Konuralp Journal of Mathematics, 7 (2) (2019) 288-291

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

A Note On Double Walsh−Fourier Coefficients of Functions of Generalized Wiener Class

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Abstract

In this note, we have estimated the order of magnitude of double Walsh−Fourier coefficients of functions of the class $(\Lambda^1,\Lambda^2)BV(p(n)\uparrow\infty,\varphi,[0,1]^2)$.

Keywords: *double Walsh−Fourier coefficients, functions of the class* $(\Lambda^1,\Lambda^2)BV(p(n)\uparrow\infty,\varphi,[0,1]^2)$.
2010 Mathematics Subject Classification: *42C10, 42B05, 26B30, 26D15.*

1. Introduction

In 2000, Akhobadze [1] introduced the generalized Wiener class $BV(p(n)\uparrow p,\varphi)$, where $1\le p\le\infty$. This class is further generalized to the class $\Lambda BV(p(n)\uparrow p,\varphi)$ in [5] and the order of magnitude of single Walsh−Fourier coefficients of functions of the class $\Lambda BV(p(n)\uparrow\infty,\varphi,[0,1])$ is estimated in [2]. Recently in [6], introducing the generalized Wiener class $(\Lambda^1,\Lambda^2)BV(p(n)\uparrow p,\varphi,[0,2\pi]^2)$, where $1\le p\le\infty$, the order of magnitude of double Fourier coefficients of functions of the class $(\Lambda^1,\Lambda^2)BV(p(n)\uparrow\infty,\varphi,[0,2\pi]^2)$ is estimated. Here, we estimate the order of magnitude of double Walsh−Fourier coefficients of functions of the class $(\Lambda^1,\Lambda^2)BV(p(n)\uparrow\infty,\varphi,[0,1]^2)$.

2. Notation and definitions

In the sequel $\mathbb{I}=[0,1)$, $\mathbb{N}_0=\{0,1,2,\cdots\}$, $\mathbb{L}$ is a class of non-decreasing sequences $\Lambda=\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers such that $\sum_n\frac{1}{\lambda_n}$ diverges, and $\varphi(n)$ is a real sequence such that $\varphi(1)\ge 2$ and $\varphi(n)\to\infty$ as $n\to\infty$.

Consider function f on $\mathbb{R}^k$. For $k=1$ and $I=[a,b]$, define $\Delta f_a^b=f(I)=f(b)-f(a)$. For $k=2$, $I=[a,b]$ and $J=[c,d]$, define

$$\Delta f_{(a,c)}^{(b,d)}=f(I\times J)=f(I,d)-f(I,c)=f(b,d)-f(a,d)-f(b,c)+f(a,c).$$

Definition 2.1. *Given $\bigwedge=(\Lambda^1,\Lambda^2)$, where $\Lambda^r=\{\lambda_k^r\}_{k=1}^{\infty}\in\mathbb{L}$, for $r=1,\ 2$, $1\le p(n)\uparrow p$ as $n\to\infty$ and $1\le p\le\infty$, a measurable function f defined on a rectangle $R^2:=[a,b]\times[c,d]$ is said to be of $p(n)-\bigwedge-$bounded variation (that is, $f\in\bigwedge BV(p(n)\uparrow p,\varphi,R^2)$) if*

$$V_{\bigwedge_{p(n)}}(f,R^2)=\sup_{n\ge1}\sup_{\{I_i\},\{I_j\}}\left\{V_{\bigwedge_{p(n)}}(f,\{I_i\},\{I_j\}):\ \delta(\{I_i\},\{I_j\})\ge\frac{(b-a)(c-d)}{\varphi(n)}\right\}<\infty,$$

where

$$V_{\bigwedge_{p(n)}}(f,\{I_i\},\{I_j\})=\left(\sum_i\sum_j\frac{|f(I_i\times I_j)|^{p(n)}}{\lambda_i^1\,\lambda_j^2}\right)^{\frac{1}{p(n)}},$$

in which $\{I_i\}$ and $\{I_j\}$ are finite collections of non-overlapping subintervals in $[a,b]$ and $[c,d]$, respectively, and

$$\delta(\{I_i\},\{I_j\})=\delta(\{[x_{i-1},x_i]\},\{[y_{j-1},y_j]\})=\inf_{i,j}|(x_i-x_{i-1})\times(y_j-y_{j-1})|.$$

