ON $I$–DEFERRED STATISTICAL CONVERGENCE IN TOPOLOGICAL GROUPS

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Abstract. In this paper, the concepts of $I$–deferred statistical convergence of order $\alpha$ and $I$–deferred statistical convergence of order $(\alpha, \beta)$ in topological groups were defined. Also some inclusion relations between $I$–statistical convergence of order $\alpha$, $I$–deferred statistical convergence of order $\alpha$, $I$–statistical convergence of order $(\alpha, \beta)$ and $I$–deferred statistical convergence of order $(\alpha, \beta)$ in topological groups are given.

1. INTRODUCTION

The idea of statistical convergence was given by Zygmund [38] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [30] and Fast [13] and later reintroduced by Schoenberg [28] independently. Later on it was further investigated from the sequence space point of view and linked with summability theory by Çakallı ([2], [3], [4], [5], [6]), Çınar et al. [7], Et et al. ([9], [10], [11], [12], [24]), Fridy [14], Fridy and Orhan [15], Işık and Akbaş [17], Salat [22], Savas [23], Sengul et. al. ([31], [32], [33], [34]), Srivastava and Et [29], Yıldız [37] and many others.

Let $X$ be a non-empty set. Then a family of sets $I \subseteq 2^X$ (power sets of $X$) is said to be an ideal if $I$ is additive i.e. $A, B \in I$ implies $A \cup B \in I$ and hereditary, i.e. $A \in I, B \subset A$ implies $B \in I$.

A non-empty family of sets $F \subseteq 2^X$ is said to be a filter of $X$ if and only if (i) $\phi \notin F$, (ii) $A, B \in F$ implies $A \cap B \in F$ and (iii) $A \in F, A \subset B$ implies $B \in F$.

An ideal $I \subseteq 2^X$ is called non-trivial if $I \neq 2^X$.

A non-trivial ideal $I$ is said to be admissible if $I \supset \{\{x\} : x \in X\}$. If $I$ is a non-trivial ideal in $X(X \neq \phi)$ then the family of sets $F(I) = \{M \subset X : (\exists A \in I)(M = X \setminus A)\}$ is a filter of $X$, called the filter associated with $I$.

Throughout the paper $I$ will stand for a non-trivial admissible ideal of $\mathbb{N}$.

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The idea of $I-$convergence of real sequences was introduced by Kostyrko et al. [19] and also independently by Nuray and Ruckle [21] (who called it generalized statistical convergence) as a generalization of statistical convergence. Later on $I-$convergence was studied in ([20], [26], [27], [25], [35], [36]).

The order of statistical convergence of a sequence of numbers was given by Gadjiev and Orhan in [16] and after then statistical convergence of order $\alpha$ and strong $p-$Cesàro summability of order $\alpha$ studied by Çolak [8].

In 1932, R.P. Agnew [1] defined the deferred Cesaro mean $D_{p,q}$ of the sequence $x = (x_k)$ by

$$\left(D_{p,q}x\right)_n = \frac{1}{q(n) - p(n)} \sum_{p(n)+1}^{q(n)} x_k$$

where $(p(n))$ and $(q(n))$ are sequences of non-negative integers satisfying $p(n) < q(n)$ and $\lim_{n \to \infty} q(n) = +\infty$. (1.1)

Let $K$ be a subset of $N$, and denote the set \( \{k : p(n) < k \leq q(n), k \in K\} \) by $K_{p,q}(n)$. Deferred density of $K$ is defined by

$$\delta_{p,q}(K) = \lim_{n \to \infty} \frac{1}{q(n) - p(n)} |K_{p,q}(n)|$$

whenever the limit exists (finite or infinite). The vertical bars in (1.2) indicate the cardinality of the set $K_{p,q}(n)$.

A real valued sequence $x = (x_k)$ is said to be deferred statistical convergent to $l$, if

$$\lim_{n \to \infty} \frac{1}{q(n) - p(n)} |\{p(n) < k \leq q(n) : |x_k - l| \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$. If $q(n) = n$, $p(n) = 0$ then deferred statistical convergence coincides statistical convergence [18].

