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Mathematical Behavior of Solutions of Fourth-Order Hyperbolic Equation with Logarithmic Source Term

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Abstract: The main goal of this paper is to study for a fourth-order hyperbolic equation with logarithmic nonlinearity. We obtain several results: Firstly, by using Feado-Galerkin method and a logaritmic Sobolev inequality, we proved local existence of solutions. Later, we proved global existence of solutions by potential well method. Finally, we showed the decay estimates result of the solutions.

Keywords: Decay of solution, Existence, Logarithmic nonlinearity.

1 Introduction

In this paper, we study the following fourth order hyperbolic equation with logarithmic nonlinearity

$$\begin{cases} u_{tt} + \triangle^2 u - \triangle u + u_t = u \ln |u|^k, & x \in \Omega,\ t > 0 \\ u(x,0) = u_0(x),\ \ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = \frac{\partial}{\partial \nu} u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0 \end{cases} \tag{1}$$

where $\Omega \subset R^3$ is a bounded domain with smooth boundary $\partial\Omega$, and k is a costant to be chosen later.

This type of problems has many applications in many branches physics, such as quantum mechanics, nuclear physics, supersymmetric field theories, optics and geophysics [2–4, 6, 11].

In [5], Cazenave and Haraux studied the existence of the solution following equation

$$u_{tt} - \triangle u + u = u \ln |u|^k \tag{2}$$

in R^3. Later, Gorka [6] studied the global existence of the solution of Eq. (2) in the one dimensional case. Furthermore, existence of the solutions were studied in [1–3].

Hiramatsu et al. [9] is introduced the following equation

$$u_{tt} - \Delta u + u + u_t + u|u|^2 = u \ln |u|^2 . \tag{3}$$

In [8], Han showed the global existence of weak solutions to the initial boundary value problem (3) in R^3.

Recently, Hu et al. [14] studied exponential growth and decay estimates of the solutions for Eq. (1), without the fourth-order term ($\triangle^2 u$). Al-Gharabli and Messaoudi [12, 13] proved existence and decay of the solutions for Eq. (1), without the $\triangle u$ term.

Motivated by the above studies, we established the local and global existence, growth and decay estimates of the solution for problem (1).

The rest of our work is organized as follows. In section 2, we gave some notations and lemmas which will be used throughout this paper. In section 3, we established the local existence of the solutions of the problem. In section 4, we established the global existence of the solutions of the problem. The decay estimates result were presented in section 5.

2 Preliminaries

In this section we will give some notations and lemmas which will be used throughout this paper. We denote $\|.\|$ and $\|.\|_p$ the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively. We denote by C and C_i $(i = 1, 2, ...)$ varius positive constants.

We define energy function as follows

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2}\|\nabla u\|^2 - \frac{1}{2}\int_\Omega \ln |u|^k u^2 dx + \frac{k}{4}\|u\|^2 . \tag{4}$$

Lemma 1. *$E(t)$ is a nonincreasing function for $t \geq 0$ and*

$$E'(t) = -\|u_t\|^2 \leq 0. \tag{5}$$

Proof: Multiplying the equation (1) by u_t and integrating on Ω, we have

$$\int_\Omega u_{tt} u_t dx + \int_\Omega \triangle^2 u u_t dx - \int_\Omega \triangle u u_t dx + \int_\Omega u_t u_t dx = \int_\Omega \ln |u|^k u u_t dx,$$

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2}\|\nabla u\|^2 - \frac{1}{2}\int_\Omega \ln|u|^k u^2 dx + \frac{k}{4}\|u\|^2\right) = -\|u_t\|^2,$$

$$E'(t) = -\|u_t\|^2.$$

□

Lemma 2. *[7] (Logarithmic Sobolev Inequality). Let u be any function $u \in H_0^1(\Omega)$ and $a > 0$ be any number. Then,*

$$\int_\Omega \ln|u| u^2 dx < \frac{1}{2}\|u\|^2 \ln\|u\|^2 + \frac{\alpha^2}{2\pi}\|\nabla u\|^2 - (1 + \ln\alpha)\|u\|^2.$$

Lemma 3. *[5] (Logarithmic Gronwall Inequality).*

Let $c > 0,\ \gamma \in L^1(0.T, R^+)$ and assume that the function $w : [0,T] \to [1,\infty]$ satisfies

$$w(t) \leq c\left(1 + \int_0^t \gamma(s) w(s) \ln w(s) ds\right),\ 0 \leq t \leq T,$$

where

$$w(t) \leq c e^{\int_0^t c\gamma(s) ds},\ 0 \leq t \leq T.$$

3 Local existence

In this section we state and prove the local existence result for problem (1). The proof is based on Faedo-Galerkin method.

