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Stability Theory and the Existence of Hilfer Type Fractional Implicit Differential Equations with Boundary Conditions

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Abstract

In this paper, we consider the existence and Ulam stability of solutions to the boundary value problem for implicit differential equations involving Hilfer fractional derivative. With the help of properties of Hilfer fractional calculus and fixed point methods, we derive existence and stability results.

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1. Introduction

In [27], Vivek et al. studied dynamics and Ulam stability of pantograph equations with Hilfer fractional derivative. Dynamics and stability results for Hilfer fractional type thermistor problem are studied in [29]. Very recently, Vivek et al. studied the existence and stability results for implicit differential equations with nonlocal initial value problem which is mixed type integral equations; see [28]. In [16], Z. Gao and X. Yu investigated the existence and uniqueness of solutions to the nonlocal BVP for semi-linear differential equations involving Hilfer fractional derivative.Unfortunately, existence, uniqueness and Ulam stability of boundary value problem (BVP) for fractional implicit differential equations with Hilfer derivative is still not studied. The problem of the existence of solutions for FDEs with boundary conditions has been recently treated in the literature in [2, 4, 20, 18].
In this paper, we go on intending to study Hilfer fractional implicit differential equation with boundary conditions of the form

$$\begin{cases} D_{0^+}^{\alpha,\beta}x(t)=f(t,x(t),D_{0^+}^{\alpha,\beta}x(t)), & t\in J:=[0,T],\\ I_{0^+}^{1-\gamma}x(0)=a, \quad I_{0^+}^{1-\gamma}x(T)=b, & \gamma=\alpha+\beta-\alpha\beta, \end{cases} \tag{1.1}$$

where $D_{0^+}^{\alpha,\beta}$ is the Hilfer fractional derivative of order α and type β, $0<\alpha<1$, $0\leq\beta\leq1$, let R be a Banach space and $f:J\times R\times R\to R$ is a given continuous function. $I_{0^+}^{1-\gamma}$ is the left-sided mixed integral of order $1-\gamma$.
It is seen that equation (1.1) is equivalent to the integral equation

$$\begin{aligned} x(t)=&\frac{a}{\Gamma(\gamma)}t^{\gamma-1}+\left(b-a-I_{0^+}^{1-\beta(1-\alpha)}f\left(T,x(T),D_{0^+}^{\alpha,\beta}x(T)\right)\right)\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{t^{\gamma+2\beta-2}}{T^{2\beta-1}}\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\left(s,x(s),D_{0^+}^{\alpha,\beta}x(s)\right)ds. \end{aligned} \tag{1.2}$$

The proposed problem has to be satisfied the following hypotheses:

(C1) There exist $l,p,q\in C_{1-\gamma}[J,R]$ with $l^*=\sup_{t\in J}l(t)<1$ such that

$$|f(t,u,v)|\leq l(t)+p(t)\,|u|+q(t)\,|v|\,,\ \text{for } t\in J,\ u,v\in R.$$

(C2) There exist two positive constants $K>0$ and $L>0$ such that

$$|f(t,u,v)-f(t,\overline{u},\overline{v})|\leq K\,|u-\overline{u}|+L\,|v-\overline{v}|\,,$$

for any $u,v,\overline{u},\overline{v}\in R$ and $t\in J$.

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(C3) There exists an increasing function $\varphi \in C_{1-\gamma}[J,R]$ and there exists $\lambda_\varphi > 0$ such that for any $t \in J$

$$I_{0^+}^{\alpha}\varphi(t) \le \lambda_\varphi \varphi(t).$$

In the past decades, fractional differential equations (FDEs) have been generally used in the fields of physics, biology and engineering. There are many remarkable results for qualitative analysis and applications. For more details on the development of this issue, one can refer to books [21, 24, 26], literatures [2, 7, 8] and the suggestions therein. Since Hilfer initiated the notion of Hilfer fractional derivative which generalized the well-known Riemann-Liouville fractional derivative in [21] (these literatures supply the information about the uses of this derivative and how it arises). Progressively, people pay interest to study the differential equation with Hilfer fractional calculus; see [9, 10, 11, 22, 23, 25].