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Consider a function $f:\bar{\mathbb{I}}^2 \to \mathbb{R}$ defined by $f(x,y)=g(x)+h(y)$, where g and h are any two arbitrary need not be bounded (or need not be measurable) functions from $\bar{\mathbb{I}}$ into $\mathbb{R}$. Then $V_{\bigwedge_{p(n)}}(f,\bar{\mathbb{I}}^2)=0$. Thus, a function $f\in \bigwedge BV(p(n)\uparrow p,\varphi,R^2)$ need not be bounded (or need not be measurable).

This class is further generalized to the class $\bigwedge^* BV(p(n)\uparrow p,\varphi,R^2)$ in the sense of Hardy as follows.

Definition 2.2. *If $f\in \bigwedge BV(p(n)\uparrow p,\varphi,R^2)$ is such that the marginal functions $f(.,c)\in \Lambda^1 BV(p(n)\uparrow p,\varphi,[a,b])$ and $f(a,.)\in \Lambda^2 BV(p(n)\uparrow p,\varphi,[c,d])$ (see [5, Definition 1.1, p. 215] for the definition of $p(n)-\Lambda-$bounded variation over $[a,b]$) then f is said to be of $p(n)-\bigwedge^*-$bounded variation (that is, $f\in \bigwedge^* BV(p(n)\uparrow p,\varphi,R^2)$).*

If $f\in \bigwedge^* BV(p(n)\uparrow p,\varphi,R^2)$ then f is bounded and each of the marginal functions $f(.,s)\in \Lambda^1 BV(p(n)\uparrow p,\varphi,[a,b])$ and $f(t,.)\in \Lambda^2 BV(p(n)\uparrow p,\varphi,[c,d])$, where $s\in[c,d]$ and $t\in[a,b]$ are fixed [6, p. 436].

Note that, for $\Lambda^1=\Lambda^2=\{1\}$ (that is, $\lambda_n^1=\lambda_n^2=1$, for all n), the classes $\bigwedge BV(p(n)\uparrow p,\varphi,R^2)$ and $\bigwedge^* BV(p(n)\uparrow p,\varphi,R^2)$ reduce to the classes $BV_V(p(n)\uparrow p,\varphi,R^2)$ and $BV_H(p(n)\uparrow p,\varphi,R^2)$, respectively. For $p(n)=p$, for all n, the classes $\bigwedge BV(p(n)\uparrow p,\varphi,R^2)$ and $\bigwedge^* BV(p(n)\uparrow p,\varphi,R^2)$ reduce to the classes $\bigwedge BV^{(p)}(R^2)$ [3, Definition 4.2, p. 54] and $\bigwedge^* BV^{(p)}(R^2)$, respectively. For $p=\infty$, the classes $\bigwedge BV(p(n)\uparrow p,\varphi,R^2)$ and $\bigwedge^* BV(p(n)\uparrow p,\varphi,R^2)$ reduce to the classes $\bigwedge BV(p(n)\uparrow\infty,\varphi,R^2)$ and $\bigwedge^* BV(p(n)\uparrow\infty,\varphi,R^2)$, respectively. For $\Lambda^1=\Lambda^2=\{1\}$ and $p=\infty$, the classes $\bigwedge BV(p(n)\uparrow p,\varphi,R^2)$ and $\bigwedge^* BV(p(n)\uparrow p,\varphi,R^2)$ reduce to the classes $BV_V(p(n)\uparrow\infty,\varphi,R^2)$ and $BV_H(p(n)\uparrow\infty,\varphi,R^2)$, respectively.

The Walsh orthonormal system $\{\psi_m(x): m\in\mathbb{N}_0\}$ on the unit interval $\mathbb{I}$ in the Paley enumeration is defined as follows.
Let

$$r_0(x)=\begin{cases} 1, & \text{if } x\in\left[0,\frac{1}{2}\right),\\ -1, & \text{if } x\in\left[\frac{1}{2},1\right); \end{cases}$$

and extend $r_0(x)$ for the half-line $[0,\infty)$ with period 1.

