A Note On Double Walsh—Fourier Coefficients of Functions of Generalized Wiener Class

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Abstract

In this note, we have estimated the order of magnitude of double Walsh—Fourier coefficients of functions of the class \((A^1, A^2)BV(p(n) \uparrow \infty, \phi, [0, 1]^2)\).

Keywords: double Walsh—Fourier coefficients, functions of the class \((A^1, A^2)BV(p(n) \uparrow \infty, \phi, [0, 1]^2)\).

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1. Introduction

In 2000, Akhobadze [1] introduced the generalized Wiener class \(BV(p(n) \uparrow p, \phi)\), where \(1 \leq p \leq \infty\). This class is further generalized to the class \(ABV(p(n) \uparrow p, \phi)\) in [5] and the order of magnitude of single Walsh—Fourier coefficients of functions of the class \(ABV(p(n) \uparrow \infty, \phi, [0, 1])\) is estimated in [2]. Recently in [6], introducing the generalized Wiener class \((A^1, A^2)BV(p(n) \uparrow p, \phi, [0, 2\pi]^2)\), where \(1 \leq p \leq \infty\), the order of magnitude of double Walsh—Fourier coefficients of functions of the class \((A^1, A^2)BV(p(n) \uparrow \infty, \phi, [0, 2\pi]^2)\) is estimated. Here, we estimate the order of magnitude of double Walsh—Fourier coefficients of functions of the class \((A^1, A^2)BV(p(n) \uparrow \infty, \phi, [0, 1]^2)\).

2. Notation and definitions

In the sequel \(I = [0, 1)\). \(N = \{0, 1, 2, \cdots\}\), \(L\) is a class of non-decreasing sequences \(\Lambda = \{\lambda_n\}_{n=1}^\infty\) of positive numbers such that \(\sum_n \frac{1}{\lambda_n}\) diverges, and \(\phi(n)\) is a real sequence such that \(\phi(1) \geq 2\) and \(\phi(n) \to \infty\) as \(n \to \infty\).

Consider function \(f\) on \(\mathbb{R}^2\). For \(k = 1\) and \(I = [a, b]\), define \(\Delta f(I) = f(b) - f(a)\). For \(k = 2, I = [a, b]\) and \(J = [c, d]\), define

\[
\Delta f_{(a,c)} = f(I \times J) - f(I, d) - f(I, c) = f(b, d) - f(a, d) - f(b, c) + f(a, c).
\]

Definition 2.1. Given \(\Lambda = \{(\lambda^1_n)_{n=1}^\infty\}, \Lambda' = \{(\lambda^2_n)_{n=1}^\infty\} \subseteq \Lambda\), \(n \geq 1\) for \(r = 1, 2, 1 \leq p(n) \uparrow p\) as \(n \to \infty\) and \(1 \leq p \leq \infty\), a measurable function \(f\) defined on a rectangle \(R^2 : = [a, b] \times [c, d]\) is said to be of \(p(n) - \Lambda\)-bounded variation (that is, \(f \in ABV(p(n) \uparrow p, \phi, R^2)\)) if

\[
V_{\Lambda, \phi(n)}(f, R^2) = \sup_{n \geq 1} \sup_{I_1, I_2} \left\{ V_{\Lambda', \phi(n)}(f, \{I_1\}, \{I_2\}) : \delta(\{I_1\}, \{I_2\}) \geq \frac{(b-a)(c-d)}{\phi(n)} \right\} < \infty,
\]

where

\[
V_{\Lambda, \phi(n)}(f, \{I_1\}, \{I_2\}) = \left( \sum_I \frac{|f(I_1 \times I_2)|^{p(n)}}{\lambda^1_I \lambda^2_J} \right)^{\frac{1}{p(n)}},
\]

in which \(I_1\) and \(I_2\) are finite collections of non-overlapping subintervals in \([a, b]\) and \([c, d]\), respectively, and