Alternatively, in 1940, Ulam offered the stability problem of the solutions of the functional equations concerning the stability of group homomorphisms. Subsequently, Hyers gave a positive response to the Ulam question in the background of Banach spaces, which was the opening significant advance in this area [17]. Since followed by, a large number of articles have been circulated in connection with many generalizations of the Ulam problem [5, 12, 13, 17]. Furthermore, some investigators used the fractional derivatives and studied the stability of Ulam-Hyers and Ulam-Hyers-Rassias [12, 19, 30, 31]. Even if there are several texts on stability through fractional derivatives, we consider development in this area is fairly hopeful.

We organize the present work as follows. In Section 2 we recall some useful preliminaries. In Section 3 we discuss the existence and uniqueness of solutions for the problem (1.1). Ulam stability type's results are discussed in Section 4. An example is constructed in Section 5 for illustrating the obtained results.

2. Prerequisites

In order to derive the existence of mild solutions of Hilfer fractional implicit differential equations with boundary conditions, we need the following basic definitions and lemmas. For more details, we refer to [1, 9, 10, 14, 21, 22, 23].

Definition 2.1. *The left-sided Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ of function $f(t)$ is defined by*

$$(I_{0^+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad (t>0),$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. *The left-sided Riemann-Liouville fractional derivative of order $\alpha \in [n-1,n)$, $n \in \mathbb{Z}^+$ of function $f(t)$ is defined by*

$$(D_{0^+}^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{\alpha-n+1} f(s)ds, \ (t>0),$$

Based on differentiating fractional integrals, a generalized definition called Hilfer fractional derivative can be introduced.

Definition 2.3. *The left-sided Hilfer fractional derivative of order $0<\alpha<1$ and $0 \le \beta \le 1$ of function $f(t)$ is defined by*

$$D_{0^+}^{\alpha,\beta} f(t) = \left(I_{0^+}^{\beta(1-\alpha)} D \left(I_{0^+}^{(1-\beta)(1-\alpha)} f\right)\right)(t),$$

where $D := \frac{d}{dt}$.

Remark 2.4. *The Hilfer fractional derivatives considered as an interpolator between the Riemann Liouville and Caputo derivative.*

1. *The operator $D_{0^+}^{\alpha,\beta}$ also can be rewritten as*

$$D_{0^+}^{\alpha,\beta} = I_{0^+}^{\beta(1-\alpha)} D I_{0^+}^{(1-\beta)(1-\alpha)} = I_{0^+}^{\beta(1-\alpha)} D_{0^+}^{\gamma}, \ \gamma = \alpha+\beta-\alpha\beta.$$

2. *Let $\beta = 0$, the left-sided Riemann-Liouville fractional derivative can be presented as $D_{0^+}^{\alpha} := D_{0^+}^{\alpha,0}$.*
3. *Let $\beta = 1$, the left-sided Caputo fractional derivative can be presented as ${}^cD_{0^+}^{\alpha} := I_{0^+}^{1-\alpha} D$.*

The following work spaces are useful.

Let $C[J,R]$ be the Banach space of all continuous functions from $[0,T]$ into R with the norm $\|x\|_c = \max\{|x(t)| : t \in [0,T]\}$. For $0 \le \gamma < 1$, we denote the space $C_\gamma[J,R]$ as

$$C_\gamma[J,R] := \{f(t) : [0,T] \to R | t^\gamma f(t) \in C[J,R]\},$$

where $C_\gamma[J,R]$ is the weighted space of the continuous functions f on the finite interval $[0,T]$.

Obviously, $C_\gamma[J,R]$ is the Banach space with the norm

$$\|f\|_{C_\gamma} = \|t^\gamma f(t)\|_C.$$

Meanwhile, $C_\gamma^n[J,R] := \left\{f \in C^{n-1}[J,R] : f^{(n)} \in C_\gamma[J,R]\right\}$ is the Banach space with the norm

$$\|f\|_{C_\gamma^n} = \sum_{i=0}^{n-1} \left\|f^k\right\|_C + \left\|f^{(n)}\right\|_{C_\gamma}, \ n \in \mathbb{N}.$$

Moreover, $C_\gamma^0[J,R] := C_\gamma[J,R]$.

