Conference Proceedings of Science and Technology, 2(1), 2019, 97–103

Conference Proceeding of 2nd International Conference on Mathematical Advances and Applications (ICOMAA-2019).

Some Pascal Spaces of Difference Sequences Spaces of Order m

ISSN: 2651-544X

http://dergipark.gov.tr/cpost

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Abstract: The main purpose of this article is to introduce new sequence spaces $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$ which are consisted by sequences whose m^{th} order differences are in the Pascal sequence spaces p_∞, p_c and p_0, respectively. Furthermore, the bases of the new difference sequence spaces $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$, and the α-, β- and γ-duals of the new difference sequence spaces $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$ have been determined. Finally, necessary and sufficient conditions on an infinite matrix belonging to the classes $(p_c\left(\Delta^{(m)}\right):l_\infty)$ and $(p_c\left(\Delta^{(m)}\right):c)$ are obtained.

Keywords: Difference operator of order m, Matrix mappings, Pascal difference sequence spaces, α-, β- and γ-duals.

1 Introduction

By w, we shall denote the space of all real or complex valued sequences. Any vector subspace of w is called as a sequence space. We shall write l_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs and l_1 we denote the spaces of all bounded, convergent and absolutely convergent series, respectively.

Let X, Y be any two sequence spaces and $A=(a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n,\ k\in N$. Then, the matrix A defines a transformation from X into Y and we denote it by $A:X\to Y$, if for every sequence $x=(x_k)\in X$ the sequence $Ax=\{(Ax)_n\}$, the A-transform of x, is in Y, where

$$(Ax)_n=\sum_k a_{nk}x_k \tag{1}$$

for each $n\in N$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(X:Y)$, we denote the class of all matrices A such that $A:X\to Y$. Thus $A\in(X:Y)$ if and only if the series on the right side of (1) converges for each $n\in N$ and every $x\in X$, and we have $Ax=\{(Ax)_n\}\in Y$ for all $x\in X$.

In the study on the sequence spaces, there are some basic approaches which are determination of topologies, matrix mapping and inclusions of sequence spaces [2]. These methods are applied to study the matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A=\{x=(x_k)\in w: Ax\in X\},$$

which is a sequence space. Although in the most cases the new sequence space X_A generated by the limitation matrix A from a sequence space X is the expansion or the contraction of the original space X, in some cases it may be observed that those spaces overlap. Indeed, one can easily see that the inclusions $X_S\subset X$ and $X\subset X_\Delta$ strictly hold for $X\in\{l_\infty,c,c_0\}$ [1]. Especially, the sequence spaces and the difference operator which are special cases for the matrix A have been studied extensively via the methods mentioned above.

Define the difference matrices $\Delta^1=(\delta_{nk})$ by

$$\delta_{nk}=\begin{cases}\{(-1)^{n-k},\ (n-1\le k\le n),\\ 0,\ (0<n-1\ or\ n>k),\end{cases}$$

for each $k,n\in N$.

In the literature, the difference sequence spaces $l_\infty(\Delta)=\{x=(x_k)\in w:\Delta x\in l_\infty\}$, $c(\Delta)=\{x=(x_k)\in w:\Delta x\in c\}$ and $c_0(\Delta)=\{x=(x_k)\in w:\Delta x\in c_0\}$ are first defined by Kızmaz [3]. Difference sequence spaces have been defined and studied by various authors [9]-[20]. The idea of difference sequences was generalized by Et and Çolak [9] and Murseelan [10]. Let λ denotes one of the sequence spaces l_∞, c, and c_0. They defined the sequence spaces $\lambda\left(\Delta^{(m)}\right)=\{x=(x_k)\in w:\Delta^{(m)}x\in\lambda\}$, where $m\in N$ and $\left(\Delta^{(m)}x\right)_n=\sum_{i=0}^m(-1)^i\binom{m}{i}x_{k-i}$. The operator $\Delta^{(m)}:w\to w$ is defined by $\left(\Delta^{(1)}x\right)_k=(x_k-x_{k+1})$ and $\Delta^{(m)}x=\left(\Delta^{(1)}x\right)_k o\left(\Delta^{(m-1)}x\right)_k$ $(m\ge$

2). Throughout the article, we shall use the convention that a term with a negative subscript is equal to naught. Also throughout this work, by F and K, respectively, we shall denote the collection of all finite subsets of N.