\[
\delta(\{I_1\}, \{I_2\}) = \delta(\{|x_{i-1}, x_i|\}, \{|y_{j-1}, y_j|\}) = \inf_{i,j} |(x_i - x_{i-1}) \times (y_j - y_{j-1})|.
\]
Consider a function $f : \mathbb{T}^2 \to \mathbb{R}$ defined by $f(x,y) = g(x) + h(y)$, where $g$ and $h$ are any two arbitrary need not be bounded (or need not be measurable) functions from $\mathbb{T}$ into $\mathbb{R}$. Then $V_{\lambda}(f, \mathbb{T}^2) = 0$. Thus, a function $f \in \Lambda^* BV(p(n) \uparrow p, \phi, R^2)$ need not be bounded (or need not be measurable).

This class is further generalized to the class $\Lambda^* BV(p(n) \uparrow p, \phi, R^2)$ in the sense of Hardy as follows.

**Definition 2.2.** If $f \in \Lambda^* BV(p(n) \uparrow p, \phi, R^2)$ is such that the marginal functions $f(.,c) \in \Lambda^* BV(p(n) \uparrow p, \phi, [a,b])$ and $f(a,.) \in \Lambda^* BV(p(n) \uparrow p, \phi, [c,d])$ (see [5, Definition 1.1, p. 215]) for the definition of $p(n) - \Lambda$–bounded variation over $[a,b]$ then $f$ is said to be of $p(n) - \Lambda^*$–bounded variation (that is, $f \in \Lambda^* BV(p(n) \uparrow p, \phi, R^2)$).

If $f \in \Lambda^* BV(p(n) \uparrow p, \phi, R^2)$ then $f$ is bounded and each of the marginal functions $f(.,s) \in \Lambda^* BV(p(n) \uparrow p, \phi, [a,b])$ and $f(t,.) \in \Lambda^* BV(p(n) \uparrow p, \phi, [c,d])$, where $s \in [c,d]$ and $t \in [a,b]$ are fixed [6, p. 436].

Note that, for $\Lambda^1 = \Lambda^2 = \{1\}$ (that is, $\lambda^1_n = \lambda^2_n = 1$, for all $n$), the classes $\Lambda^* BV(p(n) \uparrow p, \phi, R^2)$ and $\Lambda^* BV(p(n) \uparrow p, \phi, R^2)$ reduce to the classes $\Lambda^1 BV(p(n) \uparrow p, \phi, R^2)$ and $\Lambda^2 BV(p(n) \uparrow p, \phi, R^2)$, respectively. For $p(n) = p$, for all $n$, the classes $\Lambda^* BV(p(n) \uparrow p, \phi, R^2)$ and $\Lambda^1 BV(p(n) \uparrow p, \phi, R^2)$ reduce to the classes $\Lambda^1 BV(p \uparrow R^2)$ [3, Definition 4.2, p. 54] and $\Lambda^1 BV(p \uparrow R^2)$, respectively. For $p = \infty$, the classes $\Lambda^* BV(p(n) \uparrow p, \phi, R^2)$ and $\Lambda^1 BV(p(n) \uparrow p, \phi, R^2)$ reduce to the classes $\Lambda^1 BV(p \uparrow \infty, \phi, R^2)$ and $\Lambda^1 BV(p \uparrow \infty, \phi, R^2)$, respectively.

For $\Lambda^1 = \Lambda^2 = \{1\}$ and $p = \infty$, the classes $\Lambda^* BV(p(n) \uparrow p, \phi, R^2)$ and $\Lambda^1 BV(p(n) \uparrow p, \phi, R^2)$ reduce to the classes $\Lambda^1 BV(p \uparrow \infty, \phi, R^2)$ and $\Lambda^1 BV(p \uparrow \infty, \phi, R^2)$, respectively.