2. $I-$DEFERRED STATISTICAL CONVERGENCE OF ORDER $\alpha$ IN TOPOLOGICAL GROUPS

In this section, some inclusion relations between $I-$statistical convergence, $I-$statistical convergence of order $\alpha$ and $I-$deferred statistical convergence of order $\alpha$ in topological groups are given.

Definition 2.1. Let $(p(n))$ and $(q(n))$ be two sequences of non-negative integers satisfying the conditions (1.1), $X$ be an abelian topological Hausdorf group, $(x(k))$ be a sequence of real numbers and $\alpha$ be a positive real number such that $0 < \alpha \leq 1$. The sequence $x = (x(k))$ is said to be $DS_{p,q}^{\alpha}(X,I)$—statistically convergent in topological groups to $l$ (or $I-$deferred statistically convergent sequences of order $\alpha$ in topological groups to $l$) if there is a real number $l$ for each neighbourhood $U$ of $0$ such that

$$\left\{ n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^\alpha} |\{p(n) < k \leq q(n) : x(k) - l \notin U\}| \geq \delta \right\} \in I.$$ 

In this case we write $DS_{p,q}^{\alpha}(X) - \lim x(k) = l$ or $x(k) \to l \left( DS_{p,q}^{\alpha}(I) \right)$. The set of all $DS_{p,q}^{\alpha}(X,I)$—statistically convergent sequences in topological groups will be denoted by $DS_{p,q}^{\alpha}(X,I)$.

If $\alpha = 1$, then $I-$deferred statistical convergence
of order \( \alpha \) coincides then \( I \)-deferred statistical convergence in topological groups \( (DS_{p,q}^\alpha (X,I) \rightarrow \text{convergence}) \) and if \( q(n) = n \), \( p(n) = 0 \) then \( I \)-deferred statistical convergence of order \( \alpha \) coincides \( I \)-statistical convergence of order \( \alpha \) in topological groups \( (S^\alpha (X,I) \rightarrow \text{convergence}) \). If \( q(n) = n \), \( p(n) = 0 \) and \( \alpha = 1 \), then \( I \)-deferred statistical convergence of order \( \alpha \) coincides \( I \)-statistical convergence in topological groups \( (S(X,I) \rightarrow \text{convergence}) \).

**Theorem 2.1.** Let \((p(n))\) and \((q(n))\) be two sequences of non-negative integers satisfying the conditions (1.1) and \( \alpha, \beta \) be positive real numbers such that \( 0 < \alpha \leq \beta \leq 1 \) then \( DS_{p,q}^\alpha (X,I) \subseteq DS_{p,q}^\beta (X,I) \) and the inclusion is strict.

*Proof.* Omitted. \( \square \)

Theorem 2.1 yields the following corollary.

**Corollary 2.2.** If a sequence is \( DS_{p,q}^\alpha (X,I) \)-statistically convergent of order \( \alpha \) to \( l \), then it is \( DS_{p,q}^\alpha (X,I) \)-statistically convergent to \( l \).

**Theorem 2.3.** Let \((p(n))\) and \((q(n))\) be two sequences of non-negative integers satisfying the conditions (1.1) and \( \alpha \) be a positive real number such that \( 0 < \alpha \leq 1 \). If \( \lim \inf_n \frac{q(n)}{p(n)} > 1 \), then \( S^\alpha (X,I) \subseteq DS_{p,q}^\alpha (X,I) \).

