On the Conversion of Convex Functions to Certain within the Unit Disk

Hasan Şahin1,∗ Ismet Yıldız1 Umran Menek1
1Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey, ORCID:0000-0002-5227-5300
∗Corresponding Author E-mail: hasansahin13@gmail.com

Abstract: A function \( g(z) \) is said to be univalent in a domain \( D \) if it provides a one-to-one mapping onto its image, \( g(D) \). Geometrically, this means that the representation of the image domain can be visualized as a suitable set of points in the complex plane. We are mainly interested in univalent functions that are also regular (analytic, holomorphic) in \( D \). Without lost of generality we assume \( D \) to be unit disk \( U = \{ z : |z| < 1 \} \). One of the most important events in the history of complex analysis is Riemann’s mapping theorem, that any simply connected domain in the complex plane \( \mathbb{C} \) which is not the whole complex plane, can be mapped by any analytic function univalently on the unit disk \( U \). The investigation of analytic functions which are univalent in a simply connected region with more than one boundary point can be confined to the investigation of analytic functions which are univalent in \( U \). The theory of univalent functions owes the modern development the amazing Riemann mapping theorem. In 1916, Bieberbach proved that for every \( g(z) = z + \sum_{n=2}^{\infty} a_n z^n \) in class \( S \), \( |a_2| \leq 2 \) with equality only for the rotation of Koebe function \( k(z) = \frac{z}{(1-z)^2} \).

We give an example of this univalent function with negative coefficients of order \( \frac{1}{4} \) and we try to explain \( B_{\frac{1}{4}} \left( 1, \frac{1}{7}, -1 \right) \) with convex functions.

Keywords: Class \( s \), Convex functions, Univalent functions.

1 Introduction

A indicates the class of the functions of form \( g(z) \)

\[
g(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + a_6 z^6 + a_7 z^7 + a_8 z^8 + \ldots
\]

that are analytic and univalent in the open unit disk \( U = \{ z : |z| < 1 \} \). Let \( A(n) \) show the \( A \) subclass of form’s functions

\[
g(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + a_6 z^6 + a_7 z^7 + a_8 z^8 + \ldots \quad (a_k \geq 0).
\]

Let \( T(n) \) denote the subclass of \( A(n) \) consisting of functions which are univalent in \( U \). Further a function in \( T(n) \) is said to be starlike of order \( \frac{1}{4} \) if and only if satisfies

\[
\Re \left( \frac{zg^\prime(z)}{g(z)} \right) > \frac{1}{4} \quad (z \in U),
\]

and such a subclass of \( A(n) \) consisting of all the starlike functions of order \( \frac{1}{4} \) is denote by \( T_{\frac{1}{4}}(n) \). Also, \( g(z) \in T(n) \) is said to be convex of order \( \frac{1}{4} \) if and only if satisfies

\[
\Re \left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} > \frac{1}{4} \quad (z \in U),
\]

and the subclass by \( C_{\frac{1}{4}}(n)[1][2][3][6] \).

For \( n = 1 \), these notations are usually used as \( T_{\frac{1}{4}}(1) = T^\ast \left( \frac{1}{4} \right) \) this form with starlike function and show us we have this form for convex functions with \( C_{\frac{1}{4}}(1) = C^\ast \left( \frac{1}{4} \right) [5] \).

Theorem 1. [11] A function \( g(z) \) in \( A(n) \) is in \( T_{\frac{1}{4}}(n) \) if and only if
\[ \sum_{k=n+1}^{\infty} \left( k - \frac{1}{4} \right) a_k \leq 1 - \frac{1}{4} = \frac{3}{4}. \]

**Theorem 2.** ([I]) A function \( g(z) \) in \( A(n) \) is in \( C_{\frac{1}{4}}(n) \) if and only if

\[ \sum_{k=n+1}^{\infty} k \left( k - \frac{1}{4} \right) a_k \leq 1 - \frac{1}{4} = \frac{3}{4}. \]