Definition 4. *A function u defined on $[0,T]$ is called a weak solution of (1) if*

$$u \in C\left([0,T); H_0^2(\Omega)\right),\ u_t \in C\left([0,T); L^2(\Omega)\right)$$

and u satisfies

$$\begin{cases} \int_\Omega u_{tt}(x,t) w(x) dx + \int_\Omega \triangle u(x,t) \Delta w(x) dx \\ + \int_\Omega \nabla u(x,t) \nabla w(x) dx + \int_\Omega u_t(x,t) w(x) dx \\ \qquad = \int_\Omega \ln|u(x,t)|^k u(x,t) w(x) dx, \end{cases}$$

for $w \in H_0^2(\Omega)$.

Theorem 5. *Let $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$, then the problem (1) has a weak solution on $[0,T]$.*

Proof: We will use the Faedo-Galerkin method to construct approximate solutions. Let $\{w_j\}_{j=1}^\infty$ be an orthogonal basis of the "separable" space $H_0^2(\Omega)$ which is orthonormal in $L^2(\Omega)$. Let

$$V_m = \text{span}\{w_1, w_2, ..., w_m\}$$

and let the projections of the initial data on the finite dimensional subspace V_m be given by

$$\begin{aligned} u_0^m(x) &= \sum_{j=1}^m a_j w_j(x) \to u_0 \text{ in } H_0^2(\Omega), \\ u_1^m(x) &= \sum_{j=1}^m b_j w_j(x) \to u_1 \text{ in } L^2(\Omega), \end{aligned}$$

for $j = 1, 2, ..., m$.

We look for the approximate solutions

$$u^m(x,t)=\sum_{j=1}^{m}h_j^m(t)\,w_j(x)$$

of the approximate problem in V_m

$$\begin{cases} \int_\Omega (u_{tt}^m w dx + \triangle u^m \Delta w + \nabla u^m \nabla w + u_t^m w)\,dx = \int_\Omega \ln|u^m|^k u^m w dx,\ w\in V_m, \\ u^m(0)=u_0^m=\sum_{j=1}^{m}(u_0,w_j)\,w_j, \\ u_t^m(0)=u_1^m=\sum_{j=1}^{m}(u_1,w_j)\,w_j. \end{cases} \tag{6}$$

This lead to a system of ordinary differantial equations for unknown functions $h_j^m(t)$. Based on standard existence theory for ordinary differantial equation, one can obtain functions

$$h_j:[0,t_m)\to R,\ \ j=1,2,...,m,$$

which satisfy (6) in a maximal interval $[0,t_m)$, $0<t_m\le T$. Next, we show that $t_m=T$ and that the local solution is uniformly bounded independent of m and t. For this purpose, we replace w by u_t^m in (6) and integrate by parts we obtain

$$\frac{d}{dt}E^m(t)=-\left\|u_t^m\right\|^2\le 0 \tag{7}$$

where

$$E^m(t)=\frac{1}{2}\left(\left\|u_t^m\right\|^2+\left\|\Delta u^m\right\|^2+\left\|\nabla u^m\right\|^2+\frac{k}{2}\left\|u^m\right\|^2-\int_\Omega \ln|u^m|^k|u^m|^2dx\right). \tag{8}$$

Integrating (7) with respect to t from 0 to t, we obtain

$$E^m(t)\le E^m(0). \tag{9}$$

The last inequality and the Logarithmic Sobolev Inequality lead to

$$\begin{aligned} E^m(t) &\ge \frac{1}{2}\left[\left\|u_t^m\right\|^2+\left\|\Delta u^m\right\|^2+\left\|\nabla u^m\right\|^2+\frac{k}{2}\left\|u^m\right\|^2\right] \\ &\quad -\frac{k}{2}\left[\frac{1}{2}\left\|u^m\right\|^2\ln\left\|u^m\right\|^2+\frac{\alpha^2}{2\pi}\left\|\nabla u^m\right\|^2-(1+\ln\alpha)\left\|u^m\right\|^2\right], \\ &= \frac{1}{2}\left(\left\|u_t^m\right\|^2+\left\|\Delta u^m\right\|^2+\left(1-\frac{k\alpha^2}{2\pi}\right)\left\|\nabla u^m\right\|^2\right. \\ &\quad \left.+\left(\frac{k}{2}+k(1+\ln\alpha)\right)\left\|u^m\right\|^2-\frac{k}{2}\left\|u^m\right\|^2\ln\left\|u^m\right\|^2\right), \end{aligned}$$

