E', s'_1, t'_1, s'_2, t'_2) \xrightarrow{u} (E, s_1, t_1, s_2, t_2) \xrightarrow{v} (E'', s''_1, t''_1, s''_2, t''_2) \longrightarrow 0$$

of \mathbf{cat}^2 -algebras is an exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

of the underlying algebras and continuous maps compatible with the source and target maps. In this situation, we say that the \mathbf{cat}^2 -algebra $(E'', s''_1, t''_1, s''_2, t''_2)$ is the quotient of (E, s_1, t_1, s_2, t_2) by the ideal $(E', s'_1, t'_1, s'_2, t'_2)$. The latter is of finite index in (E, s_1, t_1, s_2, t_2) if E' is finite.

Given any \mathbf{cat}^2 -algebra (E, s_1, t_1, s_2, t_2) the set of its ideals $(I, s'_1, t'_1, s'_2, t'_2)$ of finite index with $E/I \in \mathcal{C}$ is directed by the inclusion so we can form an inverse system of finite quotient of (E, s_1, t_1, s_2, t_2) and take its limit within the category of pro- \mathcal{C} \mathbf{cat}^2 -algebras. (As usual one considers each finite \mathbf{cat}^2 -algebra as a pro- \mathcal{C} one having the discrete topology.)

Thus we define a pro- \mathcal{C} completion functor:

$$\widetilde{} : \mathbf{Cat}^2(\mathbf{Alg}) \rightarrow \mathbf{Pro-}\mathcal{C}.\mathbf{Cat}^2(\mathbf{Alg}) \tag{1}$$

by

$$(E, s_1, t_1, s_2, t_2) \widetilde{} = \varprojlim \{ \text{finite quotients of } (E, s_1, t_1, s_2, t_2) \text{ by } (I, s'_1, t'_1, s'_2, t'_2) \}.$$

Categorically this functor is left adjoint to the forgetful functor from $\mathbf{Pro-}\mathcal{C}.\mathbf{Cat}^2(\mathbf{Alg})$ to $\mathbf{Cat}^2(\mathbf{Alg})$. \square

Proposition 5. *A pro- \mathcal{C} completion functor from \mathbf{Crs}^2 to $\mathbf{Pro-}\mathcal{C}.\mathbf{Crs}^2$ exists (i.e. the forgetful functor $\mathcal{U}_{\mathbf{Crs}^2}$ has a left adjoint).*

Proof. In the diagram

$$\begin{array}{ccc} \mathbf{Pro-}\mathcal{C}.\mathbf{Crs}^2 & \xrightarrow{\cong} & \mathbf{Pro-}\mathcal{C}.\mathbf{Cat}^2(\mathbf{Alg}) \\ \mathcal{U}_{\mathbf{Crs}^2} \downarrow & & \downarrow \mathcal{U}_{\mathcal{C}^2\mathbf{Alg}} \\ \mathbf{Crs}^2 & \xrightarrow{\cong} & \mathbf{Cat}^2(\mathbf{Alg}) \end{array}$$

we already found that (1) is the left adjoint functor to right vertical functor. This induces a left adjoint functor to left vertical functor via the equivalence of categories. \square

Remark 6. One can attempt to use the functors defining the two equivalence to give an “explicit” description of this pro- \mathcal{C} completion functor, but in what follows we shall merely use its existence and the universal property that it satisfies to compare it with the individual algebras involved.

Notation 7. We denote the pro- \mathcal{C} completion of the crossed square $(\widetilde{(L, M, N, P)})$ or less accurately, $(\widetilde{L}, \widetilde{M}, \widetilde{N}, \widetilde{P})$, as follows:

$$\begin{array}{ccc}
 L & \xrightarrow{\lambda} & M \\
 \chi' \downarrow & & \downarrow \mu \\
 N & \xrightarrow{\mu'} & P
 \end{array}
 \quad \text{by} \quad
 \begin{array}{ccc}
 \widetilde{L} & \xrightarrow{\widetilde{\lambda}} & \widetilde{M} \\
 \widetilde{\chi}' \downarrow & & \downarrow \widetilde{\mu} \\
 \widetilde{N} & \xrightarrow{\widetilde{\mu}'} & \widetilde{P}
 \end{array}$$