Let P denote the Pascal means defined by the Pascal matrix [4] is defined by

$$P = [p_{nk}] = \begin{cases} \binom{n}{n-k}, \ 0 \le k \le n) \\ 0, \ (k > n) \end{cases}, (n, k \in N)$$

and the inverse of Pascal's matrix $P_n = (p_{nk})$ is given by

$$P^{-1} = [p_{nk}]^{-1} = \begin{cases} (-1)^{n-k}\binom{n}{n-k}, (0 \le k \le n) \\ 0, (k > n) \end{cases}, (n, k \in N).$$

There is some interesting properties of Pascal matrix. For example; we can form three types of matrices: symmetric, lower triangular, and upper triangular, for any integer $n > 0$. The symmetric Pascal matrix of order n is defined by

$$S_n = (s_{ij}) = \binom{i+j-2}{j-1}, \text{ for } i, j = 1, 2, ..., n, \tag{2}$$

we can define the lower triangular Pascal matrix of order n by

$$L_n = (l_{ij}) = \begin{cases} \binom{i-1}{j-1}, (0 \le j \le i) \\ 0, \ (j > i) \end{cases}, \tag{3}$$

and the upper triangular Pascal matrix of order n is defined by

$$U_n = (u_{ij}) = \begin{cases} \binom{j-1}{i-1}, (0 \le i \le j) \\ 0, \ (j > i) \end{cases}. \tag{4}$$

We notice that $U_n = (L_n)^T$, for any positive integer n.

i. Let S_n be the symmetric Pascal matrix of order n defined by (2), L_n be the lower triangular Pascal matrix of order n defined by (3), and U_n be the upper triangular Pascal matrix of order n defined by (4), then $S_n = L_n U_n$ and $det(S_n) = 1$ [5].

ii. Let A and B be $n \times n$ matrices. We say that A is similar to B if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = B$ [6].

iii. Let S_n be the symmetric Pascal matrix of order n defined by (2), then S_n is similar to its inverse S_n^{-1} [5].

iv. Let L_n be the lower triangular Pascal matrix of order n defined by (3), then $L_n^{-1} = ((-1)^{i-j} l_{ij})$ [7].

Recently Polat [8] has defined the Pascal sequence spaces p_∞, p_c and p_0 like as follows:

$$p_\infty = \left\{ x = (x_k) \in w : \sup_n \left| \sum_{k=0}^{n} \binom{n}{n-k} x_k \right| < \infty \right\},$$

$$p_c = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{n-k} x_k \text{ exists} \right\},$$

and

$$p_0 = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{n-k} x_k = 0 \right\}.$$

In the present paper, we define Pascal difference sequence spaces $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$ which consist of all sequences whose m^{th} order differences are in the Pascal sequence spaces p_∞, p_c and p_0, respectively. Furthermore, the Schauder bases of the sequence spaces $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$, and the α-, β- and γ- duals of the sequence spaces $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$ have been determined. The last section of the article is devoted to the characterization of some matrix mappings on the sequence space $p_c\left(\Delta^{(m)}\right)$.

2 New Pascal difference sequence spaces of order m

The triangle matrix $\Delta^{(m)} = (\delta_{nk}^{(m)})$ is defined by

$$\delta_{nk}^{(m)}=\begin{cases}(-1)^{n-k}\left(\binom{m}{n-k}\right),\ (\max\{0,n-m\}\le k\le n),\\ 0,\ (0\le k<\max\{0,n-m\}\ or\ n>k),\end{cases}$$

for all k, $n\in N$ and for any fixed $m\in N$. Using this matrix, we introduce the sequence spaces $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$ as the set of all sequences such that $\Delta^{(m)}$-transforms of them are in the Pascal sequence spaces p_∞, p_c and p_0, respectively, that is,

$$\begin{aligned} p_\infty\left(\Delta^{(m)}\right) &= \left\{x=(x_k)\in w:\Delta^{(m)}x\in p_\infty\right\},\\ p_c\left(\Delta^{(m)}\right) &= \left\{x=(x_k)\in w:\ \Delta^{(m)}x\in p_c\right\},\end{aligned}$$

and

$$p_0\left(\Delta^{(m)}\right)=\left\{x=(x_k)\in w:\ \Delta^{(m)}x\in p_0\right\}.$$