$$\begin{aligned} &\left\|u_t^m\right\|^2+\left\|\Delta u^m\right\|^2+\left(1-\frac{k\alpha^2}{2\pi}\right)\left\|\nabla u^m\right\|^2 \\ &+\left(\frac{k}{2}+k(1+\ln\alpha)\right)\left\|u^m\right\|^2 \\ \le\ & C+\frac{k}{2}\left\|u^m\right\|^2\ln\left\|u^m\right\|^2. \end{aligned} \tag{10}$$

where $C=2E^m(0)$.

Choosing

$$e^{-\frac{3}{2}}<\alpha<\sqrt{\frac{2\pi}{k}} \tag{11}$$

will make

$$1-\frac{k\alpha^2}{2\pi}>0,$$

$$\sqrt{\frac{2\pi}{k}}>\alpha$$

and

$$\frac{k+2}{2}+k(1+\ln\alpha)>0,$$

$$\alpha>e^{-\frac{3}{2}}.$$

This selection is possible thanks to (A). So, we have

$$\left\|u_t^m\right\|^2+\left\|\Delta u^m\right\|^2+\left\|\nabla u^m\right\|^2+\left\|u^m\right\|^2\leq C\left(1+\left\|u^m\right\|^2\ln\left\|u^m\right\|^2\right) \tag{12}$$

We know that

$$u^m(.,t)=u^m(.,0)+\int_0^t\frac{\partial u^m}{\partial\tau}(.,\tau)\,d\tau.$$

We make use of the following Cauchy-Schwarz inequality

$$(a+b)^2\leq 2\left(a^2+b^2\right),$$

we obtain

$$\begin{aligned}\left\|u^m(t)\right\|^2 &= \left\|u^m(.,0)+\int_0^t\frac{\partial u^m}{\partial\tau}(.,\tau)\,d\tau\right\|^2\\ &\leq 2\left\|u^m(0)\right\|^2+2\left\|\int_0^t\frac{\partial u^m}{\partial\tau}(.,\tau)\,d\tau\right\|^2\\ &\leq 2\left\|u^m(0)\right\|^2+2T\int_0^t\left\|u_t^m(\tau)\right\|^2 d\tau \end{aligned} \tag{13}$$

So if we write inequality (12) instead of inequality (13), we get

$$\left\|u^m\right\|^2\leq 2\left\|u^m(0)\right\|^2+2TC\left(1+\left\|u^m\right\|^2\ln\left\|u^m\right\|^2\right) \tag{14}$$

If we put $C_1=\max\left\{2TC,\ 2\left\|u^m(0)\right\|^2\right\}$, (14) leads to

$$\left\|u^m\right\|^2\leq 2C_1\left(1+\int_0^t\left\|u^m\right\|^2\ln\left\|u^m\right\|^2 d\tau\right).$$

Without loss of generality, we take $C_1\geq 1$, we have

$$\left\|u^m\right\|^2\leq 2C_1\left(1+\int_0^t\left(C_1+\left\|u^m\right\|^2\right)\ln\left(C_1+\left\|u^m\right\|^2\right)d\tau\right).$$

Thanks to Logarithmic Gronwall inequality, we obtain

$$\left\|u^m\right\|^2\leq 2C_1e^{2C_1T}=C_2.$$

Hence, from inequality (12), it follows that

$$\left\|u_t^m\right\|^2+\left\|\Delta u^m\right\|^2+\left\|\nabla u^m\right\|^2+\left\|u^m\right\|^2\leq C_3=C\left(1+C_2\ln C_2\right)$$

where C_3 is a positive constant independent of m and t. If these operations (12) are applied to each term of inequality, this implies

$$\sup_{t\in(0,t_m)}\left\|u_t^m\right\|^2+\sup_{t\in(0,t_m)}\left\|\Delta u^m\right\|^2+\sup_{t\in(0,t_m)}\left\|\nabla u^m\right\|^2+\sup_{t\in(0,t_m)}\left\|u^m\right\|^2\leq 4C_3. \tag{15}$$