It is natural to compare this pro- \mathcal{C} completion $(\widetilde{L}, \widetilde{M}, \widetilde{N}, \widetilde{P})$ with the pro- \mathcal{C} completions $\widehat{L}, \widehat{M}, \widehat{N}, \widehat{P}$ and $\widehat{\lambda}, \widehat{\mu}, \widehat{\lambda}', \widehat{\mu}'$ of the individual pieces of data involved. One may even wonder why

$$\begin{array}{ccc}
 \widehat{L} & \xrightarrow{\widehat{\lambda}} & \widehat{M} \\
 \widehat{\chi}' \downarrow & & \downarrow \widehat{\mu} \\
 \widehat{N} & \xrightarrow{\widehat{\mu}'} & \widehat{P}
 \end{array}$$

is not itself always the same as,

$$\begin{array}{ccc}
 \widetilde{L} & \xrightarrow{\widetilde{\lambda}} & \widetilde{M} \\
 \widetilde{\chi}' \downarrow & & \downarrow \widetilde{\mu} \\
 \widetilde{N} & \xrightarrow{\widetilde{\mu}'} & \widetilde{P}
 \end{array}$$

To start the study of this problem we first look at \widetilde{P} .

Proposition 8. For any crossed square (L, M, N, P) , we have $\widetilde{P} \cong \widehat{P}$.

Proof. This follows from an adjoint functor argument: There is a forgetful functor

$$R : \mathbf{Crs}^2 \rightarrow \mathbf{Alg}$$

given by $R(L, M, N, P) = P$ also an analogous one

$$R_{p\mathcal{C}} : \mathbf{Pro-C.Crs}^2 \rightarrow \mathbf{Pro-C.Alg}.$$

These have left adjoints L and $L_{p\mathcal{C}}$ defined by $L(P) = (P, P, P, P)$

$$\begin{array}{ccc}
 P & \xrightarrow{id} & P \\
 id \downarrow & & \downarrow id \\
 P & \xrightarrow{id} & P
 \end{array}$$

with the h -map given by $h(p, p') = pp'$ for all $p, p' \in P$ and similarly for $L_{p\mathcal{C}}$. Then, we get the diagram:

$$\begin{array}{ccc}
 \mathbf{Pro-C.Crs}^2 & \begin{array}{c} \xrightarrow{R_p} \\ \xleftarrow{L_p} \end{array} & \mathbf{Pro-C.Cat}^2(\mathbf{Alg}) \\
 (\sim) \updownarrow \mathcal{U}_{\mathbf{Crs}^2} & & (\sim) \updownarrow \mathcal{U}_{\mathbf{C}^2\mathbf{Alg}} \\
 \mathbf{Crs}^2 & \begin{array}{c} \xrightarrow{R} \\ \xleftarrow{L} \end{array} & \mathbf{Cat}^2(\mathbf{Alg})
 \end{array}$$

The right adjoint diagram commutes, so the left adjoint diagram commutes up to isomorphism, i.e.

$$(P, \widetilde{P, P, P}) \simeq (\widehat{P}, \widehat{P}, \widehat{P}, \widehat{P}),$$

but better we have a sequence of isomorphisms: for a pro- \mathcal{C} algebra H ,

$$\begin{aligned}
 & \mathbf{Pro-C.Alg}(R_{p\mathcal{C}}(\widetilde{L, M, N, P}), H) \\
 & \cong \mathbf{Pro-C.Crs}^2((L, M, N, P), L_{p\mathcal{C}}(H)) \\
 & \cong \mathbf{Crs}^2((L, M, N, P), \mathcal{U}_{\mathbf{Crs}^2} L_{p\mathcal{C}}(H)) \\
 & \cong \mathbf{Crs}^2((L, M, N, P), LU_{\mathbf{C}^2\mathbf{Alg}}(H)) \text{ by observation} \\
 & \cong \mathbf{Alg}(R(L, M, N, P), U_{\mathbf{Alg}}(H)) \\
 & \cong \mathbf{Alg}(P, U_{\mathbf{Alg}}(H)) \\
 & \cong \mathbf{Pro-C.Alg}(\widehat{P}, H)
 \end{aligned}$$

as required; hence $\widehat{P} \cong \widetilde{P}$, independent from L, M, N are. □

In order to study the conditions that yields the isomorphism between $\widetilde{L}, \widetilde{M}, \widetilde{N}$ and $\widehat{L}, \widehat{M}, \widehat{N}$, respectively; we introduce a condition called “cofinality”.