The Walsh orthonormal system $\{\psi_m(x) : m \in \mathbb{N}_0\}$ on the unit interval $\mathbb{I}$ in the Paley enumeration is defined as follows.

Let

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}), \\ -1, & \text{if } x \in \left[\frac{1}{2}, 1\right); \end{cases}$$

and extend $r_0(x)$ for the half-line $[0,\infty)$ with period 1.

The Rademacher orthonormal system $\{r_k(x) : k \in \mathbb{N}_0\}$ is defined as

$$r_k(x) = r_0(2^k x), \quad k = 1, 2, \ldots; \ x \in \mathbb{I}.$$

If

$$m = \sum_{k=0}^\infty m_k 2^k, \quad \text{each } m_k = 0 \text{ or } 1,$$

is the binary decomposition of $m \in \mathbb{N}_0$, then

$$\psi_m(x) = \prod_{k=0}^\infty r_k^m(x), \quad x \in \mathbb{I},$$

is called the $m^{th}$ Walsh function in the Paley enumeration.

In particular, we have

$$\psi_0(x) = 1 \quad \text{and} \quad \psi_{2^n}(x) = r_m(x), \ m \in \mathbb{N}_0.$$

Any $x \in \mathbb{I}$ can be written as

$$x = \sum_{k=0}^\infty x_k 2^{-(k+1)}, \quad \text{each } x_k = 0 \text{ or } 1.$$

For any $x \in \mathbb{I} \setminus Q$, there is only one expression of this form, where $Q$ is a class of dyadic rationals in $\mathbb{I}$. When $x \in Q$ there are two expressions of this form, one which terminates in 0’s and one which terminates in 1’s.

For any $x, y \in \mathbb{I}$ their dyadic sum is defined as

$$x + y = \sum_{k=0}^\infty |x_k - y_k| 2^{-(k+1)}.$$

Observe that, for each $m \in \mathbb{N}_0$, we have

$$\psi_m(x + y) = \psi_m(x) \psi_m(y), \ x, y \in \mathbb{I}, \ x + y \notin Q.$$

For a real-valued function $f \in L^1(\mathbb{T}^2)$, where $f$ is 1–periodic in each variable, its double Walsh–Fourier series is defined as

$$f(x) = \sum_{k \in \mathbb{N}_0^2} \hat{f}(k) \psi_m(x) \psi_n(y) = \sum_{m \in \mathbb{N}_0} \sum_{n \in \mathbb{N}_0} \hat{f}(m,n) \psi_m(x) \psi_n(y),$$

where

$$\hat{f}(k) = \hat{f}(m,n) = \int_{\mathbb{T}^2} f(x,y) \psi_m(x) \psi_n(y) \ dx \ dy$$

denotes the $k^{th}$ Walsh–Fourier coefficient of $f$. 

3. Results

We prove the following results.

**Theorem 3.1.** If \( f \in \mathcal{A} BV(p(n) \uparrow \infty, \varphi, \mathbb{T}^2) \cap L^\infty(\mathbb{T}^2) \), then

\[
\hat{f}(2^u, 2^v) = O\left( \frac{1}{\left( \frac{n}{2^u + 1} \right)^{\kappa(n) + \mu(n)}} \right),
\]

where

\[
\tau(r) = \min\{s : s \in \mathbb{N}, \varphi(s) \geq r\}, r \geq 1.
\]

**Corollary 3.2.** If \( f \in \mathcal{A}^* BV(p(n) \uparrow \infty, \varphi, \mathbb{T}^2) \), then (3.1) holds true.

**Corollary 3.3.** If \( f \in BV_{\varphi}(p(n) \uparrow \infty, \varphi, \mathbb{T}^2) \), then

\[
\hat{f}(2^u, 2^v) = O\left( \frac{1}{(2^{u+1})^{\kappa(n) + \mu(n)}} \right),
\]

where \( \tau(2^{u+1}) \) is defined as in (3.2).

Corollary 3.3 follows from Theorem 3.1.

