Conference Proceedings of Science and Technology, 2(1), 2019, 13–17

Conference Proceeding of 2th International Conference on Mathematical Advences and Applications (ICOMAA-2019).

On $(P,Q)-$Lucas Polynomial Coefficients for a New Class of Bi-Univalent Functions Associated with q-Analogue of Ruscheweyh Operator

ISSN: 2651-544X

http://dergipark.gov.tr/cpost

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Abstract: Recently, Fibonacci polynomials, Chebyshev polynomials, Lucas polynomials, Pell polynomials, LucasâĂŞLehmer polynomials, orthogonal polynomials and other special polynomials became more and more important in the field of Geometric Function Theory. The Theory of Geometric Functions and that of Special Functions are usually considered as very different fields. In this study, by using Lucas polynomials of the second kind, subordination and Ruschewey differential operator,these different fields were connected and a new class of bi-univalent functions was introduced. Also coefficient estimates were obtained for this new class.

Keywords: (P,Q)-Lucas polynomials, Coefficient bounds, Bi-univalent functions, Ruscheweyh differential operator, Subordination

1 Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \tag{1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$, and let $S = \{f \in A : f \text{ is univalent in } U\}$.

The Koebe one-quarter theorem [11] states that the range of every function $f \in S$ contains the disc of radius $\left\{w : |w| < \frac{1}{4}\right\}$. Thus every such function $f \in S$ has an inverse f^{-1} which satisfies

$$f^{-1}\left(f\left(z\right)\right) = z \quad \left(z \in U\right)$$

and

$$f\left(f^{-1}\left(w\right)\right) = w \quad \left(|w| < r_0\left(f\right)\ ,\ r_0\left(f\right) \geq \frac{1}{4}\right),$$

where

$$f^{-1}\left(w\right) = w \ - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

Definition 1. *If both f and f^{-1} are univalent in U, then a function $f \in A$ is said to be bi-univalent in U. We say that f is in the class Σ for such functions.*

Some functions in the class Σ are given in [23]. In 1986, Brannan and Taha [9] introduced certain subclasses of the bi-univalent function class similar to the familiar subclasses of starlike and convex functions of order. In 2012, Ali et al. [22] widen the result of Brannan and Taha by using subordination. The estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor Maclaurin series expansion (1) were found in several recent studies (see [1]-[6], [17], [19]-[20]) and still an interest to many researchers.

Definition 2. *For analytic functions f and g , f is said to be subordinate to g, denoted*

$$f(z) \prec g(z), \tag{3}$$

if there is an analytic function w such that

$$w(0) = 0\ ,\ |w(z)| < 1 \ \textit{and}\ f(z) = g\left(w(z)\right)$$

Definition 3. *([14, 15]) For $q \in (0,1)$, the q-derivative of function $f \in \mathcal{A}$ is defined by*

$$\partial_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, z \neq 0 \tag{4}$$

and

$$\partial_q f(0) = f'(0).$$

Thus we have

$$\partial_q f(z) = 1 + \sum_{k=2}^{\infty} [k,q]\, a_k z^{k-1} \tag{5}$$

where $[k,q]$ is given by

$$[k,q] = \frac{1-q^k}{1-q}, \quad [0,q] = 0 \tag{6}$$

and the q-fractional is defined by

$$[k,q]! = \begin{cases} \prod_{m=1}^{k} [m,q], & k \in \mathbb{N} \\ 1, & k = 0 \end{cases}. \tag{7}$$

Also, the $q-$generalized Pochhammer symbol for $\mathfrak{p} \geq 0$ is given by

$$[\mathfrak{p},q]_k = \begin{cases} \prod_{m=1}^{k} [\mathfrak{p}+m-1,q], & k \in \mathbb{N} \\ 1, & k = 0 \end{cases}.$$

In addition, as $q \to 1$, we have $[k,q] \to k$. If we choose the function $g(z) = z^k$, then we have

$$\partial_q g(z) = \partial_q z^k = [k,q]\, z^{k-1} = g'(z),$$

where g' is the ordinary derivative.

Now, we point out the q-analogue of Ruscheweyh operator:

Definition 4. *[10] Let $f \in \mathcal{A}$. The $q-$analogue of Ruscheweyh operator is defined by*

$$\mathcal{R}_q^{\mu} f(z) = z + \sum_{k=2}^{\infty} \frac{[k+\mu-1,q]!}{[k,q]!\,[k-1,q]!} a_k z^k, \tag{8}$$

where $[k,q]!$ is given by equation (7).

