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On the Bitsadze-Samarskii Type Nonlocal Boundary Value Problem with the Integral Condition for an Elliptic Equation

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Abstract: In the present paper, the Bitsadze-Samarskii type nonlocal boundary value problem with the integral condition for an abstract elliptic differential equation in a Hilbert space is studied. Theorem on well-posedness of this problem in Hölder spaces with a weight is established. The nonlocal boundary value problem for multidimensional elliptic equations with the Dirichlet condition is studied. The first order of accuracy difference scheme for the approximate solution of the Bitsadze-Samarskii type nonlocal boundary value problem is investigated. Theorem on well-posedness of this difference scheme in difference analogue of Hölder spaces with a weight is established.

Keywords: Bitsadze-Samarskii type nonlocal boundary value problem, Difference scheme, Elliptic equation, Well-posedness.

1 Introduction

The simply nonlocal boundary value problem was presented and investigated for the first time by A.V. Bitsadze and A.A. Samarskii in the paper [1] . Further in papers [2–13], the Bitsadze-Samarskii type nonlocal boundary value problem and its generalizations for various differential and difference equations of elliptic equations were investigated by many scientists. Coercivity inequalities in Hölder norms with a weight for the solutions of an abstract differential equation of elliptic type were established for the first Sobolevskii in the paper [12]. Further, in papers [14–25] coercive inequalities in Hölder norms with a weight were obtained for the solutions of various local and nonlocal boundary-value problems for differential and difference equations of elliptic type. In the present paper, we consider the Bitsadze-Samarskii type nonlocal boundary value problem with the integral condition

$$\begin{cases} -\frac{d^2u(t)}{dt^2}+Au(t)=f(t),\ 0<t<1, \\ u(0)=\varphi,\ u(1)=\int\limits_0^1 \rho(\lambda)u(\lambda)d\lambda+\psi \end{cases} \tag{1}$$

for the differential equation of elliptic type in a Hilbert space H with the self-adjoint positive definite operator A with a closed domain $D(A)\subset H$. Here, let $f(t)$ be a given abstract continuous function defined on $[0,1]$ with values in H, φ, and ψ are elements of $D(A)$ and $\rho(t)$ is a scalar continous function. A function $u(t)$ is called a solution of problem (1) if the following conditions are satisfied:

i. $u(t)$ is twice continuously differentiable on the segment $[0,1]$.

ii. The element $u(t)$ belongs to $D(A)$ for all $t\in[0,1]$, and the function $Au(t)$ is continuous on the segment $[0,1]$.

iii. $u(t)$ satisfies the equation and nonlocal boundary conditions (1).

A solution of problem (1) defined in this manner will from now on be referred to as a solution of problem (1) in the space $C([0,1],H)$. Here, $C([0,1],\ H)$ stands for the Banach space of all continuous functions $\varphi(t)$ defined on $[0,1]$ with values in H with the norm

$$\|\varphi\|_{C([0,1],H)}=\max_{0\le t\le 1}\|\varphi(t)\|_H .$$

We say that the problem (1) is well-posed in $C([0,1],H)$, if there exists the unique solution $u(t)$ in $C([0,1],H)$ of problem (1) for any $f(t)\in C([0,1],H)$ and the following coercivity inequality is satisfied:

$$\left\|u''\right\|_{C([0,1],H)}+\|Au\|_{C([0,1],H)}\le M_c\left[\|f\|_{C([0,1],H)}+\|A\varphi\|_H+\|A\psi\|_H\right],$$

where M_c does not depend on $f(t)$ and φ, ψ. Unfortunately, the problem (1) is ill-posed in the space $C([0,1],H)$.

In this paper, positive constants, which can differ in time (hence: not a subject of precision), will be indicated with M. On the other hand $M(\alpha;\beta;...)$ is used to focus on the fact that the constant depends only on α;β;....

Let us denote by $C_{01}^{\alpha}\left([0,1],H\right), 0<\alpha<1$, the Banach spaces obtained by completion of the set of all smooth $H-$values functions $\varphi(t)$ on $[0,1]$ in the norms

$$\|\varphi\|_{C_{01}^{\alpha}([0,1],H)}=\|\varphi\|_{C([0,1],H)}+\sup_{0\leq t<t+\tau\leq 1}\frac{(1-t)^{\alpha}\left(t+\tau\right)^{\alpha}\left\|\varphi\left(t+\tau\right)-\varphi\left(t\right)\right\|_{H}}{\tau^{\alpha}}.$$

We say that the problem (1) is well-posed in $C_{01}^{\alpha}\left([0,1],H\right)$, if there exists a unique solution $u(t)$ in $C_{01}^{\alpha}\left([0,1],H\right)$ of problem (1) for any $f(t)\in C_{01}^{\alpha}\left([0,1],H\right)$ and the following coercivity inequality is satisfied:

$$\left\|u''\right\|_{C_{01}^{\alpha}([0,1],H)}+\|Au\|_{C_{01}^{\alpha}([0,1],H)}\leq M(\delta,\alpha)\left[\|A\varphi\|_{H}+\|A\psi\|_{H}+\|\ f\ \|_{C_{01}^{\alpha}([0,1],H)}\right].$$

We will study the problem (1) under the assumption:

$$\int_{0}^{1}|\rho(\lambda)|\,d\lambda<1. \tag{2}$$

In the present paper, the well-posedness of the nonlocal boundary value problem (1) in $C_{01}^{\alpha}\left([0,1],H\right)$ spaces is established. The first order of accuracy difference scheme for the approximate solution of this problem (1) is presented. The coercive inequalities for the solution of this difference scheme in difference analogue of $C_{01}^{\alpha}\left([0,1],H\right)$ spaces are established. In applications, difference scheme for approximate nonlocal boundary value problem for elliptic equation is investigated.

2 The Bitsadze-Samarskii type nonlocal boundary value problem

In this section, let $B=A^{\frac{1}{2}}$. Then, it is clear that B is a self-adjoint positive definite operator and $B\geq\delta I$. The following lemmas will be needed below.

Lemma 1. *[8] The following estimates hold:*

$$\left\|B^{\alpha}\exp(-tB)\right\|_{H\longrightarrow H}\leq t^{-\alpha},\ 0\leq\alpha\leq 1, \tag{3}$$

$$\left\|(I-e^{-2B})^{-1}\right\|_{H\longrightarrow H}\leq M. \tag{4}$$

Lemma 2. *[17] For any $0\leq t<t+\tau\leq 1$ and $0\leq\alpha\leq 1$ one has the inequality*

$$\|\exp(-tB)-\exp\left(-(t+\tau)B\right)\|_{H\to H}\leq M\frac{\tau^{\alpha}}{(\tau+t)^{\alpha}}. \tag{5}$$

Lemma 3. *Let*

$$D=\int_{0}^{1}\rho(\lambda)(I-e^{-2B})^{-1}(e^{-(1-\lambda)B}-e^{-(1+\lambda)B})d\lambda.$$