*Proof.* Suppose that \( \lim \inf_n \frac{q(n)}{p(n)} > 1 \); then there exists an \( a > 0 \) such that \( \frac{q(n)}{p(n)} \geq 1 + a \) for sufficiently large \( n \), which implies that

\[
\frac{q(n) - p(n)}{q(n)} \geq \frac{a}{1 + a} \Rightarrow \left( \frac{q(n) - p(n)}{q(n)} \right)^\alpha \geq \left( \frac{a}{1 + a} \right)^\alpha \Rightarrow \frac{1}{q(n)^\alpha} \geq \frac{a^\alpha}{(1 + a)^\alpha} \frac{1}{(q(n) - p(n))}. 
\]

If \( S^\alpha (I) \rightarrow \text{lim}_{k \to \infty} x(k) = l \), then for each neighbourhood \( U \) of \( 0 \) and for sufficiently large \( n \), we have

\[
\frac{1}{q(n)^\alpha} \left| \{ k \leq q(n) : x(k) \not\in U \} \right| \geq \frac{1}{q(n)^\alpha} \left| \{ p(n) < k \leq q(n) : x(k) \not\in U \} \right| 
\]

\[
\geq \frac{1}{(1 + a)^\alpha} \frac{\alpha^\alpha}{(q(n) - p(n))} \left| \{ p(n) < k \leq q(n) : x(k) \not\in U \} \right|. 
\]

Therefore, we can write

\[
\left\{ n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^\alpha} \left| \{ p(n) < k \leq q(n) : x(k) \not\in U \} \right| \geq \delta \right\} \subseteq \left\{ n \in \mathbb{N} : \frac{1}{q(n)^\alpha} \left| \{ k \leq q(n) : x(k) \not\in U \} \right| \geq \delta \frac{\alpha^\alpha}{(1 + a)^\alpha} \right\} \subseteq I. 
\]

This implies that \( S^\alpha (X,I) \subseteq DS_{p,q}^\alpha (X,I) \). \( \square \)

**Theorem 2.4.** Let \((p(n))\) and \((q(n))\) be two sequences of non-negative integers satisfying the conditions (1.1) and \( \alpha \) be a positive real number such that \( 0 < \alpha \leq 1 \). If \( \lim \inf_n \frac{q(n)}{\alpha n} > 0 \) and \( q(n) < n \), then \( S(X,I) \subseteq DS_{p,q}^\alpha (X,I) \).

*Proof.* For each neighbourhood \( U \) of \( 0 \), we have

\[
\{ k \leq n : x(k) \not\in U \} \supset \{ p(n) < k \leq q(n) : x(k) \not\in U \}. 
\]
Therefore,
\[
\frac{1}{n} |\{k \leq n : x(k) - l \notin U\}| \geq \frac{1}{n} |\{p(n) < k \leq q(n) : x(k) - l \notin U\}| = \frac{(q(n) - p(n))^\alpha}{n} \frac{1}{(q(n) - p(n))^\beta} |\{p(n) < k \leq q(n) : x(k) - l \notin U\}|.
\]
Hence, we can write
\[
\left\{ n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^\alpha} |\{p(n) < k \leq q(n) : x(k) - l \notin U\}| \geq \delta \right\}
\subseteq \left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : x(k) - l \notin U\}| \geq \delta \frac{(q(n) - p(n))^\alpha}{n} \right\} \in I.
\]
Consequently, \( S(X,I) \subset DS^\alpha_{p,q}(X,I) \).

\[\square\]

**Theorem 2.5.** Let \((p(n)), (q(n)), (p'(n)), (q'(n))\) be four sequences of non-negative integers such that \(p(n) < q(n), p'(n) < q'(n)\) and \(q(n) - p(n) \leq q'(n) - p'(n)\) for all \(n \in \mathbb{N}\), let \(U\) be any neighbourhood of 0 and let \(\alpha\) and \(\beta\) be such that \(0 < \alpha \leq \beta \leq 1\).

(i) If
\[
\lim_{n \to \infty} \inf \frac{(q(n) - p(n))^{\alpha}}{(q'(n) - p'(n))^{\beta}} > 0
\]
then \(DS^\beta_{p',q'}(X,I) \subseteq DS^\alpha_{p,q}(X,I)\).

(ii) If
\[
\lim_{n \to \infty} \frac{q'(n) - p'(n)}{(q(n) - p(n))^{\beta}} = 1
\]
then \(DS^\alpha_{p,q}(X,I) \subseteq DS^\beta_{p',q'}(X,I)\).

