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Numerical Solution of Riesz Fractional Differential Equation via Meshless Method

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Abstract: In this study, we present the numerical solution of Riesz fractional differential equation with the help of meshless method. In accordance with this purpose, we benefit the radial basis functions (RBFs) interpolation method and conformable fractional calculus. We finally present the results of numerical experimentation to show that presented algorithm provide successful consequences.

Keywords: Numerical solution, Fractional calculus, RBF interpolation

1 Introduction

In this study, we provide a meshfree algorithm for the numerical solution of Riesz fractional differential equation by taking advantageous of radial basis function interpolation [5], [6]. The aim of this scheme is to acquire approximate solution of Riesz fractional differential equation with RBF collocation method. Of course this approach would provide an insight the solution of more complex cases.

The remainder of this work is organized as follows: In Section 2, the conformable derivatives are summarised, along with the conformable fractional calculus. In Section 3, the RBF interpolation method is reviewed while in Section 4 the numerical scheme of solving conformable ordinary differential equation using meshfree method is introduced and we also review the RBF collocation technique. Numerical examples are given in Section 5, while some conclusions are discussed in Section 6.

2 Conformable fractional calculus

In this manuscript, meshfree solution of Riesz fractional differential equation will be presented and tested via conformable fractional calculus. More detail, conformable derivatives for $\boldsymbol{\alpha} \in (0,1]$ and $t \in [0,\infty)$ given by

$$\mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{f})(t) = \lim_{\varepsilon \to 0} \frac{\mathfrak{f}\left(te^{\varepsilon t^{-\alpha}}\right) - \mathfrak{f}(t)}{\varepsilon}, \qquad \mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{f})(0) = \lim_{t \to 0} \mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{f})(t), \tag{1}$$

provided the limits exist (for more detail see, [1]). If $\mathfrak{f}$ is fully differentiable at t, then

$$\mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{f})(t) = t^{1-\alpha} \frac{d\mathfrak{f}}{dt}(t). \tag{2}$$

A function $\mathfrak{f}$ is $\boldsymbol{\alpha}-$differentiable at a point $t \geq 0$ if the limit in (1) exists and is finite. This yields the following consequences.

Theorem 1. *Let $\boldsymbol{\alpha} \in (0,1]$ and $\mathfrak{f}, \mathfrak{g}$ be $\boldsymbol{\alpha}-$differentiable at a point $t > 0$. Then*

1. $\mathcal{D}^{\boldsymbol{\alpha}}(a\mathfrak{f} + b\mathfrak{g}) = a\mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{f}) + bD^{\alpha}(\mathfrak{g})$, *for all* $a, b \in \mathbb{R}$,
2. $\mathcal{D}^{\boldsymbol{\alpha}}(\lambda) = 0$, *for all constant functions* $f(t) = \lambda$,
3. $\mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{f}\mathfrak{g}) = \mathfrak{f}\mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{g}) + vD^{\alpha}(\mathfrak{f})$,
4. $\mathcal{D}^{\boldsymbol{\alpha}}\left(\frac{\mathfrak{f}}{\mathfrak{g}}\right) = \frac{\mathfrak{g}\mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{f}) - \mathfrak{f}\mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{g})}{\mathfrak{g}^2}$,
5. $\mathcal{D}^{\boldsymbol{\alpha}}(t^n) = nt^{n-\alpha}$ *for all* $n \in \mathbb{R}$,
6. $\mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{f} \circ \mathfrak{g})(t) = \mathfrak{f}'(\mathfrak{g}(t))\mathcal{D}^{\boldsymbol{\alpha}}(\mathfrak{g})(t)$ *for* $\mathfrak{f}$ *is differentiable at* $\mathfrak{g}(t)$.

3 Meshless method

In this part, the basic facts of the meshless radial basis function interpolation are explained. Consider a function $\mathfrak{f}:\mathbb{R}^d\to\mathbb{R}$ a real valued function with d variables, which is to be approximated by $\mathcal{I}_{\mathcal{X}}:\mathbb{R}^d\to\mathbb{R}$, for given values $\mathfrak{f}(\mathbf{x}_i):i=1,2,\cdots,n$, where $\mathbf{x}_i:i=1,2,\cdots,n$ is a set of distinct points in $\mathbb{R}^d$, named the center set $\mathcal{X}$.