Then, under the assumption (1), the operator $I-D$ has an inverse

$$P=\left(I-D\right)^{-1}$$

and the following estimate is satisfied:

$$\|P\|_{H\to H}\leq M\left(\delta\right). \tag{6}$$

It is clear that (see [17]) the boundary value problem for elliptic equation

$$-\frac{d^{2}u(t)}{dt^{2}}+Au(t)=f(t),\ 0<t<1,u(0)=u_{0},u(1)=u_{1} \tag{7}$$

has a unique solution

$$u(t)=(I-e^{-2B})^{-1}\left\{(e^{-tB}-e^{-(2-t)B})\varphi+(e^{-(1-t)B}-e^{-(1+t)B})u(1)-(e^{-(1-t)B}-e^{-(1+t)B})\right. \tag{8}$$

$$\times(2B)^{-1}\int_{0}^{1}(e^{-(1-s)B}-e^{-(1+s)B})f(s)ds\}+(2B)^{-1}\int_{0}^{1}(e^{-|t-s|B}-e^{-(t+s)B})f(s)ds,$$

$$u(1)=P\left[\psi+\int_{0}^{1}\rho(\lambda)(I-e^{-2B})^{-1}\left\{\left(e^{-\lambda B}-e^{-(2-\lambda)B}\right)\varphi\right.\right. \tag{9}$$

$$-(e^{-(1-\lambda)B}-e^{-(1+\lambda)B})\,(2B)^{-1}\int_0^1(e^{-(1-s)B}-e^{-(1+s)B})f(s)ds\Bigg\}\,d\lambda$$

$$+\int_0^1\rho(\lambda)\,(2B)^{-1}\left(\int_0^\lambda e^{-(\lambda-s)B}f(s)ds+\int_\lambda^1 e^{-(s-\lambda)B}f(s)ds-\int_0^1 e^{-(\lambda+s)B}f(s)ds\right)d\lambda\Bigg],$$

where

$$P=\left(I-\int_0^1\rho(\lambda)\left(I-e^{-2B}\right)^{-1}\left(e^{-(1-\lambda)B}-e^{-(1+\lambda)B}\right)d\lambda\right)^{-1}.$$

Theorem 1. *Suppose $\varphi,\psi\in D(A)$, $f(t)\in C_{01}^{\alpha}([0,1],H)$ $(0<\alpha<1)$. Then, for the solution $u(t)$ of the boundary value problem (1) the coercivity inequality*

$$\|u''\|_{C_{01}^{\alpha}([0,1],H)}+\|Au\|_{C_{01}^{\alpha}([0,1],H)}\le M(\delta)\left[\|A\varphi\|_H+\|A\psi\|_H+\frac{1}{\alpha(1-\alpha)}\parallel f\parallel_{C_{01}^{\alpha}([0,1],H)}\right]$$

holds.

Proof: By [17], we had the following coercivity inequality

$$\|u''\|_{C_{01}^{\alpha}([0,1],H)}+\|Au\|_{C_{01}^{\alpha}([0,1],H)}\le\frac{M(\delta)}{\alpha(1-\alpha)}\|f\|_{C_{01}^{\alpha}([0,1],H)}+M(\delta)\left\{\|Au(0)\|_H+\|Au(1)\|_H\right\}\tag{10}$$

for the solution of boundary value problem (7). Then the proof of Theorem 1 is based on coercivity inequality (10) and on the following estimate

$$\|Au(1)\|_H\le\frac{M(\delta)}{\alpha(1-\alpha)}\|f\|_{C_{01}^{\alpha}([0,1],H)}+M(\delta)\left\{\|A\varphi\|_H+\|A\psi\|_H\right\}.\tag{11}$$

Therefore, we will prove (11). First, applying formula (9), we can write

$$Au(1)=P\left(\int_0^1\rho(\lambda)(I-e^{-2B})^{-1}\left\{\left(e^{-\lambda B}-e^{-(2-\lambda)B}\right)A\varphi+\left(I-e^{-\lambda B}\right)\left(I-e^{-(1-\lambda)B}\right)\right.\right.$$

$$\times\left(I-e^{-B}\right)f(\lambda)+\frac{B}{2}\left(I-e^{-2(1-\lambda)B}\right)\int_0^\lambda\left(e^{-(\lambda-s)B}\left(I-e^{-2sB}\right)\right)(f(s)-f(\lambda))ds$$

$$\left.\left.+\frac{B}{2}\left(I-e^{-2\lambda B}\right)\int_\lambda^1\left(e^{-(s-\lambda)B}\left(I-e^{-2(1-s)B}\right)\right)(f(s)-f(\lambda))ds\right\}d\lambda+A\psi\right)$$

$$=J_1+J_2+J_3+J_4,$$

where

$$J_1=P\left(\int_0^1\rho(\lambda)(I-e^{-2B})^{-1}(e^{-\lambda B}-e^{-(2-\lambda)B})A\varphi d\lambda+A\psi\right),$$

$$J_2=P\left(\int_0^1\rho(\lambda)(I-e^{-2B})^{-1}\left(I-e^{-\lambda B}\right)\left(I-e^{-(1-\lambda)B}\right)\left(I-e^{-B}\right)f(\lambda)d\lambda\right),$$

$$J_3=\frac{1}{2}P\int_0^1\rho(\lambda)(I-e^{-2B})^{-1}\int_0^\lambda\left(Be^{-(\lambda-s)B}\left(I-e^{-2(1-\lambda)B}\right)\left(I-e^{-2sB}\right)\right)(f(s)-f(\lambda))dsd\lambda,$$

$$J_4=\frac{1}{2}P\int_0^1\rho(\lambda)B\left(I-e^{-2B}\right)^{-1}\int_\lambda^1\left(e^{-(s-\lambda)B}\left(I-e^{-2(1-s)B}\right)\left(I-e^{-2\lambda B}\right)\right)(f(s)-f(\lambda))dsd\lambda.$$

Let us estimate J_k for $k=1,\cdots,4$, separately. First, we estimate J_1. Using estimates (4), (5) and (6), we obtain

$$\|J_1\|_H\le\|P\|_{H\to H}\left(\int_0^1|\rho(\lambda)|\left\|\left(I-e^{-2B}\right)^{-1}\right\|_{H\to H}\left\|e^{-\lambda B}-e^{-(2-\lambda)B}\right\|_{H\to H}\|A\varphi\|_H\,d\lambda+\|A\psi\|_H\right)$$

$$\leq M(\delta)\left[\int_0^1 |\rho(\lambda)|\, d\lambda \, \|A\varphi\|_H + \|A\psi\|_H\right].$$

Thus, from condition (2) it follows that

$$\|J_1\|_H \leq M_1(\delta)\left[\|A\varphi\|_H + \|A\psi\|_H\right].$$

Let us estimate J_2.

$$\|J_2\|_H \leq \|P\|_{H\to H}\int_0^1 |\rho(\lambda)| \left\|\left(I-e^{-2B}\right)^{-1}\right\|_{H\to H}\left\|I-e^{-\lambda B}\right\|_{H\to H}$$

$$\times\left\|I-e^{-(1-\lambda)B}\right\|_{H\to H}\left\|I-e^{-B}\right\|_{H\to H}\|f(\lambda)\|_H\, d\lambda.$$

Further, using estimates (3), (5), (6) and the definition of the norm of the space $C_{01}^{\alpha}([0,1],H)$, we get

$$\|J_2\|_H \leq M_2(\delta)\int_0^1 |\rho(\lambda)|\, d\lambda \max_{0\leq t\leq 1}\|f(t)\|_H.$$

Thus, from (2) it follows that

$$\|J_2\|_H \leq M_2(\delta)\|f(t)\|_{C([0,1],H)} \leq M_2(\delta)\|f\|_{C_{01}^{\alpha}([0,1],H)}.$$

To estimate J_3, we will put $J_3 = J_{3,1} + J_{3,2}$, where

$$\begin{aligned} J_{3,1} &= P\int_0^{\frac{1}{2}}\rho(\lambda)(I-e^{-2B})^{-1}\frac{1}{2}\int_0^{\lambda}\left(Be^{-(\lambda-s)B}\left(I-e^{-2(1-\lambda)B}\right)\left(I-e^{-2sB}\right)\right)(f(s)-f(\lambda))ds d\lambda, \\ J_{3,2} &= P\int_{\frac{1}{2}}^{1}\rho(\lambda)(I-e^{-2B})^{-1}\frac{1}{2}\int_0^{\lambda}\left(Be^{-(\lambda-s)B}\left(I-e^{-2(1-\lambda)B}\right)\left(I-e^{-2sB}\right)\right)(f(s)-f(\lambda))ds d\lambda. \end{aligned}$$