So, the approximate solution is uniformly bounded independent of m and t. Therefore, we can extend t_m to T. Moreover, we obtain

$$\begin{cases} u^m, \text{ is uniformly bounded in } L^\infty\left(0,T;H_0^2(\Omega)\right),\\ u_t^m, \text{ is uniformly bounded in } L^\infty\left(0,T;L^2(\Omega)\right).\end{cases} \tag{16}$$

Hence we can infer from (15) and (16) that there exists a subsequence of (u^m) (still denoted by (u^m)), such that

$$\begin{cases} u^m\to u, \text{ weakly* in } L^\infty\left(0,T;H_0^2(\Omega)\right),\\ u_t^m\to u_t, \text{ weakly* in } L^\infty\left(0,T;L^2(\Omega)\right),\\ u^m\to u, \text{ weakly in } L^2\left(0,T;H_0^2(\Omega)\right),\\ u_t^m\to u_t, \text{weakly in } L^2\left(0,T;L^2(\Omega)\right).\end{cases} \tag{17}$$

Then using (17) and Aubin–Lions' lemma, we have

$$u^m \to u, \text{ strongly in } L^2\left(0,T;L^2(\Omega)\right)$$

which implies

$$u^m \to u, \ \ \Omega\times(0,T).$$

Since the map $s \to s\ln|s|^k$ is continuous, we have the convergence

$$u^m \ln\left|u^m\right|^k \to u\ln|u|^k, \ \ \Omega\times(0,T). \tag{18}$$

By the Sobolev embedding theorem ($H_0^2(\Omega) \hookrightarrow L^\infty(\Omega)$), it is clear that $\left|u^m \ln|u^m|^k - u\ln|u|^k\right|$ is bounded in $L^\infty(\Omega\times(0,T))$. Next, taking into account the Lebesgue bounded convergence theorem, we have

$$u^m \ln\left|u^m\right|^k \to u\ln|u|^k, \text{strongly in } L^2\left(0,T;L^2(\Omega)\right). \tag{19}$$

We integrate (6) over $(0,t)$ to obtain, $\forall w\in V_m$

$$\begin{aligned}\int_0^t\int_\Omega \ln\left|u^m\right|^k u^m w dxd\tau &= \int_\Omega u_t^m w dxd\tau - \int_\Omega u_1^m w dxd\tau + \int_0^t\int_\Omega \triangle u^m \Delta w dxd\tau \\ &+ \int_0^t\int_\Omega \nabla u^m \nabla w dxd\tau + \int_0^t\int_\Omega u_t^m w dxd\tau.\end{aligned} \tag{20}$$

Convergences (17), (19) are sufficient to pass to the limit in (20)

$$\begin{aligned}\int_\Omega u_t w dx &= \int_\Omega u_1 w dx - \int_0^t\int_\Omega \triangle u \Delta w dxd\tau - \int_0^t\int_\Omega \nabla u \nabla w dxd\tau \\ &- \int_0^t\int_\Omega u_t w dxd\tau + \int_0^t\int_\Omega \ln|u|^k u w dxd\tau\end{aligned} \tag{21}$$

which implies that (20) is valid $\forall w\in H_0^2(\Omega)$ .Using the fact that the terms in the right-hand side of (21) are absolutely continuous since they are functions of t defined by integrals over $(0,t)$, hence it is differentiable for a.e. $t\in R^+$. Thus, differentiating (21), we obtain, for a.e. $t\in(0,T)$ and any $w\in H_0^2(\Omega)$,

$$\begin{aligned}&\int_0^t\int_\Omega \ln|u(x,t)|^k u(x,t)\, w(x)\, dxd\tau \\ =\ &\int_\Omega u_{tt}(x,t)\, w(x)\, dx + \int_\Omega \triangle u(x,t)\, \Delta w(x)\, dxd\tau \\ &- \int_0^t\int_\Omega \nabla u(x,t)\, \nabla w(x)\, dxd\tau - \int_0^t\int_\Omega u_t(x,t)\, w(x)\, dxd\tau.\end{aligned}$$