4. Proof of the results

**Proof of Theorem 3.1.** For fixed \( u, v \in \mathbb{N}_0 \), let \( h_1 = \frac{1}{2^u+1} \) and \( h_2 = \frac{1}{2^v+1} \). Take

\[
g(x, y) = f(x, y) - f\left(x, y + \frac{1}{2^u+1}\right) - f\left(x + \frac{1}{2^v+1}, y\right) + f\left(x + \frac{1}{2^v+1}, y + \frac{1}{2^u+1}\right),
\]

for all \( (x, y) \in \mathbb{T}^2 \).

For \( m = 2^u \) and \( n = 2^v \), \( \psi_m(h_1) = \psi_n(h_2) = -1 \) and \( \psi_m\left(\frac{1}{2^u}\right) = \psi_n\left(\frac{1}{2^v}\right) = 1 \) imply that

\[
\hat{g}(m, n) = \hat{f}(m, n) - \psi_m\left(\frac{1}{2^u+1}\right) \hat{f}(m, n) - \psi_n\left(\frac{1}{2^v+1}\right) \hat{f}(m, n) + \psi_m\left(\frac{1}{2^v+1}\right) \psi_n\left(\frac{1}{2^u+1}\right) \hat{f}(m, n) = 4\hat{f}(m, n)
\]

and

\[
|4\hat{f}(m, n)| \leq \int_{\mathbb{T}^2} \left| f(x, y) - f\left(x, y + \frac{1}{2^u+1}\right) - f\left(x + \frac{1}{2^v+1}, y\right) + f\left(x + \frac{1}{2^v+1}, y + \frac{1}{2^u+1}\right) \right| \, dx \, dy
\]

\[
= \int_{\mathbb{T}^2} \left| f\left(x + \frac{1}{2^v+1}, y + \frac{1}{2^u+1}\right) - f\left(x, y + \frac{1}{2^u+1}\right) - f\left(x, y + \frac{1}{2^v+1}\right) + f\left(x + \frac{1}{2^v+1}, y + \frac{1}{2^u+1}\right) \right| \, dx \, dy
\]

\[
= \int_{\mathbb{T}^2} \left| f\left(x + \frac{1}{2^v+1}, y + \frac{2}{2^u+1}\right) - f\left(x + \frac{2}{2^u+1}, y + \frac{1}{2^v+1}\right) - f\left(x + \frac{2}{2^u+1}, y + \frac{1}{2^v+1}\right) + f\left(x + \frac{1}{2^v+1}, y + \frac{2}{2^u+1}\right) \right| \, dx \, dy.
\]

Similarly, we get

\[
|4\hat{f}(m, n)| \leq \int_{\mathbb{T}^2} \left| f\left(x + \frac{4}{2^u+1}, y + \frac{4}{2^v+1}\right) - f\left(x + \frac{4}{2^v+1}, y + \frac{5}{2^u+1}\right) - f\left(x + \frac{5}{2^v+1}, y + \frac{4}{2^u+1}\right) + f\left(x + \frac{5}{2^u+1}, y + \frac{4}{2^v+1}\right) \right| \, dx \, dy
\]

and in general we have

\[
|4\hat{f}(m, n)| \leq \int_{\mathbb{T}^2} |\Delta f_{jk}(x, y)| \, dx \, dy,
\]

where

\[
\Delta f_{jk}(x, y) = f\left(x + \frac{2j}{2^u+1}, y + \frac{2k}{2^v+1}\right) - f\left(x + \frac{j}{2^v+1}, y + \frac{2k+1}{2^u+1}\right) - f\left(x + \frac{2j+1}{2^u+1}, y + \frac{k}{2^v+1}\right) + f\left(x + \frac{2j+1}{2^u+1}, y + \frac{2k+1}{2^v+1}\right).
\]
for all $j = 1, \ldots, 2^n$ and for all $k = 1, \ldots, 2^i$.