First, we will estimate $J_{3,1}$. Applying estimates (3), (5), (6) and the definition of the norm of the space $C_{01}^{\alpha}([0,1],H)$, we obtain

$$\|J_{3,1}\|_H \leq M(\delta)\int_0^{\frac{1}{2}}|\rho(\lambda)|\int_0^{\lambda}\frac{ds}{(\lambda-s)^{1-\alpha}\lambda^{\alpha}(1-s)^{\alpha}}d\lambda\,\|f\|_{C_{01}^{\alpha}([0,1],H)} \leq \frac{M(\delta)}{\alpha(1-\alpha)}\int_0^{\frac{1}{2}}|\rho(\lambda)|\, d\lambda\,\|f\|_{C_{01}^{\alpha}([0,1],H)}.$$

Second, we will estimate $J_{3,2}$. For $J_{3,2}$, using estimates (3), (5), (6) and the definition of the norm of the space $C_{01}^{\alpha}([0,1],H)$, we get

$$\|J_{3,2}\|_H \leq M(\delta)\int_{\frac{1}{2}}^{1}|\rho(\lambda)|\int_0^{\lambda}\frac{(2(1-\lambda))^{\alpha}\, ds}{(\lambda-s)(2-\lambda-s)^{\alpha}\lambda^{\alpha}(1-s)^{\alpha}}d\lambda\,\|f\|_{C_{01}^{\alpha}([0,1],H)}$$

$$\leq \frac{M(\delta)2^{\alpha-1}}{\alpha}\int_{\frac{1}{2}}^{1}\frac{|\rho(\lambda)|}{(1-\lambda)^{\alpha}}d\lambda\,\|f\|_{C_{01}^{\alpha}([0,1],H)} \leq \frac{M(\delta)2^{2\alpha-2}}{\alpha(1-\alpha)}\int_{\frac{1}{2}}^{1}|\rho(\lambda)|\, d\lambda\,\|f\|_{C_{01}^{\alpha}([0,1],H)}.$$

Applying estimates for $\|J_{3,1}\|_H$ and $\|J_{3,2}\|_H$, we get

$$\|J_3\|_H \leq \frac{M_3(\delta)}{\alpha(1-\alpha)}\int_0^1|\rho(\lambda)|\, d\lambda\,\|f\|_{C_{01}^{\alpha}([0,1],H)}.$$

Using condition (2), we get

$$\|J_3\|_H \leq \frac{M_4(\delta)}{\alpha(1-\alpha)}\|f\|_{C_{01}^{\alpha}([0,1],H)}.$$

Let us estimate J_4. We will put $J_4 = J_{4,1} + J_{4,2}$, where

$$\begin{aligned} J_{4,1} &= P\int_0^{\frac{1}{2}}\rho(\lambda)\frac{1}{2}\left(I-e^{-2B}\right)^{-1}\int_{\lambda}^{1}\left(Be^{-(s-\lambda)B}\left(I-e^{-2(1-s)B}\right)\left(I-e^{-2\lambda B}\right)\right)(f(s)-f(\lambda))ds d\lambda, \\ J_{4,2} &= P\int_{\frac{1}{2}}^{1}\rho(\lambda)\frac{1}{2}\left(I-e^{-2B}\right)^{-1}\int_{\lambda}^{1}\left(Be^{-(s-\lambda)B}\left(I-e^{-2(1-s)B}\right)\left(I-e^{-2\lambda B}\right)\right)(f(s)-f(\lambda))ds d\lambda. \end{aligned}$$

The estimates (3), (5), (6) and the definition of the norm of the space $C_{01}^{\alpha}([0,1],H)$ give

$$\|J_{4,1}\|_H \leq M(\delta)\int_0^{\frac{1}{2}}|\rho(\lambda)|\int_\lambda^1 \frac{(1-s)^\alpha\,ds}{(1-\lambda)^\alpha s^\alpha (s-\lambda)^{1-\alpha}(2-s-\lambda)^\alpha}d\lambda\,\|f\|_{C_{01}^{\alpha}([0,1],H)}$$

$$\leq M(\delta)\int_0^{\frac{1}{2}}\frac{|\rho(\lambda)|}{\lambda^\alpha(1-\lambda)^\alpha}\int_\lambda^1\frac{ds}{(s-\lambda)^{1-\alpha}}d\lambda\,\|f\|_{C_{01}^{\alpha}([0,1],H)} \leq \frac{M_5(\delta)}{\alpha(1-\alpha)}\int_0^{\frac{1}{2}}|\rho(\lambda)|\,d\lambda\,\|f\|_{C_{01}^{\alpha}([0,1],H)}.$$

Finally, we estimate $J_{4,2}$. For $J_{4,2}$, applying estimates (3), (5), (6) and the definition of the norm of the space $C_{01}^{\alpha}([0,1],H)$, we obtain

$$\|J_{4,2}\|_H \leq M(\delta)\int_{\frac{1}{2}}^1\frac{|\rho(\lambda)|}{\lambda^\alpha\alpha}d\lambda\,\|f\|_{C_{01}^{\alpha}([0,1],H)} \leq \frac{M(\delta)2^{\alpha-2}}{\alpha(1-\alpha)}\int_{\frac{1}{2}}^1|\rho(\lambda)|\,d\lambda\,\|f\|_{C_{01}^{\alpha}([0,1],H)}.$$

Applying estimates for $\|J_{4,1}\|_H$ and $\|J_{4,2}\|_H$, we get

$$\|J_4\|_H \leq \frac{M_6(\delta)}{\alpha(1-\alpha)}\int_0^1|\rho(\lambda)|\,d\lambda\,\|f\|_{C_{01}^{\alpha}([0,1],H)}.$$

So, from (2) it follows that

$$\|J_4\|_H \leq \frac{M_7(\delta)}{\alpha(1-\alpha)}\|f\|_{C_{01}^{\alpha}([0,1],H)}.$$

Combining estimates for $\|J_k\|_H$, $k=1,\cdots,4$, we obtain estimate (11). Theorem 1 is proved. □

Now, we consider the application of Theorem 1.

Let Ω is the unit open cube in $\mathbb{R}^n$ $\{x=(x_1,\cdots,x_n):0<x_k<1, 1\leq k\leq n\}$ with boundary S, $\overline{\Omega}=\Omega\cup S$. In $[0,1]\times\Omega$, the Dirichlet-Bitsadze-Samarskii type mixed boundary value problem for the multidimensional elliptic equation

$$\begin{cases} -u_{tt}-\sum\limits_{r=1}^{n}(a_r(x)u_{x_r})_{x_r}=f(t,x), 0<t<1,\ x=(x_1,\ldots,x_n)\in\Omega, \\ u(0,x)=\varphi(x),\ u(1,x)=\int\limits_0^1\rho(\lambda)u(\lambda,x)d\lambda+\psi(x),\ x\in\overline{\Omega}, \\ u(t,x)\mid_{x\epsilon S}=0, x\in\overline{\Omega},\ 0\leq t\leq 1 \end{cases} \tag{12}$$

is considered. We will study the problem (12) under the assumption (2). The problem has an unique smooth solution $u(t,x)$ for the smooth $f(t,x)$ $(t\in(0,1), x\in\overline{\Omega})$, $\varphi(x)$ and $\psi(x)$ functions, and $a_r(x)\geq a>0$ $(x\in\Omega)$. We introduce the Hilbert space $L_2(\overline{\Omega})$ of all square-integrable functions f defined on $\overline{\Omega}$, equipped with the norm

$$\|f\|_{L_2(\overline{\Omega})}=\left\{\int\cdots\int_{x\in\overline{\Omega}}|f(x)|^2\,dx_1\cdots dx_n\right\}^{\frac{1}{2}}.$$

We can reduce the Dirichlet-Bitsadze-Samarskii type mixed boundary value problem (12) to the nonlocal boundary problem (1) in Hilbert space $H=L_2(\overline{\Omega})$ with a self -adjoint positive definite operator A defined by (12).

