Negative Coefficient of Starlike Functions of Order 1/2

Hasan Şahin1, ∗ Ismet Yıldız1 Umran Menek1
1 Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey, ORCID:0000-0002-5227-5300
∗ Corresponding Author E-mail: hasansahin13@gmail.com

Abstract: A function $g(z)$ is said to be univalent in a domain $D$ if it provides a one-to-one mapping onto its image, $g(D)$. Geometrically, this means that the representation of the image domain can be visualized as a suitable set of points in the complex plane. We are mainly interested in univalent functions that are also regular (analytic, holomorphic) in $U$. Without lost of generality we assume $D$ to be unit disk $U = \{z : |z| < 1\}$. One of the most important events in the history of complex analysis is Riemann’s mapping theorem, that any simply connected domain in the complex plane $C$ which is not the whole complex plane, can be mapped by any analytic function univalently on the unit disk $U$. The investigation of analytic functions which are univalent in a simply connected region with more than one boundary point can be confined to the investigation of analytic functions which are univalent in $U$. The theory of univalent functions owes the modern development the amazing Riemann mapping theorem. In 1916, Bieberbach proved that for every $g(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in class $S$, $|a_2| \leq 2$ with equality only for the rotation of Koebe function $k(z) = \frac{z}{(1-z)^2}$. We give an example of this univalent function with negative coefficients of order $\frac{1}{4}$ and we try to explain $B_{\frac{1}{4}} \left(1, \frac{1}{8}, -1\right)$ with convex functions.

Keywords: Class $S$, Convex functions, Univalent functions.

1 Introduction

Let $A$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (n \in \mathbb{N} = \{1, 2, 3, \ldots\}).$$

$f(z)$ is a function in unit disk $U = \{z : |z| < 1\}$ and analytic.

Let $A(n)$ denote the subclass of $A$ consisting of functions of form

$$f(z) = z - \sum k = n + 1^{\infty} a_k z^k \quad (a_k \geq 0, n \in \mathbb{N} = \{1, 2, 3, \ldots\}).$$

Let $T(n)$ denote the subclass of $A(n)$ consisting of functions which are univalent in $U$. Further a function in $T(n)$ is said to be starlike of order $\frac{1}{2}$ if and only if

$$\frac{zf''(z)}{f'(z)} > \frac{1}{2} \quad (z \in U)$$

and such a subclass of $A(n)$ consisting of all the starlike functions of order $\frac{1}{2}$ is denote by $T_{\frac{1}{2}}(n)$. Also, $f(z) \in T(n)$ is said to be convex of order $\frac{1}{2}$ if and only if satisfies

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{1}{2} \quad (z \in U)$$

and the subclass by $C_{\frac{1}{2}}(n) \{1\} [2][3][6]$.

For $n = 1$, these notations are usually used as $T_{\frac{1}{2}}(1) = T\left(\frac{1}{2}\right)$ and $C_{\frac{1}{2}}(n) = C^{*}\left(\frac{1}{2}\right) [5]$.

Theorem 1. A function $f(z)$ in $A(n)$ is in $T_{\frac{1}{2}}(n)$ if and only if

$$\sum_{k=n+1}^{\infty} \left(k - \frac{1}{2}\right) a_k \leq 1 - \frac{1}{2} = \frac{1}{2} [1].$$
Theorem 2. A function $f(z)$ in $A(n)$ is in $C_{\nu}^2(n)$ if and only if

$$\sum_{k=n+1}^{\infty} \left( k - \frac{1}{2} \right) a_k \leq 1 - \frac{1}{2} = \frac{1}{2} [1].$$

We introduced the subclass $A(n, \theta)$ of $A$, and the subclass $T_{\nu}^2(n, \theta)$ and $C_{\nu}^2(n, \theta)$ of $A(n, \theta)$ in the following manner. Let $A(n, \theta)$ denote the subclass of $A$ consisting of function of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} a_k z^k \quad (a_k \geq 0, \; n \in \mathbb{N}) \; [4].$$

We note that $A(n, \theta) = A(n)$, that is $A(n, 0)$ is the subclass of analytic functions with negative coefficients. We denote by $T_{\nu}^2(n, \theta)$ and $C_{\nu}^2(n, \theta)$ the subclass of $A(n, \theta)$ of starlike and convex functions of order $\frac{1}{2}$ in $U$.