If we take initial data, we note that

$$\begin{aligned} u^m &\to u, \text{ weakly in } L^2\left(0,T;H_0^2(\Omega)\right) \\ u_t^m &\to u_t, \text{ weakly in } L^2\left(0,T;L^2(\Omega)\right)\end{aligned}$$

Thus, using Lion's Lemma [10], we have

$$u^m \to u, C\left(\left([0,T];L^2(\Omega)\right)\right).$$

Therefore, $u^m(x,0)$ makes sense and

$$u^m(x,0) \to u(x,0), \ \ L^2(\Omega).$$

We have

$$u^m(x,0) \to u_0(x,0)\,, \quad H_0^2(\Omega)$$

hence

$$u(x) = u_0(x)\,.$$

Now, multiply (1) by $\phi \in C_0^\infty(0,T)$ and integrate over $(0,T)$, we obtain for $\forall w \in V_m$, and because of

$$\left(u_t^m \phi(t)\right)' = u_{tt}^m \phi(t) + u_t^m \phi'(t)$$

we get

$$\begin{aligned} -\int_0^T\int_\Omega u_t^m w\phi'(t)\,dxdt &= -\int_0^T\int_\Omega \triangle u^m \Delta w\phi(t)\,dxdt - \int_0^T\int_\Omega \nabla u^m \nabla w\phi(t)\,dxdt \\ &\quad -\int_0^T\int_\Omega u_t^m w\phi(t)\,dxdt + \int_0^T\int_\Omega \ln|u^m|^k u^m w\phi(t)\,dxdt. \end{aligned}$$

As $m \to \infty$, we have for $\forall w \in H_0^2(\Omega)$ and $\phi \in C_0^\infty(0,T)$

$$\begin{aligned} -\int_0^T\int_\Omega u_t w\phi'(t)\,dxdt &= -\int_0^T\int_\Omega \triangle u \Delta w\phi(t)\,dxdt - \int_0^T\int_\Omega \nabla u \nabla w\phi(t)\,dxdt \\ &\quad -\int_0^T\int_\Omega u_t w\phi(t)\,dxdt + \int_0^T\int_\Omega \ln|u|^k u w\phi(t)\,dxdt. \end{aligned}$$

This means

$$u_{tt} \in L^2[0,T), H^{-2}(\Omega)\,,$$

on the other hand, because of

$$u_t \in \left(L^2[0,T), L^2(\Omega)\right),$$

we obtain

$$u_t \in C\left([0,T), H^{-2}(\Omega)\right).$$

So that

$$u_t^m(x,0) \to u_t(x,0)\,, \quad H^{-2}(\Omega)\,,$$

but

$$u_t^m(x,0) = u_1^m(x) \to u_1(x)\,, \quad L^2(\Omega)\,.$$

Hence

$$u_t(x,0) = u_1(x)\,.$$

This finishes the proof of the theorem. □

4 Global existence

In this section we study global existence of problem (1). We prove a global existence result using the potential wells corresponding to the logarithmic nonlinearity.

Now, we define the following functionals

$$J(t) = \frac{1}{2}\left(\|\Delta u\|^2 + \|\nabla u\|^2 + \frac{k}{2}\|u\|^2 - \int_\Omega \ln|u|^k u^2 dx\right), \tag{22}$$

$$I(t) = \|\Delta u\|^2 + \|\nabla u\|^2 - \int_\Omega \ln|u|^k u^2 dx. \tag{23}$$

Then, it is obvious that

$$J(t) = \frac{1}{2} I(u) + \frac{k}{4}\|u\|^2 \tag{24}$$

and

$$E(t) = \frac{1}{2}\|u_t\|^2 + J(u)\,. \tag{25}$$

According to the Logarithmic Sobolev inequality, $J(u)$ and $I(u)$ are well defined. The potential well depth is defined as

$$0 < d = \inf_{u} \left\{ \sup_{\lambda \geq 0} J(\lambda u) : u \in H_0^2(\Omega), \|\Delta u\| \neq 0 \right\} \tag{26}$$

and the well-known Nehari manifold

$$N = \left\{ u : u \in H_0^2(\Omega) / I(u) = 0, \|\Delta u\| \neq 0 \right\}, \tag{27}$$

$$0 < d = \inf_{u \in N} J(u). \tag{28}$$

Then, we introduce

$$W = \left\{ u : u \in H_0^2(\Omega) / I(u) > 0,\ J(u) < d \right\} \cup \{0\}.$$