Theorem 2. *The solution of the nonlocal boundary value problem* (12) *satisfies the coercivity inequality*

$$\|u_{tt}\|_{C_{01}^{\alpha}([0,1],L_2(\overline{\Omega}))}+\|u\|_{C_{01}^{\alpha}([0,1],W_2^2(\overline{\Omega}))}$$

$$\leq\frac{M(\delta)}{\alpha(1-\alpha)}\|f\|_{C_{01}^{\alpha}([0,1],L_2(\overline{\Omega}))}+M(\delta)\left[\|\varphi\|_{W_2^2(\overline{\Omega})}+\|\psi\|_{W_2^2(\overline{\Omega})}\right].$$

Here, the Sobolev space $W_2^2(\overline{\Omega})$ is defined as the set of all functions f defined on $\bar{\Omega}$ such that f and all second order partial derivative functions f_{x_r,x_r}, $r=1,...n$ is both locally integrable in $L_2(\overline{\Omega})$, equipped with the norm

$$\|f\|_{W_2^2(\overline{\Omega})}=\|f\|_{L_2(\overline{\Omega})}+\left(\int\cdots\int_{x\in\overline{\Omega}}\sum_{r=1}^{n}|f_{x_r,x_r}|^2\,dx_1\cdots dx_n\right)^{1/2}.$$

The proof of Theorem 2 is based on Theorem 1, on the symmetry properties of the space operator A generated by the problem (12), and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 3. *For the solution of the elliptic differential problem*

$$\sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r} = \omega(x), x \in \Omega,$$

$$u(x) = 0,\ x \in S,$$

the following coercivity inequality holds [22]:

$$\|u\|_{W_2^2(\overline{\Omega})} \leq M\|\omega\|_{L_2(\overline{\Omega})}.$$

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3 The first order of accuracy difference scheme

The nonlocal boundary value problem (1) is associated with the corresponding first order of accuracy difference scheme

$$\begin{cases} -\frac{1}{\tau^2}[u_{k+1} - 2u_k + u_{k-1}] + Au_k = \varphi_k, \\ \varphi_k = f(t_k),\ t_k = k\tau,\ 1 \leq k \leq N-1,\ N\tau = 1, \\ u_0 = \varphi,\ u_N = \sum\limits_{j=1}^{N} \rho\left(t_j\right) u_j\tau + \psi. \end{cases} \tag{13}$$

A study of discreatization over time of the nonlocal boundary value problem also permits one to include general difference schemes in applications, if the differential operator in space variables, A is replaced by the difference operators A_h that act in the Hilbert spaces H_h and are uniformly self-adjoint positive definite in h for $0 \leq h \leq h_0$. It is known that for a self-adjoint positive definite operator A it follows that $B = \frac{1}{2}(\tau A + \sqrt{4A + \tau^2 A^2})$ is self-adjoint positive definite and $R = (I + \tau B)^{-1}$ which defined on the whole space H is a bounded operator. Here, I is the identity operator. We will study the problem (13) under the assumption:

$$\sum_{j=1}^{N} \left|\rho\left(t_j\right)\right| \tau < 1. \tag{14}$$

Now, let us give some lemmas and theorem that will be needed below.

Lemma 4. *The estimates hold [17]*

$$\begin{cases} \left\|\left(I - R^{2N}\right)^{-1}\right\|_{H\to H} \leq M(\delta), \\ \|R^k\|_{H\to H} \leq M(\delta)(1+\delta\tau)^{-k},\ k\tau\|BR^k\|_{H\to H} \leq M(\delta), k \geq 1, \delta > 0, \\ \|B^\beta\left(R^{k+r} - R^k\right)\|_{H\to H} \leq M(\delta)\frac{(r\tau)^\alpha}{(k\tau)^{\alpha+\beta}}, 1 \leq k < k+r \leq N, 0 \leq \alpha, \beta \leq 1. \end{cases} \tag{15}$$

Lemma 5. *Suppose A is the positive operator in Hilbert space H. Then, the following estimate holds [17]:*

$$\sum_{j=1}^{N-1} \left\|(I-R)R^{j-1}\right\|_{H\to H} \leq M \min\left(\ln(\frac{1}{\tau}), 1 + \tau\left|\ln\|B\|_{H\to H}\right|\right). \tag{16}$$

Lemma 3.3. *The operator*

$$I - \sum_{j=1}^{N} \rho\left(t_j\right)\tau\left(I - R^{2N}\right)^{-1}\left(R^{N-j} - R^{N+j}\right)$$

has an inverse

$$K_\tau = \left(I - \sum_{j=1}^{N} \rho\left(t_j\right)\tau\left(I - R^{2N}\right)^{-1}\left(R^{N-j} - R^{N+j}\right)\right)^{-1}$$

and the following estimate is satisfied under the assumption (14)

$$\|K_\tau\|_{H\longrightarrow H} \leq M(\delta)\tau. \tag{17}$$

Theorem 4. *For any φ_k, $1 \le k \le N-1$, the solution of the problem (13) exists and the following formula holds for $k = 1, \cdots, N-1$,*

$$u_k = (I - R^{2N})^{-1} \left\{ \left(R^k - R^{2N-k}\right) \varphi + \left(R^{N-k} - R^{N+k}\right) u_N \right.$$

$$\left. - \left(R^{N-k} - R^{N+k}\right) (I + \tau B)(2I + \tau B)^{-1} B^{-1} \sum_{i=1}^{N-1} \left(R^{N-1-i} - R^{N-1+i}\right) \varphi_i \tau \right\} \tag{18}$$

$$+ (I + \tau B)(2I + \tau B)^{-1} B^{-1} \sum_{i=1}^{N-1} \left(R^{|k-i|-1} - R^{k+i-1}\right) \varphi_i \tau \, ,$$

$$u_N = K_\tau \left(\sum_{j=1}^{N} \rho\left(t_j\right) \tau \left(I - R^{2N}\right)^{-1} \left\{ \left(R^j - R^{2N-j}\right) \varphi - \left(R^{N-j} - R^{N+j}\right) \right. \right.$$

$$\left. \times (I + \tau B)(2I + \tau B)^{-1} \sum_{i=1}^{N-1} B^{-1} \left(R^{N-1-i} - R^{N-1+i}\right) \varphi_i \tau \right\} + (2I + \tau B)^{-1} B^{-1}$$

$$\left. \times (I + \tau B) \left(\sum_{i=1}^{j} R^{j-i-1} \varphi_i \tau + \sum_{i=j+1}^{N-1} R^{i-j-1} \varphi_i \tau - \sum_{i=1}^{N-1} R^{j+i-1} \varphi_i \tau \right) + \psi \right)$$

for $k = N$.
Let $F([0,1]_\tau, H)$ be the linear space of the mesh functions $\varphi^\tau = \{\varphi_k\}_1^{N-1}$ with values in the Hilbert space H. We denote by $C([0,1]_\tau, H)$ and $C_{01}^{\alpha}([0,1]_\tau, H)$, $0 < \alpha < 1$, Banach spaces with the norms

$$\|\varphi^\tau\|_{C([0,1]_\tau, H)} = \max_{1 \le k \le N-1} \|\varphi_k\|_H \, ,$$

$$\|\varphi^\tau\|_{C_{01}^{\alpha}([0,1]_\tau, H)} = \|\varphi^\tau\|_{C([0,1]_\tau, H)} + \sup_{1 \le k \le k+r \le N-1} \frac{((N-k)\tau)^\alpha ((k+r)\tau)^\alpha}{(r\tau)^\alpha} \|\varphi_{k+r} - \varphi_k\|_H \, .$$

