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On Generalized Sister Celine's Polynomials

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Abstract: In this research, we establish some properties for the generalized Sister Celine's polynomials. We derive various families of multilinear and multilateral generating functions for a family of generalized Sister Celine's polynomials.

Keywords: Generalized Sister Celine's polynomials, Multilinear and multilateral generating functions, Recurrence relations.

1 Introduction

Sister Celine [1] has introduced the polynomial $f_n(x)$

$$f_n\left[\begin{array}{cc} a_1,...,a_p; & \\ & x \\ b_1,...,b_q; & \end{array}\right] = {}_{p+2}F_{q+2}\left[\begin{array}{cc} -n, n+1, a_1,...,a_p; & \\ & x \\ 1, \frac{1}{2}, b_1,...,b_q; & \end{array}\right], \tag{1}$$

which is defined by the following generating function (see [2], p.290)

$$\sum_{n=0}^{\infty} f_n\left[\begin{array}{cc} a_1,...,a_p; & \\ & x \\ b_1,...,b_q; & \end{array}\right] t^n = (1-t)^{-1}\,{}_{p+2}F_{q+2}\left[\begin{array}{cc} a_1,...,a_p; & \\ & \frac{-4xt}{(1-t)^2} \\ b_1,...,b_q; & \end{array}\right] |t|<1, \tag{2}$$

where ${}_pF_q$ denotes the generalized hypergeometric function [2].

For $p=1,\ q=1,\ a_1=\frac{1}{2},\ b_1=1$ the following integral representation of Sister Celine polynomials is given by

$$f_n(\frac{1}{2};1;x) = \frac{1}{\sqrt{\pi}}\int_0^{\infty} y^{-1/2}e^{-y} f_n(-;1;xy)dy.$$

Equation (1) with no a's and no b's denotes simply

$$f_n(x) = {}_2F_2\left[-n, n+1; 1; \frac{1}{2}; x\right] = \sum_{r=0}^{n} \frac{(-1)^n (n)! x^r}{(r!)^2 (\frac{1}{2})_r (n-r)!}.$$

For the $f_n(x)$ the generating function (2) becomes

$$\sum_{n=0}^{\infty} f_n(x) t^n = (1-t)^{-1} \exp\left(\frac{-4xt}{(1-t)^2}\right),\ |t|<1. \tag{3}$$

In the view of above results, we define the generalized Sister Celine polynomial in following manner [3]

$$f_n^{(\alpha,\beta)}\left[\begin{array}{cc} a_1,...,a_p; & \\ & x \\ b_1,...,b_q; & \end{array}\right] \tag{4}$$

$$= \frac{(1+\alpha+\beta)_n}{n!}\ {}_{p+2}F_{q+2}\left[\begin{array}{cc} -n, n+\alpha+\beta+1, a_1,...,a_p; & \\ & x \\ 1+\alpha, \frac{1}{2}, b_1,...,b_q; & \end{array}\right].$$

Equation (4) with no a's and no b's denotes simply [3]

$$f_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha+\beta)_n}{n!}\ {}_2F_2\left[\begin{array}{c} -n, n+\alpha+\beta+1; \\ \\ 1+\alpha, \frac{1}{2}; \end{array} x\right] \tag{5}$$

$$= \frac{(1+\alpha+\beta)_n}{n!}\sum_{r=0}^{n}\frac{(-n)_r(n+\alpha+\beta+1)_r x^r}{(1+\alpha)_r(\frac{1}{2})_r r!}.$$

Indeed [3]

$$f_n^{(0,0)}(x) = f_n(x).$$

The following generating function can be easily obtained [3]

$$\sum_{n=0}^{\infty} f_n^{(\alpha,\beta)}(x)t^n = (1-t)^{-1-\alpha-\beta}\ {}_2F_2\left[\begin{array}{c} \frac{1+\alpha+\beta}{2}, \frac{2+\alpha+\beta}{2}; \\ \\ 1+\alpha, \frac{1}{2}; \end{array} \frac{-4xt}{(1-t)^2}\right]$$

$$\sum_{n=0}^{\infty} (C)_n f_n^{(\alpha,\beta)}(x)t^n = (1-t)^{-C-\alpha-\beta}\ {}_3F_3\left[\begin{array}{c} C, \frac{1+\alpha+\beta}{2}, \frac{2+\alpha+\beta}{2}; \\ \\ 1+\alpha, 1+\alpha+\beta, \frac{1}{2}; \end{array} \frac{-4xt}{(1-t)^2}\right]. \tag{6}$$