Lemma 6. *For any $u \in H_0^2(\Omega)$, $\|u\| \neq 0$ and let $g(\lambda) = J(\lambda u)$. Then we have*

$$I(\lambda u) = \lambda g'(\lambda) \begin{cases} > 0, & 0 \leq \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda < \lambda^* < \infty \end{cases}$$

where

$$\lambda^* = \exp\left(\frac{\|\Delta u\|^2 + \|\nabla u\|^2 - \int_\Omega \ln|u|^k u^2 dx}{k\|u\|^2} \right).$$

Proof: By the defination of $J(u)$, we obtain

$$\begin{aligned} g(\lambda) &= J(\lambda u) \\ &= \frac{1}{2}\left(\|\lambda \Delta u\|^2 + \|\lambda \nabla u\|^2 + - \int_\Omega \ln|\lambda u|^k (\lambda u)^2 dx \right) + \frac{k}{4}\|\lambda u\|^2 \\ &= \frac{\lambda^2}{2}\left(\|\Delta u\|^2 + \|\nabla u\|^2 \right) + \frac{\lambda^2}{2}\left(\frac{k}{2} - k\ln|\lambda| \right)\|u\|^2 - \frac{k\lambda^2}{2}\int_\Omega \ln|u|\,|u|^2 dx. \end{aligned}$$

Since $\|u\| \neq 0$, $\lim_{\lambda \to 0} g(\lambda) = 0$, $\lim_{\lambda \to \infty} g(\lambda) = -\infty$. Now, differentiating $g(\lambda)$ with respect to λ, we have

$$g'(\lambda) = \lambda\left(\|\Delta u\|^2 + \|\nabla u\|^2 - k\ln\lambda \|u\|^2 - k\int_\Omega \ln|u|(u)^2 dx \right).$$

We can see clearly that

$$\lambda \frac{dJ(\lambda u)}{d\lambda} = \lambda g'(\lambda) = I(\lambda u).$$

We can derive $I(\lambda u) = 0$ when

$$\lambda^* = \exp\left(\frac{\|\Delta u\|^2 + \|\nabla u\|^2 - \int_\Omega \ln|u|^k u^2 dx}{k\|u\|^2} \right).$$

Thus, we have

$$I(\lambda u) = \lambda g'(\lambda) \begin{cases} > 0, & 0 \leq \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda < \lambda^* < \infty. \end{cases}$$

□

Lemma 7. *Let $u \in H_0^2(\Omega)$ and $l = e^{\frac{2}{k}+2}\frac{2\pi}{k}$.*

i) If $0 < \|u\|^2 < l$, then $I(u) > 0$;
ii) If $I(u) = 0$ and $\|u\| \neq 0$, then $\|u\|^2 > l$;
iii)The constant d in (26) satisfies

$$d \geq \frac{\pi}{k} e^{\frac{2}{k}+2}.$$

Proof: Thanks to Logarithmic Sobolev Inequality to the last term of the $I(u)$ function, we have

$$\begin{aligned} I(u) &= \|\Delta u\|^2 + \|\nabla u\|^2 - \int_\Omega \ln|u|^k u^2 dx, \\ &\geq \|\Delta u\|^2 + \|\nabla u\|^2 - \frac{k}{2}\|u\|^2 \ln\|u\|^2 \\ &\quad -\frac{k\alpha^2}{2\pi}\|\nabla u\|^2 + k(1+\ln\alpha)\|u\|^2, \\ &\geq \left(1-\frac{k\alpha^2}{2\pi}\right)\|\nabla u\|^2 + k\left((1+\ln\alpha) - \frac{1}{2}\ln\|u\|^2\right)\|u\|^2 \end{aligned} \tag{29}$$

Taking any α satisfying $0 < \alpha \leq \sqrt{\frac{2\pi}{k}}$ in (29), we have

$$I(u) \geq \left(k(1+\ln\sqrt{\frac{2\pi}{k}}) - \frac{k}{2}\ln\|u\|^2\right)\|u\|^2. \tag{30}$$

i) If $0 < \|u\|^2 < l$, then $I(u) > 0$ from the above inequality.
ii) If $I(u) = 0$ and $\|u\| \neq 0$, then

$$\|u\|^2 \geq e^2 \frac{2\pi}{k} = l.$$

iii) Because of (26), we write

$$\sup_{\lambda\geq 0} J(\lambda u) = J(\lambda^* u) = \frac{1}{2} I(\lambda^* u) + \frac{k}{4}(\lambda^*)^2\|u\|^2 \tag{31}$$