Theorem 5. *The solution of the difference problem (13) in $C([0,1]_\tau, H)$ under the assumption (14) obeys the almost coercive inequality*

$$\| \{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1} \|_{C([0,1]_\tau, H)} + \| \{Au_k\}_1^N \|_{C([0,1]_\tau, H)} \tag{19}$$

$$\le M(\delta) \left[\min\left\{ \ln\frac{1}{\tau}, 1 + \left|\ln \| B \|_{H \to H}\right| \right\} \| \varphi^\tau \|_{C([0,1]_\tau, H)} + \|A\varphi\|_H + \|A\psi\|_H \right].$$

Proof: By [17],

$$\| \{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1} \|_{C([0,1]_\tau, H)} + \| \{Au_k\}_1^N \|_{C([0,1]_\tau, H)} \tag{20}$$

$$\le M(\delta) \left[\min\left\{ \ln\frac{1}{\tau}, 1 + \left|\ln \| B \|_{H \to H}\right| \right\} \| \varphi^\tau \|_{C([0,1]_\tau, H)} + \|A\varphi\|_H + \|Au_N\|_H \right]$$

was proved for the solution of the boundary value problem

$$\begin{cases} -\frac{1}{\tau^2}[u_{k+1} - 2u_k + u_{k-1}] + Au_k = \varphi_k, 1 \le k \le N-1, N\tau = 1, \\ u_0 = \varphi, \ u_N \text{ are given.} \end{cases} \tag{21}$$

Using the estimates (15), (17), and the formula (18), we obtain

$$\| Au_N \|_H \le M(\delta) \left(\min\left\{ \ln\frac{1}{\tau}, 1 + \left|\ln \| B \|_{H \to H}\right| \right\} \| \varphi^\tau \|_{C([0,1]_\tau, H)} + \|A\varphi\|_H + ||Au_N||_H \right) \tag{22}$$

for the solution of difference scheme (13). Applying formula (18) and $A = B^2 R$, we get

$$Au_N = J_1 + J_2,$$

where

$$J_1 = K_\tau \left(\sum_{j=1}^{N} \rho\left(t_j\right) \tau (I - R^{2N})^{-1} \left(R^j - R^{2N-j}\right) A\varphi + A\psi \right), \tag{23}$$

$$J_2 = K_\tau \sum_{j=1}^{N} \rho\left(t_j\right)\tau \left\{ (I-R^{2N})^{-1}\left(\left(-R^{N-j}+R^{N+j}\right)(I+\tau B)\right.\right. \tag{24}$$

$$\times (2I+\tau B)^{-1} B \sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\varphi_i\tau\Big) + (I+\tau B)(2I+\tau B)^{-1}B$$

$$\left.\times\left(\sum_{i=1}^{j-1} R^{j-i}\varphi_i\tau + \sum_{i=j}^{N-1} R^{i-j}\varphi_i\tau - \sum_{i=1}^{N-1} R^{j+i}\varphi_i\tau\right)\right\}.$$

To this end, it suffices to show that

$$\|J_1\|_H \le M(\delta)[\|A\varphi\|_H + \|A\psi\|_H] \tag{25}$$

and

$$\|J_2\|_H \le M(\delta)\min\left\{\ln\frac{1}{\tau}, 1+\left|\ln\| B\|_{H\to H}\right|\right\} \| \varphi^\tau \|_{C([0,1]_\tau, H)}. \tag{26}$$

The estimate (25) follows from formula (23) and estimates (15), (17). Using formula (24) and estimates (15), (16), and (17), we obtain

$$\|J_2\|_H \le \|K_\tau\|_{H\to H}\left(\sum_{j=1}^{N}|\rho(t_j)|\,\tau\left(\left\|(I-R^{2N})^{-1}\right\|_{H\to H}\left\{\left(\left\|R^{N-j}\right\|_{H\to H}+\left\|R^{N+j}\right\|_{H\to H}\right)\right.\right.\right.$$

$$\left.\times\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}\sum_{i=1}^{N-1}\left(\left\|(I-R)R^{N-i-1}\right\|_{H\to H}+\left\|(I-R)R^{N+i-1}\right\|_{H\to H}\right)\|\varphi_i\|_H\right\}$$

$$+\left\|(I+\tau B)(2I+\tau B)^{-1}\right\|_{H\to H}\left(\sum_{i=1}^{j}\left\|(I-R)R^{j-i-1}\right\|_{H\to H}\|\varphi_i\|_H\right.$$

$$\left.\left.\left.+\sum_{i=j+1}^{N-1}\left\|(I-R)R^{i-j-1}\right\|_{H\to H}\|\varphi_i\|_H+\sum_{i=1}^{N-1}\left\|(I-R)R^{j+i-1}\right\|_{H\to H}\|\varphi_i\|_H\right)\right)\right)$$

$$\le M(\delta)\min\left\{\ln\frac{1}{\tau}, 1+\left|\ln\| B\|_{H\to H}\right|\right\} \| \varphi^\tau \|_{C([0,1]_\tau, H)}.$$

So, from the last estimate and the estimate (16) it follows the estimate (26). Theorem 5 is proved. □

Theorem 6. *The difference problem* (13) *is well posed in the Hölder spaces* $C_{01}^{\alpha}([0,1]_\tau, H)$ *under the assumption* (14) *and the following coercivity inequality holds:*

$$\| \{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1} \|_{C_{01}^{\alpha}([0,1]_\tau,H)} + \| \{Au_k\}_1^{N} \|_{C_{01}^{\alpha}([0,1]_\tau,H)} \tag{27}$$

$$\le M(\delta)\left[\frac{1}{\alpha(1-\alpha)}\|\varphi^\tau\|_{C_{01}^{\alpha}([0,1]_\tau,H)} + ||A\varphi||_H + ||A\psi||_H\right].$$

Proof: By [17],

$$\| \{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1} \|_{C_{01}^{\alpha}([0,1]_\tau,H)} + \| \{Au_k\}_1^{N-1} \|_{C_{01}^{\alpha}([0,1]_\tau,H)} \tag{28}$$

$$\le M(\delta)\frac{1}{\alpha(1-\alpha)}\|\varphi^\tau\|_{C_{01}^{\alpha}([0,1]_\tau,H)} + M(\delta)[||A\varphi||_H + ||Au_N||_H]$$

was proved for the solution of difference scheme (21). Then the proof of (27) is based on (28) and on the estimate

$$\| Au_N \|_H \le M(\delta)\frac{1}{\alpha(1-\alpha)}\| \varphi^\tau \|_{C_{01}^{\alpha}([0,1]_\tau,H)} + M(\delta)[||A\varphi||_H + ||A\psi||_H].$$

Applying the triangle inequality, formulas (23), (24), and estimate (25), we get

$$\| Au_N \|_H \le \| J_1 \|_H + \| J_2 \|_H \le \| J_2 \|_H + M(\delta)[||A\varphi||_H + ||A\psi||_H].$$

To this end, it suffices to show that

$$\| J_2 \|_H \le M(\delta)\frac{1}{\alpha(1-\alpha)}\|\varphi^\tau\|_{C_{01}^{\alpha}([0,1]_\tau,H)}. \tag{29}$$