**Proof.** (i) Let (2.1) be satisfied. For each \(\varepsilon > 0\) and each neighbourhood \(W\) of 0 such that \(W \subset U\), we have
\[
\{p'(n) < k \leq q'(n) : x(k) - l \notin W\} \supseteq \{p(n) < k \leq q(n) : x(k) - l \notin U\},
\]
and so
\[
\frac{1}{(q'(n) - p'(n))^{\beta}} |\{p'(n) < k \leq q'(n) : x(k) - l \notin W\}|
\geq \frac{1}{(q(n) - p(n))^{\beta}} \frac{(q(n) - p(n))^{\alpha}}{(q'(n) - p'(n))^{\beta}} |\{p(n) < k \leq q(n) : x(k) - l \notin U\}|
\]
for all \(n \in \mathbb{N}\), where \(p(n) < q(n), p'(n) < q'(n)\) and \(q(n) - p(n) \leq q'(n) - p'(n)\).

Then we can write
\[
\left\{ n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^{\beta}} |\{p(n) < k \leq q(n) : x(k) - l \notin U\}| \geq \delta \right\}
\subseteq \left\{ n \in \mathbb{N} : \frac{1}{(q'(n) - p'(n))^{\beta}} |\{p'(n) < k \leq q'(n) : x(k) - l \notin W\}| \geq \delta \frac{(q(n) - p(n))^{\alpha}}{(q'(n) - p'(n))^{\beta}} \right\} \in I.
\]
This completes the proof.
Let $\alpha, \beta$ be positive real numbers such that $0 < \alpha \leq \beta \leq 1$. The sequence $x = (x(k))$ is said to be $I-$deferred statistical convergent of order $(\alpha, \beta)$ in topological groups to $l$ (or $DS_{p,q}^\alpha(X,I)$ statistically convergent to $l$), if there is a real number $l$, for each neighbourhood $U$ of $0$ such that

$$\left\{ n \in \mathbb{N} : \frac{1}{q(n) - p(n)} \left| \{ p(n) < k \leq q(n) : x(k) - l \notin U \} \right|^\beta \geq \delta \right\} \subseteq I.$$

In this case we write $DS_{p,q}^\alpha(X,I) - \lim x(k) = l$ or $x(k) \rightarrow l (DS_{p,q}^\alpha(X,I))$. The set of all $DS_{p,q}^\alpha(X,I)$-statistically convergent sequences in topological groups will be denoted by $DS_{p,q}^\alpha(X,I)$. If $q(n) = n$, $p(n) = 0$ and $\alpha = \beta = 1$, then $I-$deferred statistical convergence of order $(\alpha, \beta)$ coincides $I-$statistical convergence in topological groups $(S(X,I) - \text{convergence})$.

**Theorem 3.1.** Let $(p(n))$ and $(q(n))$ be two sequences of non-negative integers satisfying the conditions (1.1) and $\alpha_1, \alpha_2, \beta_1$ and $\beta_2$ be positive real numbers such that $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$, then $DS_{p,q}^{\alpha_1,\beta_1}(X,I) \subseteq DS_{p,q}^{\alpha_2,\beta_2}(X,I)$ and the inclusion is strict.

**Proof.** Omitted.

**Theorem 3.2.** Let $(p(n))$ and $(q(n))$ be two sequences of non-negative integers satisfying the conditions (1.1) and $\alpha, \beta$ be two positive real numbers such that $0 < \alpha \leq \beta \leq 1$. If $\liminf_{n} \frac{q(n)}{p(n)} > 1$, then $S_{p,q}^{\alpha,\beta}(X,I) \subset DS_{p,q}^{\alpha,\beta}(X,I)$.