In order to solve our problem, the following spaces are given

$$C_{1-\gamma}^{\alpha,\beta} = \left\{f \in C_{1-\gamma}[J,R], D_{0^+}^{\alpha,\beta} f \in C_{1-\gamma}[J,R]\right\}$$

and

$$C^{\gamma}_{1-\gamma} = \left\{ f \in C_{1-\gamma}[J,R], D^{\gamma}_{0^+} f \in C_{1-\gamma}[J,R] \right\}.$$

It is obvious that

$$C^{\gamma}_{1-\gamma}[J,R] \subset C^{\alpha,\beta}_{1-\gamma}[J,R].$$

Lemma 2.5. *Let $\alpha > 0$, $0 \le \beta \le 1$, so the homogeneous differential equation with Hilfer fractional order*

$$D^{\alpha,\beta} f(t) = 0,$$

has a solution:

$$f(t) = c_0 t^{\gamma-1} + c_1 t^{\gamma+2\beta-2} + c_2 t^{\gamma+2(2\beta)-3} + \cdots + c_n t^{\gamma+n(2\beta)-(n+1)}.$$

For the existence and Ulam stability of solutions for the problem (1.1), we need the following auxiliary lemma.

Lemma 2.6. *Let $f : J \times R \times R \to R$ be a function such that $f(\cdot, x(\cdot), D^{\alpha,\beta}_{0^+} x(\cdot)) \in C_{1-\gamma}[J,R]$ for any $x \in C_{1-\gamma}[J,R]$. A function $x \in C_{1-\gamma}[J,R]$ is a solution of the integral equation (1.2) if and only if x is a solution of the Hilfer fractional implicit fractional equation with boundary conditions*

$$D^{\alpha,\beta}_{0^+} x(t) = f\left(t, x(t), D^{\alpha,\beta}_{0^+} x(t)\right), \quad t \in [0,T], \tag{2.1}$$

$$I^{1-\gamma}_{0^+} x(0) = a, \quad I^{1-\gamma}_{0^+} x(T) = b, \quad \gamma = \alpha + \beta - \alpha\beta. \tag{2.2}$$

Proof. Assume x satisfies (1.2). Then Lemma 2.5 implies that

$$x(t) = c_0 t^{\gamma-1} + c_1 t^{\gamma+2\beta-2} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, x(s), D^{\alpha,\beta}_{0^+} x(s)\right) ds.$$

From (2.2), a simple calculation gives

$$c_0 = \frac{a}{\Gamma(\gamma)},$$
$$c_1 = \left(b - a - I^{1-\beta(1-\alpha)}_{0^+} f\left(T, x(T), D^{\alpha,\beta}_{0^+} x(T)\right)\right) \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)} \frac{1}{T^{2\beta-1}}.$$

Hence we get equation (1.2). Conversly, it is clear that if x satisfies equation (1.2), then equations (2.1)-(2.2) hold. □

For proving the existence solution of the problem (1.1), we need the following fixed point theorem.

Theorem 2.7. *(Schaefer's fixed point theorem) Let $J = [0,T]$ and $N : C_{1-\gamma}[J,R] \to C_{1-\gamma}[J,R]$ be a completely continuous operator. If the set*

$$\omega = \left\{ x \in C_{1-\gamma}[J,R] : x \in \delta(Nx) \quad \text{for some} \quad \delta \in [0,1] \right\}$$

is bounded, then N has at least a fixed point.

3. Existence theory

In this section, we are ready to present our existence results for problem (1.1) via fixed point methods.

Theorem 3.1. *Assume that (C1) holds. Then the problem (1.1) has at least one solution defined on J.*

Proof. Consider the operator $P : C_{1-\gamma}[J,R] \to C_{1-\gamma}[J,R]$ defined by

$$(Px)(t) = \begin{cases} \frac{a}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), D^{\alpha,\beta}_{0^+} x(s)) ds \\ + \left(b - a - I^{1-\beta(1-\alpha)}_{0^+} f(T, x(T), D^{\alpha,\beta}_{0^+} x(T))\right) \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)} \frac{t^{\gamma+2\beta-2}}{T^{2\beta-1}}. \end{cases} \tag{3.1}$$

The equation (3.1) can be written as

$$(Px)(t) = \begin{cases} \frac{a}{\Gamma(\gamma)} t^{\gamma-1} + \left(b - a - I^{1-\beta(1-\alpha)}_{0^+} K_x(T)\right) \\ \times \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)} \frac{t^{\gamma+2\beta-2}}{T^{2\beta-1}} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_x(s) ds, \end{cases} \tag{3.2}$$

where

$$\begin{aligned} K_x(t) &= f(t, x(t), D^{\alpha,\beta}_{0^+} x(t)) \\ &= f(t, x(t), K_x(t)) \\ &= D^{\alpha,\beta}_{0^+} x(t). \end{aligned}$$

It is obvious that the operator P is well defined.
Step 1: The operator P is continuous.