The Rademacher orthonormal system $\{r_k(x):\ k\in\mathbb{N}_0\}$ is defined as

$$r_k(x)=r_0(2^k x),\quad k=1,2,\cdots;\ x\in\mathbb{I}.$$

If

$$m=\sum_{k=0}^{\infty} m_k 2^k,\quad \text{each } m_k=0 \text{ or } 1,$$

is the binary decomposition of $m\in\mathbb{N}_0$, then

$$\psi_m(x)=\prod_{k=0}^{\infty} r_k^{m_k}(x),\quad x\in\mathbb{I},$$

is called the m^{th} Walsh function in the Paley enumeration.

In particular, we have

$$\psi_0(x)=1 \ \text{ and }\ \psi_{2^m}(x)=r_m(x),\ m\in\mathbb{N}_0.$$

Any $x\in\mathbb{I}$ can be written as

$$x=\sum_{k=0}^{\infty} x_k\, 2^{-(k+1)},\quad \text{each } x_k=0 \text{ or } 1.$$

For any $x\in\mathbb{I}\backslash Q$, there is only one expression of this form, where Q is a class of dyadic rationals in $\mathbb{I}$. When $x\in Q$ there are two expressions of this form, one which terminates in 0's and one which terminates in 1's.

For any $x,y\in\mathbb{I}$ their dyadic sum is defined as

$$x\dotplus y=\sum_{k=0}^{\infty}|x_k-y_k|\, 2^{-(k+1)}.$$

Observe that, for each $m\in\mathbb{N}_0$, we have

$$\psi_m(x\dotplus y)=\psi_m(x)\ \psi_m(y),\ \ x,y\in\mathbb{I},\ \ x\dotplus y\notin Q.$$

For a real-valued function $f\in L^1(\bar{\mathbb{I}}^2)$, where f is $1-$periodic in each variable, its double Walsh$-$Fourier series is defined as

$$f(\mathbf{x})=f(x,y)\sim\sum_{\mathbf{k}\in\mathbb{N}_0^2}\hat{f}(\mathbf{k})\ \psi_m(x)\ \psi_n(y)=\sum_{m\in\mathbb{N}_0}\sum_{n\in\mathbb{N}_0}\hat{f}(m,n)\ \psi_m(x)\ \psi_n(y),$$

where

$$\hat{f}(\mathbf{k})=\hat{f}(m,n)=\int\int_{\bar{\mathbb{I}}^2} f(x,y)\ \psi_m(x)\ \psi_n(y)\,dx\,dy$$

denotes the $\mathbf{k}$th Walsh$-$Fourier coefficient of f.

3. Results

We prove the following results.

Theorem 3.1. *If $f \in \bigwedge BV(p(n)\uparrow\infty, \varphi, \overline{\mathbb{I}}^2) \cap L^\infty(\overline{\mathbb{I}}^2)$, then*

$$\hat{f}(2^u,2^v) = O\left(\frac{1}{\left(\sum_{j=1}^{2^u}\sum_{k=1}^{2^v} \frac{1}{\lambda_j^1\,\lambda_k^2}\right)^{\frac{1}{p(\tau(2^{u+v}))}}} \right), \tag{3.1}$$

where

$$\tau(r) = \min\{s : s \in \mathbb{N},\ \varphi(s) \geq r\},\ r \geq 1. \tag{3.2}$$

Corollary 3.2. *If $f \in \bigwedge^* BV(p(n)\uparrow\infty, \varphi, \overline{\mathbb{I}}^2)$, then (3.1) holds true.*

Corollary 3.3. *If $f \in BV_H(p(n)\uparrow\infty, \varphi, \overline{\mathbb{I}}^2)$, then*

$$\hat{f}(2^u,2^v) = O\left(\frac{1}{(2^{u+v})^{\frac{1}{p(\tau(2^{u+v}))}}} \right),$$

where $\tau(2^{u+v})$ is defined as in (3.2).

