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A note on Shively's Pseudo-Laguerre Polynomials

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Abstract: In this research, we establish some properties for the Shively's Pseudo-Laguerre polynomials. We derive various families of multilinear and multilateral generating functions for a family of Shively's Pseudo-Laguerre polynomials.

Keywords: Multilinear and multilateral generating functions, Recurrence relations, Shively's Pseudo-Laguerre polynomials.

1 Introduction

Shivley (see, for example, [1]; see also [[2], p. 298, Eq. 152 (1)];[[3], p. 127, Eq. (47)] and [[4], p. 1758, Eq. (3)]) has defined the polynomial $R_n(a,x)$ by

$$R_n(a,x) := \frac{(a+n)_n}{n!}\ {}_1F_1(-n;a+n;x) \tag{1}$$

in which n is any non-negative integer, and a is independent of n.

The pseudo-Laguerre polynomial $R_n(a,x)$ may also be written as

$$R_n(a,x) = \frac{(a)_{2n}}{n!(a)_n}\ {}_1F_1(-n;a+n;x)$$

which are related to the proper simple Laguerre polynomial

$$L_n(x) = {}_1F_1(-n;1;x)$$

by

$$R_n(a,x) = \frac{1}{(a-1)_n}\sum_{k=0}^{n}\frac{(a-1)_{n-k}}{k!}L_{n-k}(x).$$

Toscano [5] had already shown that

$$\sum_{n=0}^{\infty} R_n(a,x)t^n = (1-4t)^{-1/2}\left(\frac{2}{1+\sqrt{1-4t}}\right)^{a-1}\exp\left(\frac{-4xt}{\left(1+\sqrt{1-4t}\right)^2}\right). \tag{2}$$

Shively obtained Toscano's other generating relation

$$\sum_{n=0}^{\infty}\frac{R_n(a,x)}{\left(\frac{1}{2}+\frac{1}{2}a\right)_n}t^n = e^{2t}\ {}_0F_1(-;\frac{1}{2}+\frac{1}{2}a;t^2-xt),$$

and extended Toscano's (2) to

$$\sum_{n=0}^{\infty} S_n(x)t^n = (1-4t)^{-1/2}\left(\frac{2}{1+\sqrt{1-4t}}\right)^{a-1}{}_pF_q\left[\begin{array}{cc}\alpha_1,\ldots,\alpha_p; & \\ & \frac{-4xt}{\left(1+\sqrt{1-4t}\right)^2}\\ \beta_1,\ldots,\beta_q; & \end{array}\right],$$

in which

$$S_n(x) = \frac{(a)_{2n}}{n!(a)_n}\ {}_{p+1}F_{q+1}\left[\begin{array}{cc}-n,\alpha_1,\ldots,\alpha_p; & \\ & x\\ a+n,\beta_1,\ldots,\beta_q; & \end{array}\right].$$

For the particular choice $p = 0,\ \ q = 1,\ \ b_1 = 1,\ \ a = 1$ the $S_n(x)$ becomes

$$\sigma_n(x) = \frac{(2n)!}{(n!)^2}\, {}_1F_2(-n; 1+n, 1; x)$$

for which Shively has the additional generating relation [6]

$$\sum_{n=0}^{\infty} \frac{\sigma_n(x)}{(2n)!} t^n = \ {}_0F_1\left(-; 1; \frac{t - \sqrt{4xt + t^2}}{2}\right)\ {}_0F_1\left(-; 1; \frac{t + \sqrt{4xt - t^2}}{2}\right).$$

The $R_n(a, x)$ of (1) is of Sheffer A-type zero, as pointedout by Shively. He obtains many other properties of $R_n(a, x)$. Here ${}_pF_q$ denotes, as usual, a generalized hypergeometric function with p numerator and q denominator parameters and as usual, $(\lambda)_\nu$ denotes the Pochhammer symbol or the *shifted factorial*, since

$$(1)_n = n!\ \ (n \in \mathbb{N}_0),$$

which is defined (for λ, $\nu \in \mathbb{C}$ and in terms of Gamma function) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1, & (\nu = 0;\ \lambda \in \mathbb{C}\backslash\{0\}) \\ \lambda(\lambda+1)...(\lambda+n-1), & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases}$$

it getting understood conventionally that $(0)_0 := 1$.

The main object of this paper is to study several properties of the pseudo-Laguerre polynomial $R_n(a, x)$. Various families of multilinear and multilateral generating functions, miscellaneous properties and also some special cases for these polynomials are given.

2 Generating functions

In this section, we derive several families of bilinear and bilateral generating functions for the pseudo-Laguerre polynomial $R_n(a, x)$ generated by using the similar method considered in (see, [7] - [12]).

We begin by stating the following theorem.