**Proof.** The proof is similar to that of Theorem 2.3.
Theorem 3.3. Let \((p(n)), (q(n)), (p'(n))\) and \((q'(n))\) be four sequences of non-negative integers such that \(p(n) < q(n), p'(n) < q'(n)\) and \(q(n) - p(n) \leq q'(n) - p'(n)\) for all \(n \in \mathbb{N}\), let \(U\) be any neighborhood of 0 and let \(\alpha_1, \alpha_2, \beta_1\) and \(\beta_2\) be such that \(0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1\).

(i) If

\[
\lim_{n \to \infty} \inf \frac{(q(n) - p(n))^\alpha_1}{(q'(n) - p'(n))^\alpha_2} > 0
\]  

then \(DS_{p',q'}^{\alpha_2,\beta_2}(X,I) \subseteq DS_{p,q}^{\alpha_1,\beta_1}(X,I),\)

(ii) If

\[
\lim_{n \to \infty} \frac{q'(n) - p'(n)}{(q(n) - p(n))^{\alpha_2}} = 1
\]

then \(DS_{p,q}^{\alpha_1,\beta_2}(X,I) \subseteq DS_{p',q'}^{\alpha_2,\beta_1}(X,I)\).

Proof. (i) Let \(\lim_{n \to \infty} \inf \frac{(q(n) - p(n))^\alpha_1}{(q'(n) - p'(n))^\alpha_2} > 0\). For given \(\varepsilon > 0\) and each neighborhood \(U, W\) of 0 such that \(W \subseteq U\), we have

\[
\frac{1}{(q'(n) - p'(n))^{\alpha_2}} |\{p'(n) < k \leq q'(n) : x(k) - l \notin W\}|^{\beta_2} \\
\geq \frac{(q(n) - p(n))^{\alpha_2}}{(q'(n) - p'(n))^{\alpha_2}} \frac{1}{(q(n) - p(n))^{\alpha_2}} |\{p(n) < k \leq q(n) : x(k) - l \notin U\}|^{\beta_1}
\]

for all \(n \in \mathbb{N}\).

Therefore, we can write

\[
\left\{ n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^{\alpha_2}} |\{p(n) < k \leq q(n) : x(k) - l \notin U\}|^{\beta_1} \geq \delta \right\} \\
\subseteq \left\{ n \in \mathbb{N} : \frac{1}{(q'(n) - p'(n))^{\alpha_2}} |\{p'(n) < k \leq q'(n) : x(k) - l \notin W\}|^{\beta_2} \geq \delta \frac{(q(n) - p(n))^{\alpha_1}}{(q'(n) - p'(n))^{\alpha_2}} \right\} \in I.
\]

This completes the proof.

(ii) Omitted.

\(\square\)

Corollary 3.4. Let \((p(n)), (q(n)), (p'(n))\) and \((q'(n))\) be four sequences of non-negative integers such that \(p(n) < q(n), p'(n) < q'(n)\) and \(q(n) - p(n) \leq q'(n) - p'(n)\) for all \(n \in \mathbb{N}\) and 0 \(<\alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1\).

If (3.1) holds then,

(i) \(DS_{p',q'}^{\alpha_2}(X,I) \subseteq DS_{p,q}^{\alpha_1}(X,I)\) for \(\beta_1 = \beta_2 = 1\),

(ii) \(DS_{p',q'}(X,I) \subseteq DS_{p,q}^{\alpha_1}(X,I)\) for \(\alpha_2 = \beta_1 = \beta_2 = 1\),

(iii) \(DS_{p',q'}(X,I) \subseteq DS_{p,q}(X,I)\) for \(\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1\).

If (3.2) holds then,

(i) \(DS_{p,q}(X,I) \subseteq DS_{p',q'}^{\alpha_2}(X,I)\) for \(\beta_1 = \beta_2 = 1\),

(ii) \(DS_{p,q}(X,I) \subseteq DS_{p',q'}(X,I)\) for \(\alpha_2 = \beta_1 = \beta_2 = 1\),

(iii) \(DS_{p,q}(X,I) \subseteq DS_{p',q'}(X,I)\) for \(\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1\).
References


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