Applying formula (24), we get

$$J_2=K_\tau\sum_{j=1}^{N}\rho\left(t_j\right)\tau(I-R^{2N})^{-1}\Bigg\{-\left(R^{N-j}-R^{N+j}\right)\tau^{-2}(I-R)^2\sum_{i=1}^{j-1}\tau^2\left(R^{N-i}-R^{N+i}\right)$$

$$\times\left(I-R^2\right)^{-1}(\varphi_i-\varphi_j)+\left(-\left(R^{N-j}-R^{N+j}\right)\right)\tau^{-2}(I-R)^2\sum_{i=j+1}^{N-1}\tau^2\left(R^{N-i}-R^{N+i}\right)$$

$$\times\left(I-R^2\right)^{-1}(\varphi_i-\varphi_j)+(I-R^{2N})\tau^{-2}(I-R)^2\sum_{i=1}^{j-1}\tau^2\left(R^{j-i}-R^{j+i}\right)\left(I-R^2\right)^{-1}(\varphi_i-\varphi_j)$$

$$+(I-R^{2N})\tau^{-2}(I-R)^2\sum_{i=j+1}^{N-1}\tau^2\left(R^{i-j}-R^{j+i}\right)\left(I-R^2\right)^{-1}(\varphi_i-\varphi_j)-\left(R^{N-j}-R^{N+j}\right)\tau^{-2}(I-R)^2$$

$$\times\sum_{i=1}^{j-1}\tau^2\left(R^{N-i}-R^{N+i}\right)\left(I-R^2\right)^{-1}\varphi_j-\left(R^{N-j}-R^{N+j}\right)\tau^{-2}(I-R)^2$$

$$\times\sum_{i=j+1}^{N-1}\tau^2\left(R^{N-i}-R^{N+i}\right)\left(I-R^2\right)^{-1}\varphi_j+(I-R^{2N})\tau^{-2}(I-R)^2\sum_{i=1}^{j-1}\tau^2\left(R^{j-i}-R^{j+i}\right)\left(I-R^2\right)^{-1}\varphi_j$$

$$+(I-R^{2N})\tau^{-2}(I-R)^2\sum_{i=j+1}^{N-1}\tau^2\left(R^{i-j}-R^{j+i}\right)\left(I-R^2\right)^{-1}\varphi_j\Bigg\}=\sum_{z=2}^{4}J_2^z,$$

where

$$J_2^2=K_\tau\sum_{j=1}^{N}\rho\left(t_j\right)\tau(I-R^{2N})^{-1}\left(R^{2N-j-1}\left(I-R-R^2+R^3\right)+R^{2N+j}\left(I+R-R^j-R^{-1}\right)\right)\varphi_j,$$

$$J_2^3=K_\tau\sum_{j=1}^{N}\rho\left(t_j\right)\tau(I-R^{2N})^{-1}(I-R)\left(I-R^{2N-2j}\right)$$

$$\times\sum_{i=1}^{j-1}R^{j-i}\left(I-R^{2i}\right)(I+R)^{-1}(\varphi_i-\varphi_j)=J_2^{3,1}+J_2^{3,2},$$

$$J_2^{3,1}=K_\tau\sum_{j=1}^{\left[\frac{N}{2}\right]}\rho\left(t_j\right)\tau(I-R^{2N})^{-1}(I-R)\left(I-R^{2N-2j}\right)\sum_{i=1}^{j-1}R^{j-i}\left(I-R^{2i}\right)(I+R)^{-1}(\varphi_i-\varphi_j),$$

$$J_2^{3,2}=K_\tau\sum_{j=\left[\frac{N}{2}\right]+1}^{N}\rho\left(t_j\right)\tau(I-R^{2N})^{-1}(I-R)\left(I-R^{2N-2j}\right)$$

$$\times\sum_{i=1}^{j-1}R^{j-i}\left(I-R^{2i}\right)(I+R)^{-1}(\varphi_i-\varphi_j),$$

$$J_2^4=K_\tau\sum_{j=1}^{N}\rho\left(t_j\right)\tau(I-R^{2N})^{-1}(I-R)\left(I-R^{2j}\right)\sum_{i=j+1}^{N-1}R^{i-j}\left(I-R^{2N-2i}\right)$$

$$\times(I+R)^{-1}(\varphi_i-\varphi_j)=J_2^{4,1}+J_2^{4,2},$$

$$J_2^{4,1}=K_\tau\sum_{j=1}^{\left[\frac{N}{2}\right]}\rho\left(t_j\right)\tau(I-R^{2N})^{-1}(I-R)\left(I-R^{2j}\right)\sum_{i=j+1}^{N-1}R^{i-j}\left(I-R^{2N-2i}\right)(I+R)^{-1}(\varphi_i-\varphi_j),$$

$$J_2^{4,2}=K_\tau\sum_{j=\left[\frac{N}{2}\right]+1}^{N}\rho\left(t_j\right)\tau(I-R^{2N})^{-1}(I-R)\left(I-R^{2j}\right)\sum_{i=j+1}^{N-1}R^{i-j}\left(I-R^{2N-2i}\right)(I+R)^{-1}(\varphi_i-\varphi_j).$$

Second, let us estimate J_2^m for any $m=2,\cdots,4$, separately. We start with J_2^2, using estimates (15), (17), and the definition of the norm of the space $C_{01}^\alpha([0,1]_\tau,H)$, we obtain

$$\left\|J_2^2\right\|_H \leq M_1(\delta) \sum_{j=1}^{N} \left|\rho\left(t_j\right)\right| \tau \max_{1\leq j\leq N} \left\|\varphi_j\right\|_H \leq M_1(\delta) \sum_{j=1}^{N} \left|\rho\left(t_j\right)\right| \tau \left\|\varphi^\tau\right\|_{C_{01}^\alpha([0,1]_\tau,H)}.$$

From (14) it follows that

$$\left\|J_2^2\right\|_H \leq M_2(\delta) \left\|\varphi^\tau\right\|_{C_{01}^\alpha([0,1]_\tau,H)}.$$

Now, let us estimate $J_2^{3,1}$. Using the estimates (15), (17), and the definition of the norm of the space $C_{01}^\alpha([0,1]_\tau, H)$, we obtain

$$\left\|J_2^{3,1}\right\|_H \leq \|K_\tau\|_{H\to H} \sum_{j=1}^{\left[\frac{N}{2}\right]} \left|\rho\left(t_j\right)\right| \tau \left\|(I-R^{2N})^{-1}\right\|_{H\to H}$$

$$\times \sum_{i=1}^{j-1} \left\|R^{j-i}\left(I-R^{2N-2j}\right)(I-R)\right\|_{H\to H} \left\|I-R^{2i}\right\|_{H\to H} \left\|(I+R)^{-1}\right\|_{H\to H} \left\|\varphi_i-\varphi_j\right\|_H$$

$$\leq M(\delta) \sum_{j=1}^{\left[\frac{N}{2}\right]} \frac{\left|\rho\left(t_j\right)\right|}{(j\tau)^\alpha\left((N-j)\tau\right)^\alpha} \sum_{i=1}^{j-1} \frac{\tau}{((j-i)\tau)^{1-\alpha}} \left\|\varphi^\tau\right\|_{C_{01}^\alpha([0,1]_\tau,H)}.$$

The sum

$$\sum_{i=1}^{j-1} \frac{\tau}{((j-i)\tau)^{1-\alpha}}$$

is the lower Darboux integral sum for the integral

$$\int_0^{j\tau} \frac{ds}{(j\tau-s)^{1-\alpha}}.$$

It follows that

$$\left\|J_2^{3,1}\right\|_H \leq M(\delta) \sum_{j=1}^{\left[\frac{N}{2}\right]} \frac{\left|\rho\left(t_j\right)\right|\tau}{\alpha\left((N-j)\tau\right)^\alpha} \left\|\varphi^\tau\right\|_{C_{01}^\alpha([0,1]_\tau,H)}.$$