Define the sequence $y=\{y_k\}$, which is frequently used, as the H- transform of a sequence $x=(x_k)$, i.e.,

$$\begin{aligned} y_n=(Hx)_n &= \sum_{k=0}^{n}\binom{n}{n-k}\sum_{i=0}^{m}(-1)^i\binom{m}{i}x_{k-i} \\ &= \sum_{k=0}^{n}\left[\sum_{i=k}^{n}\binom{i}{i-k}(-1)^{i-k}\binom{m}{i-k}\right]x_k \end{aligned} \tag{5}$$

for each n, $m\in N$. Here by H, we denote the matrix $H=P\Delta^{(m)}=(h_{nk})$ defined by

$$h_{nk}=\begin{cases}\sum_{i=k}^{n}\binom{i}{i-k}(-1)^{i-k}\binom{m}{i-k},\ (0\le k\le n)\\ 0,\ (k>n)\end{cases},(n,k\in N).$$

It can be easily shown that $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$ are normed linear spaces by the following norm:

$$\|x\|_\Delta=\|Hx\|_\infty=\sup_n\left|\sum_{k=0}^{n}\left[\sum_{i=k}^{n}\binom{i}{i-k}(-1)^{i-k}\binom{m}{i-k}\right]x_k\right|. \tag{6}$$

Theorem 1. *The sequence spaces $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$ are Banach spaces with the norm* (6).

Proof: Let $\left\{x^i\right\}$ be any Cauchy sequence in the space $p_\infty\left(\Delta^{(m)}\right)$, where $\left\{x^i\right\}=\left\{x_k^i\right\}=\left\{x_0^i,x_1^i,...\right\}\in p_\infty\left(\Delta^{(m)}\right)$ for each $i\in N$. Then, for a given $\varepsilon>0$ there exists a positive integer $N_0(\varepsilon)$ such that $\left\|x_i^k-x_i^n\right\|_\Delta<\varepsilon$ for all k, $n>N_0(\varepsilon)$. Hence

$$\left|H(x_i^k-x_i^n)\right|<\varepsilon$$

for all k, $n>N_0(\varepsilon)$ and for each $i\in N$. Therefore, $\left\{(Hx)_i^k\right\}=\left\{(Hx)_i^0,(Hx)_i^1,(Hx)_i^2,...\right\}$ is a Cauchy sequence of real numbers for every fixed $i\in N$. Since the set of real numbers R is complete, it converges, say

$$\lim_{i\to\infty}(Hx^i)_k\to(Hx)_k$$

for each $k\in N$. So, we have

$$\lim_{n\to\infty}\left|H(x_i^k-x_i^n)\right|=\left|H(x_i^k-x_i)\right|\le\varepsilon$$

for each $k\ge N_0(\varepsilon)$. This implies that $\left\|x^k-x\right\|_\Delta<\varepsilon$ for $k\ge N_0(\varepsilon)$, that is, $x^i\to x$ as $i\to\infty$.

Now, we must show that $x\in p_\infty\left(\Delta^{(m)}\right)$. We have

$$\begin{aligned}\|x\|_\Delta &= \|Hx\|_\infty=\sup_n\left|\sum_{k=0}^{n}\left[\sum_{i=k}^{n}\binom{i}{i-k}(-1)^{i-k}\binom{m}{i-k}\right]x_k\right| \\ &\le \sup_n\left|H(x_k^i-x_k)\right|+\sup_n\left|Hx_k^i\right| \\ &\le \left\|x^i-x\right\|_\Delta+\left|P\Delta^{(m)}x_k^i\right|<\infty\end{aligned}$$

for all $i\in N$. This implies that $x=(x_i)\in p_\infty\left(\Delta^{(m)}\right)$. Therefore $p_\infty\left(\Delta^{(m)}\right)$ is a Banach space. It can be shown that $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$ are closed subspaces of $p_\infty\left(\Delta^{(m)}\right)$, which leads us to the consequence that the spaces $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$ are also Banach spaces with the norm (6). Furthermore, since $p_\infty\left(\Delta^{(m)}\right)$ is a Banach space with continuous coordinates, i.e., $\left\|P(x^k-x)\right\|_\Delta\to 0$ implies $\left|H(x_i^k-x_i)\right|\to 0$ for all, it is a BK-space. □

3 The bases of sequence spaces $p_c(\Delta^{(m)})$ and $p_0(\Delta^{(m)})$

In this section, we shall give the Schauder bases for the spaces $p_c(\Delta^{(m)})$ and $p_0(\Delta^{(m)})$. First we define the Schauder bases. A sequence $\left\{b^{(k)}\right\}_{k\in\mathbb{N}}$ in a normed sequence space X is called a Schauder bases (or briefly bases), if for every $x \in X$ there is a unique sequence (λ_k) of scalars such that

$$\lim_{n\to\infty} \|x - (\lambda_0 x_0 + \lambda_1 x_1 + ... + \lambda_n x_n)\| = 0.$$