Dividing both sides of the above inequality by $\lambda_j^1 \lambda_k^2$ and then summing over $j = 1$ to $2^n$ and $k = 1$ to $2^i$, we get

$$4|f(2^n, 2^i)| \left( \sum_{j=1}^{2^n} \sum_{k=1}^{2^i} \frac{1}{\lambda_j^1 \lambda_k^2} \right) \leq \int \int_{\mathbb{R}^2} \left( \sum_{j=1}^{2^n} \sum_{k=1}^{2^i} \frac{|\Delta f_{jk}(x, y)|}{\lambda_j^1 \lambda_k^2} \right) \frac{1}{p(\tau(2^n+i))} \frac{1}{p(\tau(2^n+i))} \frac{1}{p(\tau(2^n+i))} \ dx \ dy,$$

where $q(\tau(2^n+i))$ is the index conjugate to $p(\tau(2^n+i))$.

Applying Hölder’s inequality on the right side of the above inequality, we get

$$4|\hat{f}(2^n, 2^i)| \left( \sum_{j=1}^{2^n} \sum_{k=1}^{2^i} \frac{1}{\lambda_j^1 \lambda_k^2} \right) \leq \int \int_{\mathbb{R}^2} \left( \sum_{j=1}^{2^n} \sum_{k=1}^{2^i} \frac{|\Delta f_{jk}(x, y)|p(\tau(2^n+i))}{\lambda_j^1 \lambda_k^2} \right) \frac{1}{p(\tau(2^n+i))} \frac{1}{p(\tau(2^n+i))} \ dx \ dy.$$

Hence,

$$4|\hat{f}(2^n, 2^i)| \left( \sum_{j=1}^{2^n} \sum_{k=1}^{2^i} \frac{1}{\lambda_j^1 \lambda_k^2} \right) \leq \int \int_{\mathbb{R}^2} \left( \sum_{j=1}^{2^n} \sum_{k=1}^{2^i} \frac{|\Delta f_{jk}(x, y)|p(\tau(2^n+i))}{\lambda_j^1 \lambda_k^2} \right) \frac{1}{p(\tau(2^n+i))} \frac{1}{p(\tau(2^n+i))} \ dx \ dy. \quad (4.2)$$

For any $x, y \in \mathbb{R}$, all these points $x + 2jh_1$, $x + (2j + 1)h_1$, for $j = 1, \ldots, 2^n$, and $y + 2kh_2$, $y + (2k + 1)h_2$, for $k = 1, \ldots, 2^i$, lie in the interval of length 1. Thus, $f \in \Lambda^+ BV(p(n) \uparrow \infty, \varphi, \tau^2)$ implies

$$\left( \sum_{j=1}^{2^n} \sum_{k=1}^{2^i} \frac{|\Delta f_{jk}(x, y)|p(\tau(2^n+i))}{\lambda_j^1 \lambda_k^2} \right) \frac{1}{p(\tau(2^n+i))} = O(1).$$

This together with above inequality (4.2) imply that

$$|\hat{f}(2^n, 2^i)| = O \left( \frac{1}{\left( \sum_{j=1}^{2^n} \sum_{k=1}^{2^i} \frac{1}{\lambda_j^1 \lambda_k^2} \right)^{p(\tau(2^n+i))}} \right).$$

This completes the proof of the theorem.

**Proof of Corollary 3.2.** Since $f \in \Lambda^+ BV(p(n) \uparrow \infty, \varphi, \tau^2)$ is bounded [6, p. 436] and $\Lambda^+ BV(p(n) \uparrow \infty, \varphi, \tau^2) \subset \Lambda BV(p(n) \uparrow \infty, \varphi, \tau^2)$, the Corollary 3.2 follows from Theorem 3.1.

One can extend these results for functions of $N$-variables ($N > 2$) analogously to the above-mentioned results for functions of two variables.

References