Let x_n be a sequence such that $x_n \to x$ in $C_{1-\gamma}[J,R]$. Then for each $t \in J$,

$$\begin{aligned}
&\left|((Px_n)(t)-(Px)(t))t^{1-\gamma}\right| \\
&\leq \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|K_{x_n}(s)-K_x(s)\right|ds + \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{t^{2\beta-1}}{T^{2\beta-1}} \\
&\times\left(\frac{1}{\Gamma(1-\beta(1-\alpha))}\int_0^T (T-s)^{(1-\beta(1-\alpha))-1}\left|K_{x_n}(s)-K_x(s)\right|ds\right) \\
&\leq \left(\frac{T^\alpha}{\Gamma(\alpha)}B(\gamma,\alpha)\right. \\
&\quad \left.+\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{T^{\beta(\alpha-1)+\gamma}}{\Gamma(1-\beta(1-\alpha))}B(\gamma,(1-\beta(1-\alpha)))\right)\left\|K_{x_n}(\cdot)-K_x(\cdot)\right\|_{C_{1-\gamma}},
\end{aligned}$$

where we use the formula

$$\begin{aligned}
\int_a^t (t-s)^{\alpha-1}|x(s)|\,ds &\leq \left(\int_a^t (t-s)^{\alpha-1}(t-s)^{\gamma-1}ds\right)\|x\|_{C_{1-\gamma}} \\
&= (t-a)^{\alpha+\gamma-1}B(\gamma,\alpha)\|x\|_{C_{1-\gamma}}.
\end{aligned}$$

Since f is continuous (i.e. K_x is continuous), by the Lebesgue Dominated Convergence Theorem, the above inequality implies

$$\|Px_n - Px\|_{C_{1-\gamma}} \to 0 \quad \text{as} \quad n\to\infty.$$

Step 2: The operator P maps bounded sets into bounded sets in $C_{1-\gamma}[J,R]$.
Indeed, it is enough to show that for $\eta>0$, there exists a positive constant l such that $x\in B_\eta = \left\{x\in C_{1-\gamma}[J,R] : \|x\|\leq\eta\right\}$, we have $\|(Px)\|_{C_{1-\gamma}} \leq l$.

$$\begin{aligned}
&\left|(Px)(t)t^{1-\gamma}\right| \\
&\leq \frac{a}{\Gamma(\gamma)} + \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{t^{2\beta-1}}{T^{2\beta-1}} \\
&\quad\left(|b-a| + \frac{1}{\Gamma(1-\beta(1-\alpha))}\int_0^T (T-s)^{(1-\beta(1-\alpha))-1}|K_x(s)|\,ds\right) \\
&\quad + \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|K_x(s)|\,ds \\
&:= A_1 + A_2,
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
|K_x(t)| &= |f(t,x(t),K_x(t))| \\
&\leq l(t) + p(t)|x(t)| + q(t)|K_x(t)| \\
&\leq l^* + p^*|x(t)| + q^*|K_x(t)| \\
&\leq \frac{l^*+p^*|x(t)|}{1-q^*}.
\end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we estimate A_1 and A_2 as follows:

$$\begin{aligned}
A_1 &= \frac{a}{\Gamma(\gamma)} + \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{t^{2\beta-1}}{T^{2\beta-1}} \\
&\quad\left(|b-a| + \frac{1}{\Gamma(1-\beta(1-\alpha))}\int_0^T (T-s)^{(1-\beta(1-\alpha))-1}|K_x(s)|\,ds\right) \\
&= \frac{a}{\Gamma(\gamma)} + \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}|b-a| + \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{T^{(1-\beta(1-\alpha))}}{\Gamma(2+\beta(\alpha-1))}\left(\frac{l^*}{1-q^*}\right) \\
&\quad + \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{T^{\beta(\alpha-1)+\gamma}}{\Gamma(1-\beta(1-\alpha))}\left(\frac{p^*}{1-q^*}\right)B(\gamma,(1-\beta(1-\alpha)))\|x\|_{C_{1-\gamma}}. \\
A_2 &= \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|K_x(s)|\,ds \\
&= \frac{T^{\alpha+1-\gamma}}{\Gamma(\alpha+1)}\left(\frac{l^*}{1-q^*}\right) + \frac{T^\alpha}{\Gamma(\alpha)}B(\gamma,\alpha)\left(\frac{p^*}{1-q^*}\right)\|x\|_{C_{1-\gamma}}.
\end{aligned}$$