Theorem 2. *Define the sequence* $b^{(k)} = \left\{b_n^{(k)}\right\}_{n\in N}$ *of the elements of the space* $p_0(\Delta^{(m)})$ *for every fixed* $k \in \mathbb{N}$ *by*

$$b_n^{(k)} = \begin{cases} 0, \ (n < k) \\ \sum_{i=k}^{n} \binom{m+n-i-1}{n-i} (-1)^{i-k} \binom{i}{i-k}, \ (n \geq k) \end{cases} . \tag{7}$$

Then, the following assertions are true:

i. *The sequence* $\left\{b^{(k)}\right\}_{k\in\mathbb{N}}$ *is bases for the space* $p_0\left(\Delta^{(m)}\right)$ *and for any* $x \in p_0\left(\Delta^{(m)}\right)$ *has a unique representation of the form*

$$x = \sum_k (Hx)\, b^{(k)}.$$

ii. *The set* $\left\{t, b^{(1)}, b^{(2)}, ...\right\}$ *is a basis for the space* $p_c\left(\Delta^{(m)}\right)$ *and for any* $x \in p_c\left(\Delta^{(m)}\right)$ *has a unique representation of form*

$$x = lt + \sum_k \left[(Hx)_k - l\right] b^{(k)},$$

where $t = \{t_n\} = \sum_{k=0}^{n} \sum_{i=k}^{n} \binom{i}{i-k} (-1)^{i-k} \binom{m}{i-k}$, $(m, n \in \mathbb{N})$, $l = \lim_{k\to\infty} (Hx)_k$ *and* $H = P\Delta^{(m)}$.

Theorem 3. *The sequence spaces* $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ *and* $p_0\left(\Delta^{(m)}\right)$ *are linearly isomorphic to the spaces* l_∞, c *and* c_0 *respectively, i.e.,* $p_\infty\left(\Delta^{(m)}\right) \cong l_\infty$, $p_c\left(\Delta^{(m)}\right) \cong c$ *and* $p_0\left(\Delta^{(m)}\right) \cong c_0$.

Proof: To prove the fact $p_0\left(\Delta^{(m)}\right) \cong c_0$, we should show the existence of a linear bijection between the spaces $p_0(\Delta)$ and c_0. Consider the transformation T defined, with the notation (5), from $p_0\left(\Delta^{(m)}\right)$ to c_0 by $x \to y = Tx$. The linearity of T is clear. Further, it is trivial that $x = 0$ whenever $Tx = 0$ and hence T is injective. Let $y \in c_0$ and define the sequence $x = \{x_n\}$ by

$$x_n = \sum_{k=0}^{n} \left[\sum_{i=k}^{n} \binom{m+n-i-1}{n-i} (-1)^{i-k} \binom{i}{i-k}\right] y_k \tag{8}$$

for each m, $n \in \mathbb{N}$. Then,

$$\lim_{n\to\infty} (Hx)_k = \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{n-k} \Delta^{(m)} x_k$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{n-k} \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k-i}$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \left[\sum_{i=k}^{n} \binom{i}{i-k} (-1)^{i-k} \binom{m}{i-k}\right] x_k = \lim_{n\to\infty} y_n = 0$$

Thus, we have that $x \in p_0\left(\Delta^{(m)}\right)$. Consequently, T is surjective and is norm preserving. Hence, T is a linear bijection which implies that the spaces $p_0\left(\Delta^{(m)}\right)$ and c_0 are linearly isomorphic. In the same way, it can be shown that $p_\infty\left(\Delta^{(m)}\right)$ and $p_c\left(\Delta^{(m)}\right)$ are linearly isomorphic to l_∞ and c, respectively, and so we omit the detail. □

4 The α-, β- and γ- duals of the sequence spaces $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$

In this section, we state and prove the theorems determining the α-, β- and γ- duals of Pascal difference sequence spaces $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$. For the sequence spaces λ and μ , define the set $S(\lambda, \mu)$ by

$$S(\lambda, \mu) = \{z = (z_k) \in w : xz = (x_k z_k) \in \mu \text{ for all } x \in \lambda\}. \tag{9}$$

The α-, β- and γ- duals of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ are defined

$$\lambda^\alpha = S(\lambda, l_1), \lambda^\beta = S(\lambda, cs) \text{ and } \lambda^\gamma = S(\lambda, bs).$$

We shall begin with some lemmas due to Stieglitz and Tietz [21] that are needed in proving (4)-(6).