Theorem 1. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1, ..., y_r)$ of r complex variables $y_1, ..., y_r$ $(r \in \mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\psi}(y_1, ..., y_r; \zeta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, ..., y_r)\zeta^k\ \ (a_k \neq 0,\ \mu, \psi \in \mathbb{C}),$$

and

$$\Theta_{n,p}^{\mu,\psi}(a, x; y_1, ..., y_r; \xi) := \sum_{k=0}^{[n/p]} a_k R_{n-pk}(a, x)\Omega_{\mu+\psi k}(y_1, ..., y_r)\xi^k.$$

Then, for $p \in \mathbb{N}$, we have

$$\sum_{n=0}^{\infty} \Theta_{n,p}^{\mu,\psi}(a, x; y_1, ..., y_r; \eta)\, t^n = (1-4t)^{-1/2}\left(\frac{2}{1+\sqrt{1-4t}}\right)^{a-1} \exp\left(\frac{-4xt}{\left(1+\sqrt{1-4t}\right)^2}\right)\ \Lambda_{\mu,\psi}(y_1, ..., y_r; \eta) \tag{3}$$

provided that each member of (3) *exists.*

Proof: For convenience, let S denote the first member of the assertion (3). Then,

$$S = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k R_{n-pk}(a, x)\Omega_{\mu+\psi k}(y_1, ..., y_r)\eta^k t^n.$$

Replacing n by $n + pk$, we may write that

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k\, R_n(a, x) y_n(x, \alpha - n, \beta)\Omega_{\mu+\psi k}(y_1, ..., y_r)\eta^k t^n \\ &= \sum_{n=0}^{\infty} R_n(a, x)t^n \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, ..., y_r)\eta^k \\ &= (1-4t)^{-1/2}\left(\frac{2}{1+\sqrt{1-4t}}\right)^{a-1} \exp\left(\frac{-4xt}{\left(1+\sqrt{1-4t}\right)^2}\right)\ \Lambda_{\mu,\psi}(y_1, ..., y_r; \eta), \end{aligned}$$

which completes the proof. □

If we set

$$\Omega_{\mu+\psi k}(y_1,...,y_r) = \Phi^{(\alpha)}_{\mu+\psi k}(y_1,...,y_r)$$

in Theorem 1 , where the multivariable polynomials $\Phi^{(\alpha)}_{\mu+\psi k}(x_1,...,x_r)$, generated by [10]

$$\begin{gathered}\sum_{n=0}^{\infty} \Phi^{(\alpha)}_n (x_1,...,x_r)\, t^n = (1-x_1 t)^{-\alpha} e^{(x_2+...+x_r)\, t}, \\ \left(\alpha \in \mathbb{C}\,;\ |t| < \left\{|x_1|^{-1}\right\}\right).\end{gathered} \tag{4}$$

Thus, we have the following result which provides a class of bilateral generating functions for the multivariable polynomials $\Phi^{(\alpha)}_{\mu+\psi k}(x_1,...,x_r)$ and the pseudo-Laguerre polynomial $R_n(a,x)$ as follows:

Corollary 1. *If*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;w) := \sum_{k=0}^{\infty} a_k \Phi^{(\alpha)}_{\mu+\psi k}(y_1,...,y_r) w^k \ \ (a_k \neq 0\, \mu,\psi \in \mathbb{C}),$$

then, we have

$$\begin{aligned} &\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k R_{n-pk}\,(a,x)\, \Phi^{(\alpha)}_{\mu+\psi k}(y_1,...,y_r) w^k t^n \\ = \ & (1-4t)^{-1/2}\left(\frac{2}{1+\sqrt{1-4t}}\right)^{a-1} \exp\left(\frac{-4xt}{\left(1+\sqrt{1-4t}\right)^2}\right) \Lambda_{\mu,\psi}(y_1,...,y_r;\eta), \end{aligned} \tag{5}$$

provided that each member of (5) *exists.*

Remark 1. *Using the generating relation* (4) *for the multivariable polynomials* $\Phi^{(\alpha)}_n(x_1,...,x_r)$ *and getting* $a_k = 1,\ \mu = 0,\ \psi = 1$ *in Corollary 1 , we find that*

$$\begin{aligned} &\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} R_{n-pk}\,(a,x)\, \Phi^{(\alpha)}_{k}(x_1,...,x_r) w^k t^n \\ = \ & (1-4t)^{-1/2}\left(\frac{2}{1+\sqrt{1-4t}}\right)^{a-1} \exp\left(\frac{-4xt}{\left(1+\sqrt{1-4t}\right)^2}\right) \\ & \times (1-x_1 w)^{-\alpha} e^{(x_2+...+x_r)w}, \\ & \left(\alpha_j \in \mathbb{C},\ \ |w| < \left\{|x_1|^{-1}\right\}, |t| < \frac{1}{4}\right). \end{aligned}$$

3 Conclusion

In this paper, we esteblish some properties for the Shively's Pseudo-Laguerre polynomials. Various families of multilinear and multilateral generating functions and their miscellaneous properties are obtained. With the method used here, it is possible to obtain bilinear and bilateral generating functions for other polynomials.

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