Substituting A_1, A_2 in equation (3.3), we have

$$\begin{aligned}
&\left|(Px)(t)t^{1-\gamma}\right| \\
&\leq \frac{a}{\Gamma(\gamma)} + \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}|b-a| + \left(\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{T^{(1-\beta(1-\alpha))}}{\Gamma(2+\beta(\alpha-1))} + \frac{T^{\alpha+1-\gamma}}{\Gamma(\alpha+1)}\right) \\
&\times\left(\frac{l^*}{1-q^*}\right) + \left(\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{T^{\beta(\alpha-1)+\gamma}}{\Gamma(1-\beta(1-\alpha))}B(\gamma,(1-\beta(1-\alpha)))\right. \\
&\left.+\frac{T^\alpha}{\Gamma(\alpha)}B(\gamma,\alpha)\right)\times\left(\frac{p^*}{1-q^*}\right)\|x\|_{C_{1-\gamma}} \\
&:= l.
\end{aligned}$$

Step 3: The operator P maps bounded sets into equicontinuous set of $C_{1-\gamma,\log}[J,R]$.
Let $t_1, t_2 \in J, t_1 < t_2$ and $x \in B_\eta$. Using the fact f is bounded on the compact set $J \times B_\eta$.

$$\begin{aligned}
&\left|(Px)(t_2)(t_2)^{1-\gamma} - (Px)(t_1)(t_1)^{1-\gamma}\right| \\
&\leq \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\left[\frac{t_2^{2\beta-1}-t_1^{2\beta-1}}{T^{2\beta-1}}\right]\left((b-a)+\left|I_{0^+}^{1-\beta(-\alpha)}K_x(T)\right|\right) \\
&+\left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[t_1^{1-\gamma}(t_1-s)^{\alpha-1}-t_2^{1-\gamma}(t_2-s)^{\alpha-1}\right]K_x(s)ds\right| \\
&+\left|\frac{t_2^{1-\gamma}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}K_x(s)ds\right|
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. As a consequence of Step 1 to 3, together with Arzela-Ascoli theorem, we can conclude that $P : C_{1-\gamma}[J,R] \to C_{1-\gamma}[J,R]$ is continuous and completely continuous.
Step 4: A priori bounds.
Now it remains to show that the set

$$\omega = \left\{x \in C_{1-\gamma}[J,R] : x = \delta(Px), \quad 0 < \delta < 1\right\}$$

is bounded.
Let $x \in \omega$, $x = \delta(Px)$ for some $0 < \delta < 1$. Thus for each $t \in J$, we have,

$$\begin{aligned}
x(t) = \delta&\left[\frac{a}{\Gamma(\gamma)}t^{\gamma-1} + \left(b-a-I_{0^+}^{1-\beta(1-\alpha)}K_x(T)\right)\right. \\
&\left.\times\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{t^{\gamma+2\beta-2}}{T^{2\beta-1}} + \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}K_x(s)ds\right].
\end{aligned}$$

This implies by (C1) that for each $t \in J$, we have

$$\begin{aligned}
&\left|x(t)t^{1-\gamma}\right| \\
&\leq \left|(Px)(t)t^{1-\gamma}\right| \\
&\leq \frac{a}{\Gamma(\gamma)} + \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}|b-a| + \left(\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{T^{(1-\beta(1-\alpha))}}{\Gamma(2+\beta(\alpha-1))} + \frac{T^{\alpha+1-\gamma}}{\Gamma(\alpha+1)}\right) \\
&\times\left(\frac{l^*}{1-q^*}\right) + \left(\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{T^{\beta(\alpha-1)+\gamma}}{\Gamma(1-\beta(1-\alpha))}B(\gamma,(1-\beta(1-\alpha)))\right. \\
&\left.+\frac{T^\alpha}{\Gamma(\alpha)}B(\gamma,\alpha)\right)\times\left(\frac{p^*}{1-q^*}\right)\|x\|_{C_{1-\gamma}}
\end{aligned}$$

This shows that the set ω is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that P has a fixed point which is a solution of problem (1.1). □

The following lemma is based on Banach contraction principle.