Lemma 1. $A \in (c_0 : l_1)$ if and only if

$$\sup_{K\in F}\sum_n \left|\sum_{k\in K} a_{nk}\right| < \infty. \tag{10}$$

Lemma 2. $A \in (c_0 : c)$ if and only if

$$\sup_n \sum_k |a_{nk}| < \infty, \tag{11}$$

$$\lim_{n\to\infty} a_{nk} - \alpha_k = 0. \tag{12}$$

Lemma 3. $A \in (c_0 : l_\infty)$ if and only if

$$\sup_n \sum_k |a_{nk}| < \infty. \tag{13}$$

Theorem 4. *Let $a = (a_k) \in w$ and the matrix $B = (b_{nk})$ by*

$$b_{nk} = \begin{cases} \sum_{i=k}^{n} \binom{m+n-i-1}{n-i}(-1)^{i-k}\binom{i}{i-k}a_n, & (0 \le k \le n) \\ 0, & (k > n) \end{cases}$$

for all m, $n \in \mathbb{N}$. Then the α- dual of the spaces $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$ is the set

$$D = \left\{a = (a_n) \in w : \sup_{K\in F}\sum_n \left|\sum_{k\in K} b_{nk}\right| < \infty\right\}.$$

Proof: Let $a = (a_n) \in w$ and consider the matrix B whose rows are the products of the rows of the matrix $H^{-1} = \left(P\Delta^{(m)}\right)^{-1} = (\Delta^{(m)})^{-1}P^{-1}$ and sequence $a = (a_n)$. Bearing in mind the relation (5), we immediately derive that

$$a_n x_n = \sum_{k=0}^{n}\left[\sum_{i=k}^{n}\binom{m+n-i-1}{n-i}(-1)^{i-k}\binom{i}{i-k}a_n\right] y_k = \sum_{k=1}^{n} b_{nk}y_k = (By)_n \tag{14}$$

$m, n \in \mathbb{N}$, we therefore observe Lemma 1 and by (14) that $ax = (a_n x_n) \in l_1$ whenever $x \in p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ and $p_0\left(\Delta^{(m)}\right)$ if and only if $By \in l_1$ whenever $y = (y_k) \in l_\infty$, c and c_0. This means that $a = (a_n) \in \left[p_\infty\left(\Delta^{(m)}\right)\right]^\alpha$, $\left[p_c\left(\Delta^{(m)}\right)\right]^\alpha$ and $\left[p_0\left(\Delta^{(m)}\right)\right]^\alpha$ if and only if $By \in (\left[p_\infty\left(\Delta^{(m)}\right)\right]^\alpha : l_1)$, $(\left[p_\infty\left(\Delta^{(m)}\right)\right]^\alpha : l_1)$ and $(\left[p_\infty\left(\Delta^{(m)}\right)\right]^\alpha : l_1)$ which yields the consequence that $\left[p_\infty\left(\Delta^{(m)}\right)\right]^\alpha = \left[p_c\left(\Delta^{(m)}\right)\right]^\alpha = \left[p_0\left(\Delta^{(m)}\right)\right]^\alpha = D$. □

Theorem 5. *Let $a = (a_k) \in w$ and the matrix $C = (c_{nk})$ by*

$$c_{nk} = \begin{cases} \sum_{i=k}^{n}\left[\sum_{j=k}^{n}\binom{m+i-j-1}{i-j}(-1)^{j-k}\binom{j}{j-k}\right] a_i, & (0 \le k \le n) \\ 0, & (k > n) \end{cases}$$

and define the sets c_1, c_2, c_3 and c_4 by

$$\begin{aligned} c_1 &= \left\{a = (a_k) \in w : \sup_n \sum_k |c_{nk}| < \infty\right\}, \\ c_2 &= a = (a_k) \in w : \lim_{n\to\infty} c_{nk} \text{ exists for each } k \in N, \\ c_3 &= \left\{a = (a_k) \in w : \lim_{n\to\infty}\sum_k |c_{nk}| = \sum_k \left|\lim_{n\to\infty} c_{nk}\right|\right\}, \end{aligned}$$

and

$$c_4 = \left\{ a = (a_k) \in w : \lim_{n\to\infty} \sum_k c_{nk} \text{ exists} \right\}.$$