Then the approximation to the function $\mathfrak{f}$ is of the form

$$\mathfrak{I}_{\mathfrak{X}}(\mathbf{x})=\sum_{k=1}^{N}\mathfrak{a}_k\tau_k(\|\mathbf{x}-\mathbf{x}_k\|),\tag{3}$$

where $\tau_k:\mathbb{R}^d\to\mathbb{R}$ is a univariate radial basis function. Then the interpolation condition can be constructed as $\mathfrak{I}_{\mathfrak{X}}(\mathbf{x}_m)=\mathfrak{f}(\mathbf{x}_m)$, $m=1,2,\cdots,N$. Namely, the interpolation condition is

$$\sum_{k=1}^{N}\mathfrak{a}_k\tau_k(\|\mathbf{x}_m-\mathbf{x}_k\|)=\mathfrak{f}(\mathbf{x}_m),\qquad m=1,2,\cdots,N\tag{4}$$

In other words the system of matrix for interpolation condition can be written as

$$[A]\{\mathfrak{a}\}=\{\mathfrak{f}\}\tag{5}$$

where the entries of the matrix A are $A_{m,k}=\tau_k(\|\mathbf{x}_m-\mathbf{x}_k\|)$ such that $m,k=1,2,\cdots,N$, $\mathfrak{a}=\{\mathfrak{a}_1,\mathfrak{a}_2,\cdots,\mathfrak{a}_N\}^T$ and $\mathfrak{f}=\{\mathfrak{f}_1,\mathfrak{f}_2,\cdots,\mathfrak{f}_N\}^T$. The interpolant of $\mathfrak{f}(\mathbf{x})$ is unique if and only if the matrix X is non-singular. It has been discussed about sufficient conditions for $\tau(r)$ to guarantee non-singularity of the a matrix [2], [3]

Well known radial basis functions are given below:

RBFs	$\tau(r)$
Piecewise Polynomial (R_n)	$\|r\|^n$, n odd
Thin Plate Spline (TPS_n)	$\|r\|^n ln\|r\|$, n even
Multiquadric (MQ)	$\sqrt{1+r^2}$
Inverse Multiquadric (IMQ)	$\dfrac{1}{\sqrt{1+r^2}}$
Inverse Quadratic (IQ)	$\dfrac{1}{1+r^2}$
Gaussian (GA)	e^{-r^2}
Bessel (BE)	$J_0(2r)$

4 Numerical approach

In this section, we provide a numerical scheme to solve Riesz fractional differential equation via meshless method. To begin with, consider the following Riesz fractional differential equation [4],

$$\frac{\partial\mathfrak{f}(x,t)}{\partial t}=-\kappa_{\boldsymbol{\alpha}}\frac{\partial^{\boldsymbol{\alpha}}}{\partial|x|^{\boldsymbol{\alpha}}}\mathfrak{f}(x,t),\qquad x\in[0,\pi],\qquad t\in(0,T],$$

$$\mathfrak{f}(x,0)=\mathfrak{f}_0(x),$$

$$\mathfrak{f}(0,t)=\mathfrak{f}(\pi,t)=0,$$

where $\kappa_{\boldsymbol{\alpha}}$ represents the dispersion coefficient and $\mathfrak{f}(x,t)$ is a solute concentration. Let x_i be equally spaced grid points in the interval $0\le x_i\le\pi$ such that $1\le i\le N$, $x_1=0$ and $x_N=\pi$. Then by solving the equation $\omega(x)=\tau(x)\lambda^{-1}$ for $\omega(x)=(\omega_k(x))_{1\le k\le N}$, $\tau(x)=(\tau(|x-x_k|))_{1\le k\le N}$ and $\lambda=(\tau(|x_i-x_k|))_{1\le i\le N,1\le k\le N}$ one can construct the Lagrange basis $\omega_1(x),\omega_1(x),\cdots,\omega_N(x)$ of the span of the functions $(\tau(|x-x_k|))_{1\le k\le N}$. If $\mathfrak{D}$ is a differential operator, and if the radial basis function τ is sufficiently smooth to to allow implementation of $\mathfrak{D}$, the desired derivatives $\mathfrak{D}\omega_k$ of the Lagrange basis ω_k come via solving $(\mathfrak{D}\omega)(x)=(\mathfrak{D}\tau)(x)\lambda^{-1}$. So one can write the approximation solution as

$$\mathfrak{f}(x,t)=\sum_{k=2}^{N-1}\varphi_k(t)\omega_k(x)$$

where $\varphi(t)=\varphi_k(t), 2\leq k\leq N-1$, which yields

$$\sum_{k=2}^{N-1}\varphi_k'(t)\omega_k(x_i)=-\kappa_{\boldsymbol{\alpha}}\sum_{k=2}^{N-1}\varphi_k(t)\frac{\partial^{\boldsymbol{\alpha}}}{\partial|x|^{\boldsymbol{\alpha}}}\omega_k(x_i),\qquad \varphi_k(0)=\mathfrak{f}_0(x_k),\quad 2\leq k\leq N-1.$$

