A Convergence Result for a Three-Step Iterative Algorithm

Harun Polat¹, Faik Gürsoy²,*
¹Department of Mathematics, Faculty of Science and Arts, Muş Alparslan University, Muş, Turkey, ORCID:0000-0003-3955-9197
²Department of Mathematics, Faculty of Science and Arts, Adıyaman University, Adıyaman, Turkey, ORCID:0000-0002-7118-9088
* Corresponding Author E-mail: faikgursoy02@hotmail.com

Abstract: We prove under some mild conditions that iterative algorithm (1.7) of [1] converges strongly to the fixed point of a member in the class of weak contraction mappings.

Keywords: Fixed point, Iterative algorithm, Strong convergence.

1 Introduction

Let \( C \) be a nonempty closed convex subset of a Banach space \( X \) and \( T : C \to C \) a mapping. An element \( x \) in \( C \) is said to be a fixed point of \( T \) if \( Tx = x \).

Iterative approximation of fixed points has become a useful tool for solving many problems which arise in various branches of science and engineering.

Recently, Karakaya et al. [1] introduced a three-step iterative algorithm as follows:

\[
\begin{align*}
    x_1 & \in C, \\
    x_{n+1} & = Ty_n, \\
    y_n & = (1 - \alpha_n) z_n + \alpha_n Tz_n, \\
    z_n & = Tx_n, \quad n \in \mathbb{N},
\end{align*}
\]  

(1)

where \( \{\alpha_n\}_{n=1}^{\infty} \) is a real sequence in \([0, 1]\).

**Definition 1.** (\cite{2}) Let \((M, d)\) be a metric space. A mapping \( T : M \to M \) is said to be weak-contraction if there exist \( \delta \in [0, 1) \) and \( L \geq 0 \) such that

\[ d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx), \quad \text{for all } x, y \in M. \]

**Theorem 1.** (\cite{2}) Let \((M, d)\) be a complete metric space and \( T : M \to M \) a weak-contraction for which there exist \( \delta \in [0, 1) \) and \( L_1 \geq 0 \) such that

\[ d(Tx, Ty) \leq \delta d(x, y) + L_1 d(x, Tx), \quad \text{for all } x, y \in M. \]

(2)

Then, \( T \) has a unique fixed point.

Karakaya et al. [1] showed that iterative algorithm (1) strongly converges to the fixed points of weak-contraction mappings. More precisely, they proved the following result.

**Theorem 2.** (\cite{1}) Let \( C \) be a nonempty closed convex subset of a Banach space \( X \) and \( T : C \to C \) a weak-contraction satisfying condition (2). Let \( \{x_n\}_{n=1}^{\infty} \) be an iterative sequence generated by (1) with real sequence \( \{\alpha_n\}_{n=1}^{\infty} \subseteq [0, 1] \) satisfying \( \sum_{n=1}^{\infty} \alpha_n = \infty \). Then, \( \{x_n\}_{n=1}^{\infty} \) converges to a unique fixed point \( p^* \) of \( T \).

2 Main result

**Theorem 3.** Let \( C \) be a nonempty closed convex subset of a Banach space \( X \) and \( T : C \to C \) with \( p^* =Tp^* \) a weak-contraction satisfying condition (2). Let \( \{x_n\}_{n=1}^{\infty} \) be an iterative sequence generated by (1) with real sequence \( \{\alpha_n\}_{n=1}^{\infty} \subseteq [0, 1] \). Then, the sequence \( \{x_n\}_{n=1}^{\infty} \) converges to a unique fixed point \( p^* \) of \( T \).
Proof: The following inequality was obtained in ([1], Theorem 2.1):

\[ \|x_{n+1} - p^*\| \leq \|x_1 - p^*\| \delta^{2n} \prod_{i=1}^{n} [1 - \alpha_i (1 - \delta)], \text{ for all } n \in \mathbb{N}. \]  

(3)

As \( \delta \in [0, 1) \) and \( \{\alpha_n\}_{n=1}^{\infty} \subseteq [0, 1] \) implies \( 1 - \alpha_n (1 - \delta) < 1 \) for all \( n \in \mathbb{N} \), so inequality (3) becomes

\[ \|x_{n+1} - p^*\| \leq \delta^{2n} \|x_1 - p^*\|, \text{ for all } n \in \mathbb{N}. \]  

(4)

Taking limit on both sides of inequality (4), we have \( \lim_{n \to \infty} \|x_n - p^*\| = 0. \)

\[ \square \]

3 Conclusion

Theorem 2 was proven under the condition \( \sum_{n=1}^{\infty} \alpha_n = \infty \). In Theorem 3, we remove this condition. Therefore, Theorem 3 is an improvement of Theorem 2.

4 References