Obviously for $C = 1$, $\alpha = \beta = 0$, equation (6) reduces to the generating function (3), and (see, [3])

$$\sum_{n=0}^{\infty} f_n^{(\alpha,\beta)}\left[\begin{array}{c} a_1, ..., a_p; \\ \\ b_1, ..., b_q; \end{array} x\right] t^n = (1-t)^{-1-\alpha-\beta}$$

$$\times\ {}_{p+2}F_{q+2}\left[\begin{array}{c} a_1, ..., a_p, \frac{1+\alpha+\beta}{2}, \frac{2+\alpha+\beta}{2}; \\ \\ b_1, ..., b_q, 1+\alpha, \frac{1}{2}, \end{array} \frac{-4xt}{(1-t)^2}\right].$$

The main object of this paper to study several properties of the Sister Celine polynomials $f_n(x)$ and the generalized Sister Celine's polynomials $f_n^{(\alpha,\beta)}(x)$. Various families of multilinear and multilateral generating functions, miscellaneous properties and also some special cases for these polynomials are given.

2 Generating and Special Functions

In this section, we derive several families of bilinear and bilateral generating functions for the Sister Celine polynomials $f_n(x)$ and the generalized Sister Celine's polynomials $f_n^{(\alpha,\beta)}(x)$ generated by using the similar method considered in (see, [4] - [10]).

We begin by stating the following theorem.

Theorem 1. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1, ..., y_r)$ of r complex variables $y_1, ..., y_r$ $(r \in \mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\psi}(y_1, ..., y_r; \zeta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, ..., y_r)\zeta^k,\ (a_k \neq 0,\ \mu, \psi \in \mathbb{C})$$

and

$$\Theta_{n,p}^{\mu,\psi}(x; y_1, ..., y_r; \xi) := \sum_{k=0}^{[n/p]} a_k f_{n-pk}(x)\Omega_{\mu+\psi k}(y_1, ..., y_r)\xi^k.$$

Then, for $p \in \mathbb{N}$, we have

$$\sum_{n=0}^{\infty} \Theta_{n,p}^{\mu,\psi}(x; y_1, ..., y_r; \eta)\, t^n = (1-t)^{-1}\exp\left(\frac{-4xt}{(1-t)^2}\right)\Lambda_{\mu,\psi}(y_1, ..., y_r; \eta t^p), \tag{7}$$

provided that each member of (7) exists.

Proof: For convenience, let H denote the first member of the assertion (7). Then,

$$H=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k f_{n-pk}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k t^n.$$

Replacing n by $n+pk$, we may write that

$$\begin{aligned}H &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k\, f_n(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k t^{n+pk}\\ &= \sum_{n=0}^{\infty}f_n(x)t^n\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)(\eta t^p)^k\\ &= (1-t)^{-1}\exp\left(\frac{-4xt}{(1-t)^2}\right)\Lambda_{\mu,\psi}(y_1,...,y_r;\eta t^p),\end{aligned}$$

which completes the proof. □

If we set $r=1$ and

$$\Omega_{\mu+\psi k}(y_1)=f_{\mu+\psi k}(y_1)$$

in Theorem 1 , where the Sister Celine's polynomials $f_n(x)$, generated by [3]

$$\sum_{n=0}^{\infty}f_n(x)t^n=(1-t)^{-1}\exp\left(\frac{-4xt}{(1-t)^2}\right),\ |t|<1.$$

Thus, we have the following result which provides a class of bilinear generating functions for the Sister Celine's polynomials $f_n(x)$, as follows:

Corollary 1. *If*

$$\Lambda_{\mu,\psi}(y_1;w):=\sum_{k=0}^{\infty}a_k f_{\mu+\psi k}(y_1)w^k\quad(a_k\neq 0\,,\ \ \mu,\psi\in\mathbb{C})\,,$$

then, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k f_n(x)f_{\mu+\psi k}(y_1)w^k t^n \qquad (8)\\ &= (1-t)^{-1}\exp\left(\frac{-4xt}{(1-t)^2}\right)\Lambda_{\mu,\psi}(y_1,...,y_r;w),\end{aligned}$$

provided that each member of (8) *exists.*

Theorem 2. *Corresponding to an identically non-vanishing function* $\Omega_\mu(y_1,...,y_r)$ *of* r *complex variables* $y_1,...,y_r$ $(r\in\mathbb{N})$ *and of complex order* μ, *let*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\zeta^k\ \ (a_k\ \ \neq\ \ 0\ \ \mu,\psi\in\mathbb{C}),$$

and

$$\Theta_{n,p}^{\mu,\psi}(x;y_1,...,y_r;\xi):=\sum_{k=0}^{[n/p]}a_k f_{n-pk}^{(\alpha,\beta)}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\xi^k.$$

Then, for $p\in\mathbb{N}$, *we have*

$$\begin{aligned}&\sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\psi}(x;y_1,...,y_r;\eta)\,t^n \qquad (9)\\ &= (1-t)_2^{-1-\alpha-\beta}F_2\left[\begin{array}{cc}\frac{1+\alpha+\beta}{2},\frac{2+\alpha+\beta}{2}; & \\ & \frac{-4xt}{(1-t)^2}\\ 1+\alpha,\frac{1}{2}; & \end{array}\right]\Lambda_{\mu,\psi}(y_1,...,y_r;\eta t^p),\end{aligned}$$

provided that each member of (9) *exists.*

Proof: For convenience, let S denote the first member of the assertion (9). Then,

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k f_{n-pk}^{(\alpha,\beta)}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k t^n.$$

Replacing n by $n+pk$, we may write that

$$\begin{aligned}
S &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k\, f_n^{(\alpha,\beta)}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k t^{n+pk}\\
&= \sum_{n=0}^{\infty}f_n^{(\alpha,\beta)}(x)t^n\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)(\eta t^p)^k\\
&= (1-t)^{-1-\alpha-\beta}\;{}_2F_2\left[\begin{array}{cc}\frac{1+\alpha+\beta}{2},\frac{2+\alpha+\beta}{2}; & \\ & \frac{-4xt}{(1-t)^2}\\ 1+\alpha,\frac{1}{2}; & \end{array}\right]\Lambda_{\mu,\psi}(y_1,...,y_r;\eta t^p),
\end{aligned}$$

which completes the proof. □

If we set

$$\Omega_{\mu+\psi k}(y_1,...,y_r)=\Phi_{\mu+\psi k}^{(\alpha)}(y_1,...,y_r)$$

in Theorem 2 , where the multivariable polynomials $\Phi_{\mu+\psi k}^{(\alpha)}(x_1,...,x_r)$, generated by [9]

$$\begin{gathered}\sum_{n=0}^{\infty}\Phi_n^{(\alpha)}(x_1,...,x_r)\,t^n=(1-x_1t)^{-\alpha}e^{(x_2+...+x_r)\,t},\\ \left(\alpha\in\mathbb{C}\,;\;\;|t|<\left\{|x_1|^{-1}\right\}\right).\end{gathered}\tag{10}$$

Thus, we have the following result which provides a class of bilateral generating functions for the multivariable polynomials $\Phi_{\mu+\psi k}^{(\alpha)}(x_1,...,x_r)$ and the generalized Sister Celine's polynomials as follows:

Corollary 2. *If*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;w):=\sum_{k=0}^{\infty}a_k\Phi_{\mu+\psi k}^{(\alpha)}(y_1,...,y_r)w^k\;\;(a_k\neq 0\,,\;\;\mu,\psi\in\mathbb{C}),$$

then, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k f_n^{(\alpha,\beta)}(x)\Phi_{\mu+\psi k}^{(\alpha)}(y_1,...,y_r)w^k t^n\\
&= (1-t)^{-1-\alpha-\beta}\;{}_2F_2\left[\begin{array}{cc}\frac{1+\alpha+\beta}{2},\frac{2+\alpha+\beta}{2}; & \\ & \frac{-4xt}{(1-t)^2}\\ 1+\alpha,\frac{1}{2}; & \end{array}\right]\Lambda_{\mu,\psi}(y_1,...,y_r;w),
\end{aligned}\tag{11}$$

provided that each member of (11) exists.