From the definition we observe that if $q \to 1$, we have

$$\lim_{q\to 1} \mathcal{R}_q^{\mu} f(z) = z + \lim_{q\to 1} \sum_{k=2}^{\infty} \frac{[k+\mu-1,q]!}{[\mu,q]!\,[k-1,q]!} a_k z^k = z + \sum_{k=2}^{\infty} \frac{[k+\mu-1]!}{[\mu]!\,[k-1]!} a_k z^k = \mathcal{R}^{\mu} f(z), \tag{9}$$

where $\mathcal{R}_q^{\mu} f(z)$ is Ruscheweyh differential operator defined in [29].

Some of special polynomials, for example Fibonacci polynomials, Lucas polynomials, Chebyshev polynomials, Pell polynomials, Lucas–Lehmer polynomials, orthogonal polynomials and the other special polynomials, are of great importance in several papers from a theoretical point of view (see, for example [7, 8, 12, 13, 18, 24–28]).

Definition 5. *[16] Let calP(x) and calQ(x) are polynomials with real coefficients. The (calP,calQ) Lucas polynomials $L_{\mathcal{P},\mathcal{Q},m}(x)$ are defined by the reccurence relation*

$$L_{\mathcal{P},\mathcal{Q},m}(x) = \mathcal{P}(x)L_{\mathcal{P},\mathcal{Q},m-1}(x) + \mathcal{Q}(x)L_{\mathcal{P},\mathcal{Q},m-2}(x) \quad (m \geq 2), \tag{10}$$

from which the first few Lucas polynomials can be found as

$$\begin{aligned} L_{\mathcal{P},\mathcal{Q},0}(x) &= 2, \\ L_{\mathcal{P},\mathcal{Q},1}(x) &= \mathcal{P}(x), \\ L_{\mathcal{P},\mathcal{Q},2}(x) &= \mathcal{P}^2(x) + 2\mathcal{Q}(x), \\ L_{\mathcal{P},\mathcal{Q},3}(x) &= \mathcal{P}^3(x) + 3\mathcal{P}(x)\mathcal{Q}(x) \end{aligned} \tag{11}$$

In this article, we aim at introducing a new class of bi-univalent functions defined through the $(P,Q)-$Lucas polynomials of the second kind.

Definition 6. *[16] Let $\mathcal{G}_{\{L_m(x)\}}(z)$ be the generating function of the $(P,Q)-$Lucas polynomial sequence $L_{\mathcal{P},\mathcal{Q},m}(x)$. Then*

$$\mathcal{G}_{\{L_m(x)\}}(z)=\sum_{m=0}^{\infty}L_{\mathcal{P},\mathcal{Q},m}(x)z^m=\frac{2-\mathcal{P}(x)z}{1-\mathcal{P}(x)z-\mathcal{Q}(x)z^2}. \tag{12}$$

2 The class $\mathbf{Q}^{\Sigma}(q,\mu;x)$

We begin this section by defining the class calQ$^{\Sigma}(q,\mu;x)$ and by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this class.

Definition 7. *The function f is said to be in the class $\mathcal{Q}^{\Sigma}(q,\mu;x)$ if the following conditions are satisfied:*

$$\frac{z\partial_q(\mathcal{R}_q^{\mu}f(z))}{\mathcal{R}_q^{\mu}f(z)}\prec\mathcal{G}_{\{L_{\mathcal{P},\mathcal{Q},m}(x)\}}(z)-1$$

and

$$\frac{w\partial_q(\mathcal{R}_q^{\mu}f(w))}{\mathcal{R}_q^{\mu}f(w)}\prec\mathcal{G}_{\{L_{\mathcal{P},\mathcal{Q},m}(x)\}}(w)-1 \tag{13}$$

where $\mathcal{R}_q^{\mu}f(z)$ is Ruscheweyh differential operator defined in [29].