Let (L, P, ∂) be a crossed module and write $\Omega_P(L)$ for the directed subset of $\Omega(L)$ the set of finite index ideals of L , consisting of those $L_1 \in \Omega(L)$, $L/L_1 \in \mathcal{C}$, which are P -invariant. We will say that (L, M, N, P) satisfies the cofinal condition if $\Omega_P(L)$ is cofinal in $\Omega(L)$. It was shown in [6] that if (L, P, ∂) satisfies the cofinality condition, then $\widetilde{L} \cong \widehat{L}$.

Let

$$\begin{array}{ccc}
 L & \xrightarrow{\lambda} & M \\
 \chi' \downarrow & & \downarrow \mu \\
 N & \xrightarrow{\mu'} & P
 \end{array}$$

be a crossed square and write $\Omega_P(L)$ for the directed subset of $\Omega(L)$ the set of finite index ideals of L , given above.

We say that (L, M, N, P) satisfies the cofinal condition if $\Omega_P(L)$ is cofinal in $\Omega(L)$, $\Omega_P(M)$ is cofinal in $\Omega(M)$ and $\Omega_P(N)$ is cofinal in $\Omega(N)$. Note that $\Omega_P(L) \subseteq \Omega_M(L)$ and $\Omega_P(L) \subseteq \Omega_N(L)$ so if $\Omega_P(L)$ is cofinal in $\Omega(L)$, then $\Omega_M(L)$ and $\Omega_N(L)$ are cofinal in $\Omega(L)$.

Proposition 9. *If $P \in \mathcal{C}$, then any crossed square*

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ \lambda' \downarrow & & \downarrow \mu \\ N & \xrightarrow{\mu'} & P \end{array}$$

satisfies the cofinality condition.

Proof. Given any $L_1 \in \Omega(L)$, let

$$L'_1 = \bigcap_{p \in P} pL_1$$

Then, L'_1 is P -invariant and as P in \mathcal{C} , L'_1 is of finite index $L/L'_1 \in \mathcal{C}$. As L'_1 is contained in L_1 , so $\Omega_P(L)$ is cofinal in $\Omega(L)$. Similarly, we can show that $\Omega_P(M)$ and $\Omega_P(N)$ are cofinal in $\Omega(L)$. This completes the proof. \square

Remark 10. *Let C be k -algebra, let $\mathcal{M}(C)$ be the commutative k -algebra of multipliers of C . Recall that a multiplier of C is a linear mapping $\lambda : C \rightarrow C$ such that for all $c, c' \in C$; $\lambda(cc') = \lambda(c)c'$, see [2] for more details.*

Theorem 11. *If (L, M, N, P)*

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ \lambda' \downarrow & & \downarrow \mu \\ N & \xrightarrow{\mu'} & P \end{array}$$

satisfies the cofinality condition, then $\widetilde{L} \cong \widehat{L}, \widetilde{M} \cong \widehat{M}, \widetilde{N} \cong \widehat{N}$.

Proof. Since $\Omega_P(M)$ is cofinal in $\Omega(M)$ and $\Omega_P(N)$ is cofinal in $\Omega(N)$, $\widehat{M} \cong \widetilde{M}$ and $\widehat{N} \cong \widetilde{N}$. On the other hand since $\Omega_P(L)$ is cofinal in $\Omega(L)$, $\widehat{L} \cong \widetilde{L}$.