Lemma 3.2. *Assume that (C2) holds. If*

$$\left(\frac{K}{1-L}\right)\left(\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{T^{\beta(\alpha-1)+\gamma}}{\Gamma(1-\beta(1-\alpha))}B(\gamma,(1-\beta(1-\alpha)))+\frac{T^\alpha}{\Gamma(\alpha)}B(\gamma,\alpha)\right)<1, \tag{3.5}$$

then the problem (1.1) *has a unique solution.*

4. Stability testing

In this section, we will study the Ulam-Hyers stability for the fractional implicit differential equations by means of Hilfer calculus. We adopt the definition in [19, 27] of the Ulam-Hyers (UH)stability, generalized UH stability, Ulam-Hyers-Rassias (UHR)stability and generalized UHR stability.

Definition 4.1. *Equation (1.1) is UH stable if there exists a real number $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in C^{\gamma}_{1-\gamma}[J,R]$ of the inequality*

$$\left| D^{\alpha,\beta}_{0^+} z(t) - f(t, z(t), D^{\alpha,\beta}_{0^+} z(t)) \right| \leq \varepsilon, \quad t \in J, \tag{4.1}$$

there exists a solution $x \in C^{\gamma}_{1-\gamma}[J,R]$ of equation (1.1) with

$$|z(t) - x(t)| \leq C_f \varepsilon, \quad t \in J.$$

Definition 4.2. *Equation (1.1) is generalized UH stable if there exists $\psi_f \in C([0,\infty),[0,\infty))$, $\psi_f(0) = 0$ such that for each solution $z \in C^{\gamma}_{1-\gamma}[J,R]$ of the inequality*

$$\left| D^{\alpha,\beta}_{0^+} z(t) - f(t, z(t), D^{\alpha,\beta}_{0^+} z(t)) \right| \leq \varepsilon, \quad t \in J, \tag{4.2}$$

there exists a solution $x \in C^{\gamma}_{1-\gamma}[J,R]$ of equation (1.1) with

$$|z(t) - x(t)| \leq \psi_f \varepsilon, \quad t \in J.$$

Definition 4.3. *Equation (1.1) is UHR stable with respect to $\varphi \in C_{1-\gamma}[J,R]$ if there exists a real number $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in C^{\gamma}_{1-\gamma}[J,R]$ of the inequality*

$$\left| D^{\alpha,\beta}_{0^+} z(t) - f(t, z(t), D^{\alpha,\beta}_{0^+} z(t)) \right| \leq \varepsilon \varphi(t), \quad t \in J, \tag{4.3}$$

there exists a solution $x \in C^{\gamma}_{1-\gamma}[J,R]$ of equation (1.1) with

$$|z(t) - x(t)| \leq C_f \varepsilon \varphi(t), \quad t \in J.$$

Definition 4.4. *Equation (1.1) is generalized UHR stable with respect to $\varphi \in C_{1-\gamma}[J,R]$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $z \in C^{\gamma}_{1-\gamma}[J,R]$ of the inequality*

$$\left| D^{\alpha,\beta}_{0^+} z(t) - f(t, z(t), D^{\alpha,\beta}_{0^+} z(t)) \right| \leq \varphi(t), \quad t \in J, \tag{4.4}$$

there exists a solution $x \in C^{\gamma}_{1-\gamma}[J,R]$ of equation (1.1) with

$$|z(t) - x(t)| \leq C_{f,\varphi} \varphi(t), \quad t \in J.$$

Remark 4.5. *A function $z \in C^{\gamma}_{1-\gamma}[J,R]$ is a solution of the inequality*

$$\left| D^{\alpha,\beta}_{0^+} z(t) - f(t, z(t), D^{\alpha,\beta}_{0^+} z(t)) \right| \leq \varepsilon, \quad t \in J,$$

if and only if there exists a function $g \in C^{\gamma}_{1-\gamma}[J,R]$ (which depend on solution x) such that