Remark 1. *Using the generating relation (10) for the multivariable polynomials $\Phi_n^{(\alpha)}(x_1,...,x_r)$ and getting $a_k=1,\ \mu=0,\ \psi=1$ in Corollary 2, we find that*

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}f_n^{(\alpha,\beta)}(x)\Phi_k^{(\alpha)}(x_1,...,x_r)w^k t^n\\
&= (1-t)^{-1-\alpha-\beta}\;{}_2F_2\left[\begin{array}{cc}\frac{1+\alpha+\beta}{2},\frac{2+\alpha+\beta}{2}; & \\ & \frac{-4xt}{(1-t)^2}\\ 1+\alpha,\frac{1}{2}; & \end{array}\right](1-x_1w)^{-\alpha}e^{(x_2+...+x_r)w},\\
&\left(\alpha_j\in\mathbb{C},\;\;|w|<\left\{|x_1|^{-1}\right\},\;|t|<1\right).
\end{aligned}$$

Theorem 3. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,...,y_r)$ of r complex variables $y_1,...,y_r$ $(r\in\mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\zeta^k\ (a_k\neq 0,\ \mu,\psi\in\mathbb{C}),$$

and

$$\Theta_{n,p}^{\mu,\psi}(x;y_1,...,y_r;\xi):=\sum_{k=0}^{[n/p]}a_k(C)_{n-pk}f_{n-pk}^{(\alpha,\beta)}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\xi^k.$$

Then, for $p\in\mathbb{N}$, we have

$$\sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\psi}(x;y_1,...,y_r;\eta)\,t^n \tag{12}$$
$$=(1-t)^{-C-\alpha-\beta}\ {}_3F_3\left[\begin{array}{c}C,\frac{1+\alpha+\beta}{2},\frac{2+\alpha+\beta}{2};\\ 1+\alpha,1+\alpha+\beta,\frac{1}{2};\end{array}\ \frac{-4xt}{(1-t)^2}\right]\Lambda_{\mu,\psi}(y_1,...,y_r;\eta t^p),$$

provided that each member of (12) exists.

Proof: For convenience, let K denote the first member of the assertion (12). Then,

$$K=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k(C)_{n-pk}f_{n-pk}^{(\alpha,\beta)}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k t^n.$$

Replacing n by $n+pk$, we may write that

$$\begin{aligned}K&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k\,(C)_nf_n^{(\alpha,\beta)}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k t^{n+pk}\\&=\sum_{n=0}^{\infty}(C)_nf_n^{(\alpha,\beta)}(x)t^n\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)(\eta t^p)^k\\&=(1-t)^{-C-\alpha-\beta}\ {}_3F_3\left[\begin{array}{c}C,\frac{1+\alpha+\beta}{2},\frac{2+\alpha+\beta}{2};\\ 1+\alpha,1+\alpha+\beta,\frac{1}{2};\end{array}\ \frac{-4xt}{(1-t)^2}\right]\Lambda_{\mu,\psi}(y_1,...,y_r;\eta t^p),\end{aligned}$$

which completes the proof. □

Furthermore, for every suitable choice of the coefficients a_k $(k\in\mathbb{N}_0)$, if the multivariable functions $\Omega_{\mu+\psi k}(y_1,...,y_r)$, $r\in\mathbb{N}$, are expressed as an appropriate product of several simpler functions, the assertions of Theorem 1, Theorem 2 and Theorem 3 can be applied in order to derive various families of multilinear and multilateral generating functions for the family of the Sister Celine's polynomials and the generalized Sister Celine's polynomials given explicitly by (1) and (5).

3 Conclusion

In this paper, we establish some properties for the generalized Sister Celine's polynomials. Various families of multilinear and multilateral generating functions and their miscellaneous properties are obtained. With the method used here, it is possible to obtain bilinear and bilateral generating functions for other polynomials.

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