By the Lemma 7 and (30), we obtain

$$0 = I(\lambda^* u) \geq \left(k(1+\ln\sqrt{\frac{2\pi}{k}}) - \frac{k}{2}\ln\|\lambda^* u\|^2\right)\|\lambda^* u\|^2.$$

Therefore; we have

$$0 \geq k(1+\ln\sqrt{\frac{2\pi}{k}}) - \frac{k}{2}\ln\|\lambda^* u\|^2,$$

$$\ln\|\lambda^* u\|^2 \geq 2 + 2\ln\sqrt{\frac{2\pi}{k}}),$$

$$\|\lambda^* u\|^2 \geq e^2\frac{2\pi}{k} = l. \tag{32}$$

Thus, by using of (26), (31) and (32) , we obtain

$$d \geq \frac{\pi}{2k}e^2.$$

□

Lemma 8. *Let $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ and $l = e^2\frac{2\pi}{k}$ such that $0 < E(0) < \frac{k}{4}l < d$ and $I(u_0) > 0$. Then any solution of (1), $u \in W$.*

Proof: Let T be maximal existence time of weak solution of u. From (25) and (9), we have

$$\frac{1}{2}\|u_t\|^2 + J(u) \leq \frac{1}{2}\|u_1\|^2 + J(u_0) < d, \forall t \in [0,T). \tag{33}$$

Then we claim that $u(t) \in W$ for all $t \in [0,T)$. If it is false, then there is a $t_0 \in [0,T)$ such that $u(t_0) \in \partial W$, so we have
(a) either $I(u(t_0)) = 0$ and $\|\Delta u(t_0)\| \neq 0$, or (b) $J(u(t_0)) = d$.
By (33), (b) is impossible, thus we have $I(u(t_0)) = 0$ and $\|\Delta u(t_0)\| \neq 0$. However, at least one $J(u(t_0)) \geq d$ exists if $0 < d = \inf_{u\in N} J(u)$. Because of this contradiction, $u(t) \in W$ is found for $\forall t \in [0,T)$. □

5 Decay of solution

In this section, we will prove decay of solutions to problem (1).
For this purpose, we use the Lyapunov functional

$$L(t) = E(t) + \varepsilon\int_\Omega u u_t dx + \frac{\varepsilon}{2}\int_\Omega u^2 dx \tag{34}$$

where ϵ is a positive constant. We will show the $L(t)$ and $E(t)$ are equivalent:

Lemma 9. *For $\epsilon > 0$ small enough, the relation*

$$\beta_1 L(t) \leq E(t) \leq \beta_2 L(t) \tag{35}$$

holds for two positive constants β_1 and β_2.

We can choose ε small enough such that $L \sim E$.

Theorem 10. *Let $u_0 \in W$, $u_1 \in L^2(\Omega)$. Assume further $0 < E(0) < \alpha l < d$, where*

$$l = \frac{2\pi}{k} e^2 \quad \text{and } 0 < \alpha \leq \frac{\sqrt{2\pi}}{k} \alpha^{\frac{1}{2}} e$$

then there exist two positive constants c_1 and c_2 such that

$$0 < E(t) \leq c_1 e^{-c_2 t}, \; t \geq 0.$$

Proof: By taking the time derivative of the $L(t)$ and using Eq. (1), we obtain

$$\begin{aligned} L'(t) &= E'(t) + \varepsilon \int_\Omega \left(u_{tt} u + u_t^2\right) dx + \varepsilon \int_\Omega u u_t dx \\ &= (\varepsilon - 1) \|u_t\|^2 - \varepsilon \left(\|\triangle u\|^2 + \|\nabla u\|^2\right) + \varepsilon \int_\Omega \ln |u|^k u^2. \end{aligned} \tag{36}$$