By the lower Darboux integral sum for the integral, it concludes that

$$\left\|J_2^{3,1}\right\|_H \leq M(\delta) \frac{2^{\alpha-1}}{\alpha(1-\alpha)} \sum_{j=1}^{\left[\frac{N}{2}\right]} \left|\rho\left(t_j\right)\right| \tau \left\|\varphi^\tau\right\|_{C_{01}^\alpha([0,1]_\tau,H)}.$$

For $J_2^{3,2}$, applying (15), (17), and the definition of the norm of the space $C_{01}^\alpha([0,1]_\tau, H)$, we get

$$\left\|J_2^{3,2}\right\|_H \leq M(\delta) \sum_{j=\left[\frac{N}{2}\right]+1}^{N} \frac{\left|\rho\left(t_j\right)\right| 2^\alpha((N-j)\tau)^\alpha}{((N-j)\tau)^\alpha (j\tau)^\alpha}$$

$$\times \sum_{i=1}^{j-1} \frac{\tau}{((j-i)\tau)^{1-\alpha}((N-j-i+N)\tau)^\alpha} \left\|\varphi^\tau\right\|_{C_{01}^\alpha([0,1]_\tau,H)}.$$

The sum

$$\sum_{i=1}^{j-1} \frac{\tau}{((j-i)\tau)^{1-\alpha}((N-j-i+N)\tau)^\alpha}$$

is the lower Darboux integral sum for the integral

$$\int_0^{j\tau} \frac{ds}{(j\tau-s)^{1-\alpha}(1-j\tau-s+1)^\alpha}.$$

Since

$$\int_0^{j\tau} \frac{ds}{(j\tau-s)^{1-\alpha}(N\tau-j\tau-s+N\tau)^\alpha} \leq \frac{1}{(1-j\tau)^\alpha} \int_0^{j\tau} \frac{ds}{(j\tau-s)^{1-\alpha}} \leq \frac{M}{\alpha(j\tau)^\alpha},$$

it follows that

$$\left\|J_2^{3,2}\right\|_H \leq M(\delta) \sum_{j=\left[\frac{N}{2}\right]+1}^{N} \left|\rho\left(t_j\right)\right| \tau \frac{2^{\alpha}}{(j\tau)^{\alpha}\left(N\tau - j\tau\right)^{\alpha} \alpha (j\tau)^{-\alpha}} \left\|\varphi^{\tau}\right\|_{C_{01}^{\alpha}\left([0,1]_{\tau}, H\right)}.$$

By the lower Darboux integral sum for the integral it follows that

$$\left\|J_2^{3,2}\right\|_H \leq \frac{M(\delta) 2^{2\alpha-1}}{\alpha(1-\alpha)} \sum_{j=\left[\frac{N}{2}\right]+1}^{N} \left|\rho\left(t_j\right)\right| \tau \left\|\varphi^{\tau}\right\|_{C_{01}^{\alpha}\left([0,1]_{\tau}, H\right)}.$$

Applying estimates for $\left\|J_2^{3,1}\right\|_H$ and $\left\|J_2^{3,2}\right\|_H$, we get

$$\left\|J_2^{3}\right\|_H \leq \frac{M_3(\delta)}{\alpha(1-\alpha)} \sum_{j=1}^{N} \left|\rho\left(t_j\right)\right| \tau \left\|\varphi^{\tau}\right\|_{C_{01}^{\alpha}\left([0,1]_{\tau}, H\right)}.$$

From (14) it follows that

$$\left\|J_2^{3}\right\|_H \leq \frac{M_4(\delta)}{\alpha(1-\alpha)} \left\|\varphi^{\tau}\right\|_{C_{01}^{\alpha}\left([0,1]_{\tau}, H\right)}.$$

Next, let us estimate $J_2^{4,1}$. Using estimates (15), (17), and the definition of the norm space $C_{01}^{\alpha}\left([0,1]_{\tau}, H\right)$, we obtain

$$\left\|J_2^{4,1}\right\|_H \leq M(\delta) \sum_{j=1}^{\left[\frac{N}{2}\right]} \frac{\left|\rho\left(t_j\right)\right| (N-j)^{\alpha}}{\left((N-j)\tau\right)^{\alpha} (j\tau)^{\alpha}} \sum_{i=j+1}^{N-1} \frac{\tau}{\left((i-j)\tau\right)^{1-\alpha} \left((N-j-i+N)\tau\right)^{\alpha}} \left\|\varphi^{\tau}\right\|_{C_{01}^{\alpha}\left([0,1]_{\tau}, H\right)}.$$

The sum

$$\sum_{i=j+1}^{N-1} \frac{\tau}{\left((i-j)\tau\right)^{1-\alpha}}$$

is the lower Darboux integral sum for the integral

$$\int_{j\tau}^{1} \frac{ds}{(s-j\tau)^{1-\alpha}}.$$

Since

$$\int_{j\tau}^{1} \frac{ds}{(s-j\tau)^{1-\alpha}(2N-j\tau-s)^{\alpha}} \leq \frac{1}{(N\tau-j\tau)^{\alpha}} \int_{j\tau}^{1} \frac{ds}{(s-j\tau)^{1-\alpha}}$$

$$\leq \frac{(N\tau-j\tau)^{\alpha}}{\alpha(N\tau-j\tau)^{\alpha}},$$

we have that

$$\left\|J_2^{4,1}\right\|_H \leq M(\delta) \sum_{j=1}^{\left[\frac{N}{2}\right]} \frac{\left|\rho\left(t_j\right)\right| \tau}{(j\tau)^{\alpha}\alpha} \left\|\varphi^{\tau}\right\|_{C_{01}^{\alpha}\left([0,1]_{\tau}, H\right)}.$$

By the lower Darboux integral sum for the integral, it follows that

$$\left\|J_2^{4,1}\right\|_H \leq \frac{M(\delta) 2^{2\alpha}}{\alpha(1-\alpha)} \sum_{j=1}^{\left[\frac{N}{2}\right]} \left|\rho\left(t_j\right)\right| \tau \left\|\varphi^{\tau}\right\|_{C_{01}^{\alpha}\left([0,1]_{\tau}, H\right)}.$$

Finally, let us estimate $J_2^{4,2}$. Using estimates (15), (17), and the definition of the norm space $C_{01}^{\alpha}\left([0,1]_{\tau}, H\right)$, we get

$$\left\|J_2^{4,2}\right\|_H \leq M(\delta) \sum_{j=\left[\frac{N}{2}\right]+1}^{N} \frac{\left|\rho\left(t_j\right)\right| \tau}{\left((N-j)\tau\right)^{\alpha} (j\tau)^{\alpha}} \sum_{i=j+1}^{N-1} \frac{\tau}{\left((i-j)\tau\right)^{1-\alpha}} \left\|\varphi^{\tau}\right\|_{C_{01}^{\alpha}\left([0,1]_{\tau}, H\right)}.$$

The sum

$$\sum_{i=j+1}^{N-1} \frac{\tau}{((i-j)\tau)^{1-\alpha}}$$

is the lower Darboux integral sum for the integral

$$\int_{j\tau}^{1} \frac{ds}{(s-j\tau)^{1-\alpha}}.$$

Thus, we show that

$$\left\| J_2^{4,2} \right\|_H \leq M(\delta) \sum_{j=\left[\frac{N}{2}\right]+1}^{N} \frac{\left|\rho\left(t_j\right)\right| \tau}{(j\tau)^{\alpha}\alpha} \left\|\varphi^{\tau}\right\|_{C_{01}^{\alpha}([0,1]_{\tau},H)}.$$