1. *$|g(t)| \leq \varepsilon$, $t \in J$;*
2. *$D^{\alpha,\beta}_{0^+} z(t) = f(t, z(t), D^{\alpha,\beta}_{0^+} z(t)) + g(t)$, $t \in J$.*

Remark 4.6. *Noticeably*

1. *Definition 4.1$\Rightarrow$ Definition 4.2.*
2. *Definition 4.3$\Rightarrow$ Definition 4.4.*

Lemma 4.7. *Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$, if a function $z \in C^{\gamma}_{1-\gamma}[J,R]$ is a solution of the inequality (4.1), then z is a solution of the following integral inequality*

$$\left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha} K_z(s) ds \right| \leq \left(\frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta - 1)} \frac{T^{\beta(\alpha+1)+\gamma-1}}{T^{2\beta-1}} + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \right) \varepsilon, \tag{4.5}$$

where

$$A_z = \frac{a}{\Gamma(\gamma)} t^{\gamma-1} + \left(b - a - I^{1-\beta(1-\alpha)}_{0^+} K_x(T) \right) \times \frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta - 1)} \frac{t^{\gamma+2\beta-2}}{T^{2\beta-1}}.$$

Proof. Indeed by Remark 4.5, we have that

$$\begin{aligned} D^{\alpha,\beta}_{0^+} z(t) &= f(t,z(t),\ D^{\alpha,\beta}_{0^+} z(t)) + g(t), \\ &= K_z(t) + g(t). \end{aligned}$$

From (1.2), it follows that

$$\begin{aligned} &\left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha} K_z(s)ds \right| \\ &\leq \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)} \frac{t^{\gamma+2\beta-2}}{T^{2\beta-1}} \left| I^{1-\beta(1-\alpha)}_{0^+} g(T) \right| + \left| I^{\alpha}_{0^+} g(t) \right| \\ &\leq \left(\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)} \frac{T^{\beta(\alpha+1)+\gamma-1}}{T^{2\beta-1}} + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \right) \varepsilon. \end{aligned}$$

□

We have analogous observation for the solutions of the inequality (4.3) and (4.4). In order to derive the stability results, we need the following lemma.

Lemma 4.8. *[28] Let $v : [0,T] \to [0,\infty)$ be a real function and $w(\cdot)$ is a nonnegative, locally integrable function on $[0,T]$ and there are constants $a > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \leq w(t) + a \int_0^t \frac{v(s)}{(t-s)^{\alpha}} ds.$$

Then there exists a constant $K = K(\alpha)$ such that

$$v(t) \leq w(t) + Ka \int_0^t \frac{w(s)}{(t-s)^{\alpha}} ds,$$

for every $t \in [0,T]$.

We are ready to prove our stability results for problem (1.1).

Theorem 4.9. *Assume that (C2) and (3.5) hold. Then the problem (1.1) is UH stable.*

Proof. Let $\varepsilon > 0$ and let $z \in C^{\gamma}_{1-\gamma}[J,R]$ be a function which satisfies the inequality (4.1) and let $x \in C^{\gamma}_{1-\gamma}[J,R]$ be the unique solution of the following Hilfer fractional implicit BVP

$$\begin{aligned} &D^{\alpha,\beta}_{0^+} x(t) = f(t,x(t),D^{\alpha,\beta}_{0^+} x(t)),\ t \in J := [0,T], \\ &I^{1-\gamma}_{0^+} x(0) = I^{1-\gamma}_{0^+} z(0) = a, \quad I^{1-\gamma}_{0^+} x(T) = I^{1-\gamma}_{0^+} z(T) = b, \end{aligned}$$

where $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta - \alpha\beta$.
Using Lemma 2.6, we obtain

$$x(t) = A_x + \frac{1}{\Gamma(\alpha)} \int_0^T (t-s)^{\alpha-1} K_x(s)ds$$

where

$$A_x = \frac{a}{\Gamma(\gamma)} t^{\gamma-1} + \left(b - a - I^{1-\beta(1-\alpha)}_{0^+} K_x(T) \right) \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)} \frac{t^{\gamma+2\beta-2}}{T^{2\beta-1}}.$$