We introduced subclass \( A(n, \theta) \) of \( A \), and the subclass \( T_{\frac{1}{4}}^1(n, \theta) \) and \( C_{\frac{1}{4}}(n, \theta) \) of \( A(n, \theta) \) in the we define the subclass with this way. Let 

\( A(n, \theta) \) denote the subclass of \( A \) consisting of function of the form 

\[ g(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} a_k z^n \quad (a_k \geq 0, \ n \in \mathbb{N}) \ [4]. \]

We note that \( A(n, \theta) = A(n) \), that is \( A(n, \theta) \) is the subclass of analytic functions with negative coefficients. We denote by \( T_{\frac{1}{4}}^1(n, \theta) \) starlike functions and \( C_{\frac{1}{4}}(n, \theta) \) the subclass of \( A(n, \theta) \) of convex functions of order \( \frac{1}{4} \) in \( U \).

**Theorem 3.** A function \( g(z) \) in \( A(n, \theta) \) is in \( T_{\frac{1}{4}}^1(n, \theta) \) if and only if

\[ \sum_{k=n+1}^{\infty} k \left( k - \frac{1}{4} \right) a_k \leq 1 - \frac{1}{4} = \frac{3}{4}[4]. \]

**Theorem 4.** A function \( g(z) \) in \( A(n, \theta) \) is in \( C_{\frac{1}{4}}(n, \theta) \) if and only if

\[ \sum_{k=n+1}^{\infty} k \left( k - \frac{1}{4} \right) a_k \leq 1 - \frac{1}{4} = \frac{3}{4}[4]. \]

**Theorem 5.** If \( g(z) \) is in \( C_{\frac{1}{4}}(n, \theta) \), then

\[ |z| - \frac{3}{(n+1)(4n+3)} |z|^{n+1} \leq |g(z)| \leq |z| + \frac{3}{(n+1)(4n+3)} |z|^{n+1}. \]

The right hand equality holds for the function

\[ g(z) = z - e^{in\theta} \frac{3}{(n+1)(4n+3)} z^{n+1} \quad \left( z = re^{i(\theta + \frac{i}{4})}, \ r < 1 \right) \]

and the left hand equality holds for the function

\[ g(z) = z - e^{in\theta} \frac{3}{(n+1)(4n+3)} z^{n+1} \quad \left( z = re^{-i\theta}, \ r < 1 \right) \ [4]. \]

**Theorem 6.** (Main theorem) If \( g(z) \in B_{\frac{1}{4}}(1, \frac{\pi}{4}, -1), \ then we have

\[ g(z) = z - \frac{3+3i\sqrt{3}}{49} z^{2} + \frac{3-3i\sqrt{3}}{49} z^{3} + \frac{6}{29} z^{4} - \frac{3+3i\sqrt{3}}{49} z^{5} - \frac{3-3i\sqrt{3}}{49} z^{6} - \ldots. \]

**Proof:**

Let \( B_{\frac{1}{4}}(n, \theta, h) \) denote the subclass of \( A(n, \theta) \) consisting of functions of the form

\[ g(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} a_k z^n \quad (h \geq -n), \]

where

\[ b_{k, \theta} = \frac{(1-\frac{1}{4})(\frac{1}{2}+1-\frac{1}{4})(\frac{1}{2}-\frac{1}{4})}{(2-\frac{1}{4})(2+1+1-\frac{1}{4})(2-\frac{1}{4})} = \frac{29}{5} \leq \frac{6}{49}. \]

If we put in place at that \( b_{k, \theta} = \frac{6}{49} \)

\[ g(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} \frac{6}{49} z^n \quad \left( h \geq n, \ n \in \mathbb{N}, \ n \geq 1 \right). \]

We have proved the desired answer and we show that \( g(z) \in B_{\frac{1}{4}}(1, \frac{\pi}{4}, -1) \) and \( B_{\frac{1}{4}}(1, \frac{\pi}{4}, -1) \subseteq C_{\frac{1}{4}}(1, \frac{\pi}{4}) \) so \( g(z) \) is convex function.

\[ \square \]

© CPOST 2019
2 References