By the lower Darboux integral sum for the integral, it follows that

$$\left\| J_2^{4,2} \right\|_H \leq M(\delta) \frac{2^{\alpha-1}}{\alpha(1-\alpha)} \sum_{j=\left[\frac{N}{2}\right]+1}^{N} \left|\rho\left(t_j\right)\right| \tau \left\|\varphi^{\tau}\right\|_{C_{01}^{\alpha}([0,1]_{\tau},H)}.$$

Applying estimates for $\left\| J_2^{4,1} \right\|_H$ and $\left\| J_2^{4,2} \right\|_H$, we get

$$\left\| J_2^{4} \right\|_H \leq M(\delta) \left(\frac{2^{\alpha-1}}{\alpha(1-\alpha)} + \frac{2^{2\alpha}}{\alpha(1-\alpha)} \right) \sum_{j=1}^{N} \left|\rho\left(t_j\right)\right| \tau \left\|\varphi^{\tau}\right\|_{C_{01}^{\alpha}([0,1]_{\tau},H)}.$$

From (14) it follows that

$$\left\| J_2^{4} \right\|_H \leq \frac{M_5(\delta)}{\alpha(1-\alpha)} \left\|\varphi^{\tau}\right\|_{C_{01}^{\alpha}([0,1]_{\tau},H)}.$$

Combining the estimates for $\|J_2^m\|_H, m = 2, \cdots, 4$ we get the estimate (29). Theorem 6 is proved. □

Now, we consider the applications of Theorems 3.2- 3.3.

The Bitsadze-Samarskii type nonlocal boundary value problem for the multidimensional elliptic equation (12) is considered. The discretization of problem (12) is carried out in two steps. In the first step, let us define the grid sets

$$\overline{\Omega}_h = \{x = x_m = (h_1 m_1, \cdots, h_n m_n), m = (m_1, \cdots, m_n), 0 \leq m_r \leq N_r,$$

$$h_r N_r = 1,\ r = 1, \cdots, n\}, \Omega_h = \widetilde{\Omega}_h \cap \Omega, S_h = \widetilde{\Omega}_h \cap S.$$

We introduce the Hilbert spaces $L_{2h} = L_2(\widetilde{\Omega}_h)$ and $W_{2h}^2(\widetilde{\Omega}_h)$of the grid functions $\varphi^h(x) = \{\varphi(h_1 m_1, \cdots, h_n m_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norms

$$\left\|\varphi^h\right\|_{L_{2h}(\widetilde{\Omega}_h)} = (\sum_{x \in \overline{\Omega_h}} \left|\varphi^h(x)\right|^2 h_1 \cdots h_n)^{1/2},$$

$$\left\|\varphi^h\right\|_{W_{2h}^2(\widetilde{\Omega}_h)} = \left\|\varphi^h\right\|_{L_{2h}(\widetilde{\Omega}_h)} + \left(\sum_{x \in \widetilde{\Omega}_h} \sum_{r=1}^{n} \left|\varphi^h_{x_r \overline{x_r},\, m_r}\right|^2 h_1 \cdots h_n \right)^{1/2}.$$

To the differential operator A generated by the problem (12), we assign the difference operator A_h^x by the formula

$$A_h^x u^h = -\sum_{r=1}^{n} (a_r(x) u^h_{\overline{x_r}})_{x_r, m_r}, \tag{30}$$

acting in the space of the grid functions $u^h(x)$, satisfying the conditions $u^h = 0$ for all $x \in S_h$. It is known that A_h^x is a self-adjoint positive definite operator in $L_{2h}(\widetilde{\Omega}_h)$. With the help of A_h^x, we arrive at the nonlocal boundary value problem for an infinite system of ordinary differential equations

$$\begin{cases} -\frac{d^2 u^h(t,x)}{dt^2} + A_h^x u^h(t,x) = f^h(t,x),\ 0 < t < 1,\ x \in \Omega_h, \\ u^h(0,x) = \varphi^h(x);\ u^h(1,x) = \int_0^1 \rho(t) u^h(t,x) dt + \psi^h(x),\ x \in \widetilde{\Omega}_h. \end{cases} \tag{31}$$

In the second step, (31) is replaced by the difference scheme (13), and we get the following difference scheme:

$$\begin{cases} -\frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_k^h(x) = \varphi_k^h(x), \\ \varphi_k^h(x) = f^h(t_k, x),\ x \in \Omega_h, t_k = k\tau, 1 \leq k \leq N-1,\ N\tau = 1, \\ u_0^h(x) = \varphi^h(x), x \in \widetilde{\Omega}_h, \\ u_N^h(x) = \sum_{j=1}^{N} \rho\left(t_j\right) \tau u_j^h(x) + \psi^h(x),\ x \in \widetilde{\Omega}_h. \end{cases} \tag{32}$$

Theorem 7. *Let τ and $|h|$ be sufficiently small positive numbers. Under the assumption (14), the solution of the difference scheme (32) satisfies the following almost coercivity estimate:*

$$\max_{1\leq k\leq N-1}\left\|\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\|_{L_{2h}}+\max_{1\leq k\leq N-1}\left\|u_k^h\right\|_{W_{2h}^2}$$

$$\leq M(\delta)\left[\ln\frac{1}{\tau+|h|}\max_{1\leq k\leq N-1}\left\|\varphi_k^h\right\|_{L_{2h}}+\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^2}\right].$$

The proof of Theorem 3.4 is based on Theorem 3.2 on the estimate

$$\min\left\{\ln\frac{1}{\tau},1+\left|\ln\|\ B_h^x\|_{L_{2h}\to L_{2h}}\right|\right\}\ \leq M\ln\ \frac{1}{\tau+|h|},$$

on the symmetry properties of the difference operator A_h^x defined by (30) in L_{2h}, and on the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Theorem 8. *For the solution of the elliptic difference problem*

$$A_h^x u^h(x)=\omega^h(x), x\in\Omega_h, \tag{33}$$

$$u^h(x)=0, x\in S_h$$

the following coercivity inequality holds [22]:

$$\left\|u^h\right\|_{W_{2h}^2}\leq M\left(\delta\right)||\omega^h||_{L_{2h}}.$$

Theorem 9. *τ and $|h|$ be sufficiently small positive numbers. Then under the assumption (14) the solution of the difference scheme (32) satisfies the following coercivity stability estimate*

$$\|\left\{\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\}_1^{N-1}\|_{C_{01}^{\alpha}([0,1]_\tau,L_{2h})}+\|\left\{u_k^h\right\}_1^{N-1}\|_{C_{01}^{\alpha}([0,1]_\tau,W_{2h}^2)}$$

$$\leq M(\delta)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^2}+\frac{1}{\alpha(1-\alpha)}\left\|\left\{\varphi_k^h\right\}_1^{N-1}\right\|_{C_{01}^{\alpha}\left([0,1]_\tau L_{2h}\right)}\right].$$

The proof of Theorem 9 is based on Theorem 6, on the symmetry properties of the difference operator A_h^x defined by the formula (30), and on Theorem 8 on the coercivity inequality for the solution of the elliptic difference equation (13) in L_{2h}.

4 Conclusion

In this paper, the well-posedness of problem (1) in Hölder spaces with a weight is established. The coercivity inequality for the solution of the nonlocal boundary value problem for elliptic equation is obtained. The first order of accuracy difference scheme for the approximate solution of the Bitsadze-Samarskii type nonlocal boundary value problem with integral condition for elliptic equation is studied. Theorems on the almost coercive stability estimates and coercive stability estimates for the solution of difference scheme for elliptic equations are proved.

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5 References

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