Then $\left[p_\infty\left(\Delta^{(m)}\right)\right]^\beta$, $\left[p_c\left(\Delta^{(m)}\right)\right]^\beta$ *and* $\left[p_0\left(\Delta^{(m)}\right)\right]^\beta$ *is* $c_2 \cap c_3$, $c_1 \cap c_2 \cap c_4$ *and* $c_1 \cap c_2$, *respectively.*

Proof: We only give the proof the space $p_0\left(\Delta^{(m)}\right)$. Since the rest of the proof can be obtained by the same way for the spaces $p_c\left(\Delta^{(m)}\right)$ and $p_\infty\left(\Delta^{(m)}\right)$. Consider the equation

$$\begin{aligned} \sum_{k=1}^{n} a_k x_k &= \sum_{k=0}^{n}\sum_{i=0}^{k}\left[\sum_{j=i}^{k}\binom{m+k-j-1}{k-j}(-1)^{j-i}\binom{j}{j-i}\right] a_k y_i \\ &= \sum_{k=0}^{n}\left[\sum_{i=k}^{n}\left[\sum_{j=k}^{i}\binom{m+i-j-1}{i-j}(-1)^{j-k}\binom{j}{j-k}\right] a_i\right] y_k \\ &= (Cy)_n\,. \end{aligned} \tag{15}$$

Thus , we deduce from Lemma 2 and (15) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in h_0\left(\Delta^{(m)}\right)$ if and only if $Cy \in c$ whenever $y = (y_k) \in c_0$. That is to say that $a = (a_k) \in \left[p_0\left(\Delta^{(m)}\right)\right]^\beta$ if and only if $C \in (c_0 : c)$ which yields us $[p_0\left(\Delta^{(m)}\right)]^\beta = c_1 \cap c_2$. The β- dual of the sequence spaces $[p_c\left(\Delta^{(m)}\right)]$ and $[p_\infty\left(\Delta^{(m)}\right)]$ may be obtained in the similiar way, we omit their proofs. □

Theorem 6. *The* γ*- dual of the spaces* $p_\infty\left(\Delta^{(m)}\right)$, $p_c\left(\Delta^{(m)}\right)$ *and* $p_0\left(\Delta^{(m)}\right)$ *is the set* c_1.

Proof: This may be obtained in the similiar way used in the prof of Theorem (5) together with Lemma 3 instead of Lemma 2. So, we omit the detail. □

5 Matrix transformations on the sequence space $p_c\left(\Delta^{(m)}\right)$

We shall write throughout for brevity that

$$\tilde{a}_{nk} = \sum_{j=k}^{\infty}\binom{m+n-j-1}{n-j}(-1)^{j-k}\binom{j}{j-k} a_{nj},$$

and

$$\hat{g}_{nk} = \sum_{j=k}^{s}\binom{m+n-j-1}{n-j}(-1)^{j-k}\binom{j}{j-k} a_{nj}$$

for all $m, n, s \in \mathbb{N}$.

In this section, we give the characterization of the classes $\left(p_c\left(\Delta^{(m)}\right) : l_\infty\right)$ and $\left(h_c\left(\Delta^{(m)}\right) : c\right)$. Following theorems can be proved using standart methods, we omit the detail.

Theorem 7. $A \in \left(p_c\left(\Delta^{(m)}\right) : l_\infty\right)$ *if and only if*

$$\sup_n \sum_k |\hat{g}_{nk}| < \infty, \tag{16}$$

$$\lim_{n\to\infty}\sum_k \hat{g}_{nk} \text{ exists for all } m \in N, \tag{17}$$

$$\sup_{n\in N}\sum_k |\tilde{a}_{nk}| < \infty,\ (n \in N)\,, \tag{18}$$

and

$$\lim_{n\to\infty} \tilde{a}_{nk} \text{ exists for all } n \in N. \tag{19}$$

Theorem 8. $A \in \left(p_c\left(\Delta^{(m)}\right) : c\right)$ *if and only if* (16)-(19) *hold, and*

$$\lim_{n\to\infty} \sum_k \tilde{a}_{nk} = \alpha, \tag{20}$$

$$\lim_{n\to\infty} (\tilde{a}_{nk}) = \alpha_k,\ (k \in N) \tag{21}$$

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