Theorem 1. *Let f given by (1) be in the class $\mathcal{Q}^{\Sigma}(q,\mu;x)$. Then,*

$$|a_2|\leq\frac{|\mathcal{P}(x)|\sqrt{2|\mathcal{P}(x)|}}{\sqrt{q[q+1,q]\left|\left\{2[\mu+2]_q-[\mu+1]_q(3q+1)\right\}\mathcal{P}^2(x)-4q[\mu+1]_q\mathcal{Q}(x)\right|}} \tag{14}$$

and

$$|a_3|\leq\frac{\mathcal{P}^2(x)}{(1+\delta)^{2m}(1+\zeta)^2}+\frac{|\mathcal{P}(x)|}{(1+2\delta)^m(1+2\zeta)}. \tag{15}$$

Proof: Let $f\in\mathcal{Q}^{\Sigma}(q,\mu;x)$. Then from Definition 7, for some analytic functions Ω,Λ such that $\Omega(0)=\Lambda(0)=0$ and $|\Omega(z)|<1,|\Lambda(w)|<1$ for all $z,w\in U$, we can write

$$\frac{z\partial_q(\mathcal{R}_q^{\mu}f(z))}{\mathcal{R}_q^{\mu}f(z)}=\mathcal{G}_{\{L_{\mathcal{P},\mathcal{Q},m}(x)\}}(\Omega(z))-1 \tag{16}$$

and

$$\frac{w\partial_q(\mathcal{R}_q^{\mu}f(w))}{\mathcal{R}_q^{\mu}f(w)}=\mathcal{G}_{\{L_{\mathcal{P},\mathcal{Q},m}(x)\}}(\Lambda(w))-1 \tag{17}$$

or equivalently

$$\frac{z\partial_q(\mathcal{R}_q^{\mu}f(z))}{\mathcal{R}_q^{\mu}f(z)}=-1+L_{\mathcal{P},\mathcal{Q},0}(x)+L_{\mathcal{P},\mathcal{Q},1}(x)\Omega(z)+L_{\mathcal{P},\mathcal{Q},2}(x)\Omega^2(z)+\cdots \tag{18}$$

and

$$\frac{w\partial_q(\mathcal{R}_q^{\mu}f(w))}{\mathcal{R}_q^{\mu}f(w)}=-1+L_{\mathcal{P},\mathcal{Q},0}(x)+L_{\mathcal{P},\mathcal{Q},1}(x)\Lambda(w)+L_{\mathcal{P},\mathcal{Q},2}(x)\Lambda^2(w)+\cdots \tag{19}$$

From the equalities (18)and (19), we obtain that

$$\frac{z\partial_q(\mathcal{R}_q^{\mu}f(z))}{\mathcal{R}_q^{\mu}f(z)}=1+L_{\mathcal{P},\mathcal{Q},1}(x)l_1z+\left[L_{\mathcal{P},\mathcal{Q},1}(x)l_2+L_{\mathcal{P},\mathcal{Q},2}(x)l_1^2\right]z^2+\cdots \tag{20}$$

and

$$\frac{w\partial_q(\mathcal{R}_q^{\mu}f(w))}{\mathcal{R}_q^{\mu}f(w)}=1+L_{\mathcal{P},\mathcal{Q},1}(x)r_1w+\left[L_{\mathcal{P},\mathcal{Q},1}(x)r_2+L_{\mathcal{P},\mathcal{Q},2}(x)r_1^2\right]w^2+\cdots \tag{21}$$

It is known before that if for $z,w\in U$,

$$\Omega(z)=\left|\sum_{i=1}^{m}l_iz^i\right|<1$$

and

$$\Lambda(w)=\left|\sum_{i=1}^{m} r_i w^i\right|<1$$

than

$$|l_i|<1$$

and

$$\Lambda(w)=|r_i|<1$$

where $i\in\mathbb{N}$. Also, we can write

$$\frac{z\partial_q(\mathcal{R}_q^{\mu}f(z))}{\mathcal{R}_q^{\mu}f(z)}=1+q\left[\mu+1\right]_q a_2 z+\left\{q\left[\mu+1\right]_q\left[\mu+2\right]_q a_3-q\left[\mu+1\right]_q^2 a_2^2\right\}z^2+\cdots,$$

and

$$\frac{w\partial_q(\mathcal{R}_q^{\mu}f(w))}{\mathcal{R}_q^{\mu}f(w)}=1-q\left[\mu+1\right]_q a_2 w+\left\{-q\left[\mu+1\right]_q\left[\mu+2\right]_q a_3+q\left[\mu+1\right]_q\left(2\left[\mu+2\right]_q-\left[\mu+1\right]_q\right)a_2^2\right\}w^2+\cdots.$$