Adding and subtracting $\varepsilon \beta E(t)$ into (36) where β is a positive constant, we get

$$\begin{aligned} L'(t) &= \left(\varepsilon + \frac{\varepsilon}{2}\beta - 1\right) \|u_t\|^2 + \varepsilon \left(\frac{\beta}{2} - 1\right) \|\triangle u\|^2 + \varepsilon \left(\frac{\beta}{2} - 1\right) \|\nabla u\|^2 \\ &+ \varepsilon \left(1 - \frac{\beta}{2}\right) \int_\Omega \ln |u|^k u^2 + \frac{k}{4} \varepsilon \beta \|u\|^2 - \varepsilon \beta E(t). \end{aligned} \tag{37}$$

By the Logarithmic Sobolev inequality and embedding theorems and choosing c_p is smallest enough positive constant, we have

$$\begin{aligned} L'(t) &\leq \left(\varepsilon + \frac{\varepsilon}{2}\beta - 1\right) \|u_t\|^2 + \varepsilon \left(\frac{\beta}{2} - 1\right) \|\triangle u\|^2 \\ &+ \varepsilon \left(\frac{\beta}{2} - 1\right) \|\nabla u\|^2 + \frac{k}{4} \varepsilon \beta \|u\|^2 - \varepsilon \beta E(t) \\ &+ \varepsilon \left(1 - \frac{\beta}{2}\right) k \left(\frac{1}{2} \|u\|^2 \ln \|u\|^2 + \frac{\alpha^2}{2\pi} \|\nabla u\|^2 - (1 + \ln \alpha) \|u\|^2\right), \\ &= -\varepsilon \beta E(t) + \left(\varepsilon + \frac{\varepsilon}{2}\beta - 1\right) \|u_t\|^2 \\ &+ \varepsilon \left(\frac{\beta}{2} - 1\right) . \left(1 - k \frac{\alpha^2}{2\pi}\right) \|\nabla u\|^2 + \varepsilon \left(\frac{\beta k c_p}{4} + \frac{\beta}{2} - 1\right) \|\triangle u\|^2 \\ &+ \varepsilon k \left[\left(1 - \frac{\beta}{2}\right) . \left(\frac{1}{2} \ln \|u\|^2 - (1 + \ln \alpha)\right)\right] \|u\|^2. \end{aligned}$$

Noting that since $\beta = \min\left\{2, \frac{4}{2 + k c_p}\right\}$, and $\varepsilon > 0$ sufficiently small so that

$$\varepsilon + \frac{\varepsilon}{2}\beta - 1 < 0,$$

we get

$$\begin{aligned} L'(t) &\leq -\varepsilon \beta E(t) - \varepsilon \left(1 - \frac{\beta}{2}\right) \left(1 - k \frac{\alpha^2}{2\pi}\right) \|\nabla u\|^2 - \varepsilon \left(1 - \frac{\beta}{2} - \frac{\beta k c_p}{4}\right) \|\triangle u\|^2 \\ &+ \varepsilon k \left[\left(1 - \frac{\beta}{2}\right) \left(\frac{1}{2} \ln \|u\|^2 - (1 + \ln \alpha)\right)\right] \|u\|^2. \end{aligned} \tag{38}$$

Using (4), (5), (22) and assumption in the Theorem 10, we have

$$
\begin{aligned}
\ln \|u\|^2 &\leq \ln\left(\frac{4}{k}E(t)\right) \\
&\leq \ln\left(\frac{4}{k}E(t)\right) \\
&\leq \ln\left(\frac{4}{k}E(0)\right) \\
&\leq \ln\left(\frac{4}{k}\alpha l\right) \\
&= \ln\left(\frac{2\pi}{k^2}\alpha e^2\right).
\end{aligned}
$$

Taking α satisfying

$$\frac{\sqrt{2\pi}}{k}\alpha^{\frac{1}{2}}e < \alpha \leq \sqrt{\frac{2\pi}{k}}$$

we guarantee

$$\frac{1}{2}\ln \|u\|^2 - (1 + \ln \alpha) < 0.$$

Consequently, inequlatiy (38) becomes

$$L'(t) \leq -\varepsilon\beta E(t).$$

By (35), we have

$$L'(t) \leq -\varepsilon\beta\beta_2 L(t) \tag{39}$$

setting $c_2 = \varepsilon\beta\beta_2 > 0$ and integrating (39) between $(0, t)$ gives the following estimate

$$L(t) \leq c_1 e^{-c_2 t}$$

Consequently, by using (35) once again. This completes the proof. □

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