Theorem 3. A function $f(z)$ in $A(n, \theta)$ is in $T_{\nu}^2(n, \theta)$ if and only if

$$\sum_{k=n+1}^{\infty} \left( k - \frac{1}{2} \right) a_k \leq 1 - \frac{1}{2} = \frac{1}{2} [4].$$

Theorem 4. A function $f(z)$ in $A(n, \theta)$ is in $C_{\nu}^2(n, \theta)$ if and only if

$$\sum_{k=n+1}^{\infty} k \left( k - \frac{1}{2} \right) a_k \leq 1 - \frac{1}{2} = \frac{1}{2} [4].$$

Theorem 5. $f(z) \in A_{\frac{1}{2}}(n, \theta, h)$ then $f(z) \in T_{\nu}^2(n, \theta)$.

Proof:

$$\sum_{k=n+1}^{\infty} \left( k - \frac{1}{2} \right) a_k, h = \sum_{k=n+1}^{\infty} \left( k - \frac{1}{2} \right) \left( \frac{1}{4} \right)^{\frac{k}{2+n+k+\frac{1}{2}}} = \frac{1}{2} \sum_{k=n+1}^{\infty} \left( k - \frac{1}{2} \right) \left( \frac{1}{4} \right)^{\frac{k}{2+n+k+\frac{1}{2}}}$$

$$= \left\{ \begin{array}{ll}
\left( \frac{1}{2} \right)^2 = \frac{1}{4}, & h = -n, \\
\left( \frac{1}{n+1} \right)^2 \leq \left( \frac{1}{2} \right)^2 = \frac{1}{4}, & h > -n.
\end{array} \right.$$

Hence we know that $f(z)$ is an element of $T_{\nu}^2(n, \theta)$.

\[ \Box \]

Theorem 6. (Main theorem) If $f(z) \in A_{\frac{1}{2}} \left( 1, \frac{\pi}{3}, 0 \right)$, then we have starlike function and $A_{\frac{1}{2}} \left( 1, \frac{\pi}{3}, 0 \right) \in S^*.$

$$f(z) = z - \frac{1 + i\sqrt{3}}{45} z^2 + \frac{1 - i\sqrt{3}}{45} z^3 - \frac{2}{45} z^4 - \frac{1 + i\sqrt{3}}{45} z^5 - \ldots$$

Proof: Let $f(z) \in A_{\frac{1}{2}} \left( 1, \frac{\pi}{3}, 0 \right)$ denote the subclass of $A \left( 1, \frac{\pi}{3} \right)$ consisting of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} a_k z^k \quad (h \geq -n, \; n \in \mathbb{N} = \{1,2,3,\ldots\})$$

where

$$a_{k, h} = a_{2,0} = \frac{1}{2} \left( \frac{1}{2} - \frac{1}{2} \right) = \frac{1}{45} = \frac{2}{45}.$$
We show the results we’ve achieved our proof.

\[\begin{aligned}
 &= z - \frac{2e^{i\frac{\pi}{3}}}{45} z^2 + \frac{2e^{i\frac{\pi}{3}}}{45} z^3 - \frac{2e^{i\frac{\pi}{3}}}{45} z^4 - \frac{2e^{i\frac{\pi}{3}}}{45} z^5 - \ldots \\
&= z - \frac{1 + i\sqrt{3}}{45} z^2 + \frac{1 - i\sqrt{3}}{45} z^3 - \frac{2}{45} z^4 - \frac{1 + i\sqrt{3}}{45} z^5 - \ldots
\end{aligned}\]

2 References