Now, comparing the corresponding coefficients in (20)and (21), we get

$$q\left[\mu+1\right]_q a_2 = L_{\mathcal{P},\mathcal{Q},1}(x)l_1, \tag{22}$$

$$q\left[\mu+1\right]_q\left[\mu+2\right]_q a_3-q\left[\mu+1\right]_q^2 a_2^2 = L_{\mathcal{P},\mathcal{Q},1}(x)l_2+L_{\mathcal{P},\mathcal{Q},2}(x)l_1^2, \tag{23}$$

$$-q\left[\mu+1\right]_q a_2 = L_{\mathcal{P},\mathcal{Q},1}(x)r_1, \tag{24}$$

$$\begin{aligned}&-\ q\left[\mu+1\right]_q\left[\mu+2\right]_q a_3+q\left[\mu+1\right]_q\left(2\left[\mu+2\right]_q-\left[\mu+1\right]_q\right)a_2^2\\ &\quad= L_{\mathcal{P},\mathcal{Q},1}(x)r_2+L_{\mathcal{P},\mathcal{Q},2}(x)r_1^2.\end{aligned} \tag{25}$$

From (22)and (24)

$$l_1=-r_1, \tag{26}$$

$$2q^2\left[\mu+1\right]_q^2 a_2^2=L_{\mathcal{P},\mathcal{Q},1}^2(x)\left(l_1^2+r_1^2\right). \tag{27}$$

Adding (23)and (25) we get

$$2q^{\mu+2}\left[\mu+1\right]_q a_2^2=L_{\mathcal{P},\mathcal{Q},1}(x)\left(l_2+r_2\right)+L_{\mathcal{P},\mathcal{Q},2}(x)\left(l_1^2+r_1^2\right) \tag{28}$$

By using (27) in (28) we have

$$\left[2L_{\mathcal{P},\mathcal{Q},1}^2(x)q^{\mu+2}\left[\mu+1\right]_q-2L_{\mathcal{P},\mathcal{Q},2}(x)q^2\left[\mu+1\right]_q^2\right]a_2^2=L_{\mathcal{P},\mathcal{Q},1}^3(x)\left(l_2+r_2\right) \tag{29}$$

which gives

$$|a_2|\leq\frac{|\mathcal{P}(x)|\sqrt{|\mathcal{P}(x)|}}{\sqrt{\left|\left[q^{\mu+2}\left[\mu+1\right]_q-q^2\left[\mu+1\right]_q^2\right]\mathcal{P}^2(x)-q^2\left[\mu+1\right]_q^2\mathcal{Q}(x)\right|}}.$$

Also, by subtructing (25) from (23) , we get

$$\left(2q\left[\mu+1\right]_q\left[\mu+2\right]_q\right)\left(a_3-a_2^2\right)=L_{\mathcal{P},\mathcal{Q},1}(x)\left(l_2-r_2\right). \tag{30}$$

Then, by using (26) and (27) in (30), we have

$$a_3=\frac{L_{\mathcal{P},\mathcal{Q},1}^2(x)\left(l_1^2+r_1^2\right)}{2q^2\left[\mu+1\right]_q^2}+\frac{L_{\mathcal{P},\mathcal{Q},1}(x)\left(l_2-r_2\right)}{2q\left[\mu+1\right]_q\left[\mu+2\right]_q}$$

and by the help of (9), we conclude that

$$|a_3|\leq\frac{\mathcal{P}^2(x)}{q^2\left[\mu+1\right]_q^2}+\frac{|\mathcal{P}(x)|}{q\left[\mu+1\right]_q\left[\mu+2\right]_q}.$$

□

Remark 1. *Choosing $\mu=0$ in Theorem 8, we obtain following corollary*

Corollary 1. *Let* $f \in \mathcal{Q}^{\Sigma}(q,0;x) = \mathcal{Q}^{\Sigma}(q;x)$. *Then,*

$$|a_2| \leq \frac{|\mathcal{P}(x)|\sqrt{|\mathcal{P}(x)|}}{\sqrt{q^2\left|-2\mathcal{Q}(x)\right|}} \tag{31}$$

and

$$|a_3| \leq \frac{\mathcal{P}^2(x)}{q^2} + \frac{|\mathcal{P}(x)|}{q(1+q)}. \tag{32}$$

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