To check the axioms we need an explicit description of $\widehat{\lambda} : \widehat{L} \rightarrow \widehat{M}, \widehat{\mu} : \widehat{M} \rightarrow \widehat{P}, \widehat{\lambda}' : \widehat{L} \rightarrow \widehat{N}, \widehat{\mu}' : \widehat{N} \rightarrow \widehat{P}, \widehat{\mu\lambda} = \widehat{\mu'}\widehat{\lambda}' : \widehat{M} \rightarrow \widehat{P}$ and the h -map $\widehat{h} : \widehat{M} \times \widehat{N} \rightarrow \widehat{L}$. Given $U_{\text{fin}}P$, there is a composed homomorphism $M \rightarrow P \rightarrow P/U$.

Take K_U to be its kernel then since μ is P -equivariant and P/U is finite, it follows that K_U is in $\Omega_P(M)$ and that $U \subseteq \mathcal{M}_P(M/K_U)$. Similarly, there is a

composed homomorphism $L \rightarrow M \rightarrow M/K_U$. Take T_U to be its kernel then since λ is M -equivariant and P/U is finite, it follows that T_U is in $\Omega_M(L)$ and that $K_U \subseteq \mathcal{M}_P(L/T_U)$. On the other hand there is also composed homomorphism $L \rightarrow P \rightarrow P/U$.

Take H_U to be its kernel then since $\mu\lambda$ is P -equivariant and P/U is finite, it follows that H_U is in $\Omega_P(L)$ and that $U \subseteq \mathcal{M}_P(L/H_U)$. Similarly there are composed homomorphisms $N \rightarrow P \rightarrow P/U$, $L \rightarrow N \rightarrow N/K'_U$ and $L \rightarrow P \rightarrow P/U$ and kernels K'_U, T'_U, H'_U of these morphisms respectively. It is easy to show that $H_U = T_U = T'_U = H'_U$. These observations readily imply that $\widehat{\lambda}, \widehat{\mu}, \widehat{\lambda}', \widehat{\mu}', \widehat{\mu\lambda} = \widehat{\mu'\lambda'}$ and the h -map \widehat{h} , defined by

$$\begin{aligned} \widehat{\lambda}_U(lT_U) &= \lambda l_U K_U, & \widehat{\mu}_U(mK_U) &= \mu m_U U, \\ \widehat{\lambda}'_U(lT_U) &= \lambda' l_U K'_U, & \widehat{\mu}'_U(nK'_U) &= \mu n_U U, \\ \widehat{\mu\lambda}_U(lH_U) &= (\mu\lambda) l_U U, & \widehat{\mu'\lambda'}_U(lH_U) &= (\mu'\lambda') l_U U, \\ \widehat{h}_U(mK_U, nK'_U) &= h(m, n)_U H_U. \end{aligned}$$

It is clear that $\widehat{\mu\lambda}_U = \widehat{\mu}_U \widehat{\lambda}_U = \widehat{\mu}'_U \widehat{\lambda}'_U = \widehat{\mu'\lambda'}_U$. Rest of the proof follows from the crossed square axioms of (L, M, N, P) and the descriptions of $\widehat{\lambda}, \widehat{\mu}, \widehat{\lambda}', \widehat{\mu}', \widehat{\mu\lambda} = \widehat{\mu'\lambda'}$, \widehat{h} and the \widehat{P} -action. \square

Corollary 12. *If P is in \mathcal{C} , and*

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ \lambda' \downarrow & & \downarrow \mu \\ N & \xrightarrow{\mu'} & P \end{array}$$

is a crossed square then

$$\begin{array}{ccc} \widehat{L} & \xrightarrow{\widehat{\lambda}} & \widehat{M} \\ \widehat{\lambda}' \downarrow & & \downarrow \widehat{\mu} \\ \widehat{N} & \xrightarrow{\widehat{\mu}'} & \widehat{P} \end{array}$$

is a crossed square, which is the pro- \mathcal{C} completion of (L, M, N, P) .

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Current address: Hatice Gülsün Akay: Department of Mathematics and Computer Science, Eskişehir Osmangazi University, Turkey.

E-mail address: hgulsun@ogu.edu.tr

ORCID Address: <http://orcid.org/0000-0001-7983-6852>