Hence one obtain the following system of ordinary differential equation:

$$\varphi'(t)=-\kappa_{\boldsymbol{\alpha}}\left[\frac{\partial^{\boldsymbol{\alpha}}}{\partial|x|^{\boldsymbol{\alpha}}}\omega*\varphi(t)\right],\varphi(0)=\Delta_0.$$

where

$$\frac{\partial^{\boldsymbol{\alpha}}}{\partial|x|^{\boldsymbol{\alpha}}}\omega=\left[\frac{\partial^{\boldsymbol{\alpha}}}{\partial|x|^{\boldsymbol{\alpha}}}\omega_k(x_i)\right]_{2\leq i\leq N-1,2\leq k\leq N-1},$$

and

$$\Delta_0=\begin{pmatrix}\mathfrak{f}_0(x_2)\\ \vdots\\ \vdots\\ \mathfrak{f}_0(x_{N-1})\end{pmatrix}.$$

5 Numerical experiments

Now, in order to validate our approach, we present a numerical experiment which performed by MATLAB. Here, let consider the parameter $\alpha=0.5, \kappa_{\boldsymbol{\alpha}}=0.25, T=0.4$ and $\mathfrak{f}_0(x)=\sin(\pi-3x)$. In Figure, we present the numerical solutions of given fractional differential equations by using the Gaussian RBF with $\varepsilon=1$, and taking 101 discretization grids.

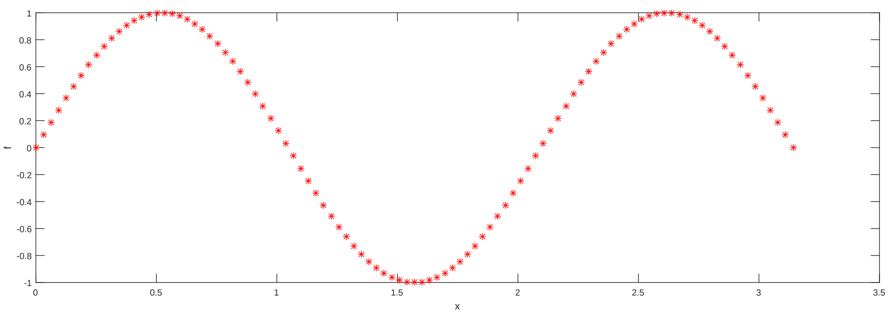

Fig. 1: $\mathfrak{f}$ versus x using Gaussain RBF with $\varepsilon=1$ taking 101 discretization grids

6 Conclusion

In this article, we presented and tested the numerical solution of Riesz fractional differential equation with the help of meshless method. Here, we benefit the radial basis functions (RBFs) interpolation method and conformable fractional calculus. We finally present the results of numerical experimentation to show that presented algorithm provide successful consequences.

7 References

[1] U. Katugampola, *A new fractional derivative with classical properties*, ArXiv:1410.6535v2.
[2] MD. Buhmann, *Radial Basis Functions: Theory and Implementations*, Cambridge University Press, 2003.
[3] W. Cheney and W. Light, *A Course in Approximation Theory*, William Allan, New York, 1999.
[4] Q. Yang., F. Liu, and I. Turner, *Numerical methods for fractional partial differential equations with Riesz space fractional derivatives*, Appl. Math. Modelling., **34**(200-218) (2010).
[5] C. Franke and R. Schaback, *Solving partial differential equations by collocation using radial basis functions*, Appl. Math. Comput. , 93 (1998) 73-82.
[6] E. J. Kansa, *Multiquadrics a scattered data approximation scheme with applications to computational filuid-dynamics. I. Surface approximations and partial derivative estimates*, Comput. Math. Appl. **19(8-9)** (1990) 127-145.