On the other hand, if $I^{1-\gamma}_{0^+} x(T) = I^{1-\gamma}_{0^+} z(T)$ and $I^{1-\gamma}_{0^+} x(0) = I^{1-\gamma}_{1^+} x(0)$, then $A_x = A_z$.
Indeed,

$$\begin{aligned} |A_x - A_z| &\leq \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)} \frac{t^{\gamma+2\beta-2}}{T^{2\beta-1}} \left(\frac{K}{1-L} \right) I^{1-\beta(1-\alpha)}_{0^+} |x(T) - z(T)| \\ &= 0. \end{aligned}$$

Thus, $A_x = A_z$.
Then, we have

$$x(t) = A_z + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_x(s)ds.$$

By integration of the inequality (4.1) and using Lemma 4.7, we obtain

$$\left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha} K_z(s)ds \right| \leq \left(\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)} \frac{T^{\beta(\alpha+1)+\gamma-1}}{T^{2\beta-1}} + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \right) \varepsilon,$$

For sake of brevity, we take $U := \left(\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{T^{\beta(\alpha+1)+\gamma-1}}{T^{2\beta-1}}+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right)$. We have

$$
\begin{aligned}
|z(t)-x(t)| &\leq \left|z(t)-A_z-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}K_z(s)ds\right| \\
&\quad + \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[K_z(s)-K_x(s)\right]ds\right| \\
&\leq U\varepsilon + \left(\frac{K}{1-L}\right)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|z(s)-x(s)|\,ds,
\end{aligned}
$$

and applying Lemma 4.8, we obtain

$$
\begin{aligned}
|z(t)-x(t)| &\leq U\left[1+\frac{KT^{\alpha}\nu}{(1-L)\Gamma(\alpha+1)}\right]\varepsilon \\
&:= C_f\varepsilon,
\end{aligned}
$$

where $\nu = \nu(\alpha)$ is a constant which completes the proof of the theorem. Moreover, if we set $\psi(\varepsilon) = C_f\varepsilon$; $\psi(0) = 0$, then the problem (1.1) is generalized UH stable. □

Remark 4.10. *Under the assumptions of Theorem 4.9, we consider problem (1.1) and hypothesis (C3). One can repeat the same process to verify that problem (1.1) is UHR stable.*

In this section, we present an to indicate how our theorems can be applied to concrete problems.

5. An illustrative example

Let us consider consider the following Hilfer fractional implicit BVP

$$
D_{0^+}^{\alpha,\beta}x(t) = \frac{1}{10}\left(\frac{1}{1+|x(t)|+D_{0^+}^{\alpha,\beta}|x(t)|}\right), \quad t \in J := [0,1], \tag{5.1}
$$

$$
I_{0^+}^{1-\gamma}x(0) = 1, \quad I_{0^+}^{1-\gamma}x(1) = 2, \quad \gamma = \alpha+\beta-\alpha\beta. \tag{5.2}
$$

Notice that this problem is a particular case of (1.1).
Set

$$
f(t,u,v) = \frac{1}{10}\left(\frac{1}{1+|u|+|v|}\right),
$$

for $u, v \in R$, and $t \in J$.
Clearly, the function f satisfies condition of Theorem 3.1.
For each $u, v, \overline{u}, \overline{v} \in R$ and $t \in J$.

$$
|f(t,u,v)-f(t,\overline{u},\overline{v})| \leq \frac{1}{10}|u-\overline{u}|+\frac{1}{10}|v-\overline{v}|.
$$

Hence, the condition (C2) is satisfied with $K = L = \frac{1}{10}$. Here $T = 1$.
If $\alpha = \frac{2}{3}$, $\beta = \frac{1}{2}$ and choose $\gamma = \frac{5}{6}$.
Thus, condition from (3.5)

$$
\left(\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{T^{\beta(\alpha-1)+\gamma}}{\Gamma(1-\beta(1-\alpha))}B(\gamma,(1-\beta(1-\alpha)))+\frac{T^{\alpha}}{\Gamma(\alpha)}B(\gamma,\alpha)\right) \approx 0.2646 < 1.
$$

It follows from Lemma 3.2 that the problem (5.1)-(5.2) has a unique solution. It is obvious that all the assumptions in Theorem 4.9 are satisfied. Thus, the problem (5.1)-(5.2) is UH stable.

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