Corollary 3.3 follows from Theorem 3.1.

4. Proof of the results

Proof of Theorem 3.1. For fixed $u, v \in \mathbb{N}_0$, let $h_1 = \frac{1}{2^{u+1}}$ and $h_2 = \frac{1}{2^{v+1}}$. Take

$$g(x,y) = f(x,y) - f\left(x, y \dotplus \frac{1}{2^{v+1}}\right) - f\left(x \dotplus \frac{1}{2^{u+1}}, y\right) + f\left(x \dotplus \frac{1}{2^{u+1}}, y \dotplus \frac{1}{2^{v+1}}\right),$$

for all $(x,y) \in \overline{\mathbb{I}}^2$.

For $m = 2^u$ and $n = 2^v$, $\psi_m(h_1) = \psi_n(h_2) = -1$ and $\psi_m\left(\frac{1}{2^u}\right) = \psi_n\left(\frac{1}{2^v}\right) = 1$ imply that

$$\begin{aligned}\hat{g}(m,n) &= \hat{f}(m,n) - \psi_n\left(\frac{1}{2^{v+1}}\right)\hat{f}(m,n) - \psi_m\left(\frac{1}{2^{u+1}}\right)\hat{f}(m,n) + \psi_m\left(\frac{1}{2^{u+1}}\right)\psi_n\left(\frac{1}{2^{v+1}}\right)\hat{f}(m,n) \\ &= 4\hat{f}(m,n)\end{aligned}$$

and

$$\begin{aligned}4|\hat{f}(m,n)| &\leq \int\int_{\overline{\mathbb{I}}^2}\left| f(x,y) - f\left(x, y \dotplus \frac{1}{2^{v+1}}\right) - f\left(x \dotplus \frac{1}{2^{u+1}}, y\right) + f\left(x \dotplus \frac{1}{2^{u+1}}, y \dotplus \frac{1}{2^{v+1}}\right)\right| dx\,dy \\ &= \int\int_{\overline{\mathbb{I}}^2}\left| f\left(x \dotplus \frac{1}{2^u}, y \dotplus \frac{1}{2^v}\right) - f\left(x \dotplus \frac{1}{2^u}, y \dotplus \frac{1}{2^v} \dotplus \frac{1}{2^{v+1}}\right)\right. \\ &\qquad \left. - f\left(x \dotplus \frac{1}{2^u} \dotplus \frac{1}{2^{u+1}}, y \dotplus \frac{1}{2^v}\right) + f\left(x \dotplus \frac{1}{2^u} \dotplus \frac{1}{2^{u+1}}, y \dotplus \frac{1}{2^v} \dotplus \frac{1}{2^{v+1}}\right)\right| dx\,dy \\ &= \int\int_{\overline{\mathbb{I}}^2}\left| f\left(x \dotplus \frac{2}{2^{u+1}}, y \dotplus \frac{2}{2^{v+1}}\right) - f\left(x \dotplus \frac{2}{2^{u+1}}, y \dotplus \frac{3}{2^{v+1}}\right)\right. \\ &\qquad \left. - f\left(x \dotplus \frac{3}{2^{u+1}}, y \dotplus \frac{2}{2^{v+1}}\right) + f\left(x \dotplus \frac{3}{2^{u+1}}, y \dotplus \frac{3}{2^{v+1}}\right)\right| dx\,dy.\end{aligned}$$

Similarly, we get

$$\begin{aligned}4|\hat{f}(m,n)| &\leq \int\int_{\overline{\mathbb{I}}^2}\left| f\left(x \dotplus \frac{4}{2^{u+1}}, y \dotplus \frac{4}{2^{v+1}}\right) - f\left(x \dotplus \frac{4}{2^{u+1}}, y \dotplus \frac{5}{2^{v+1}}\right)\right. \\ &\qquad \left. - f\left(x \dotplus \frac{5}{2^{u+1}}, y \dotplus \frac{4}{2^{v+1}}\right) + f\left(x \dotplus \frac{5}{2^{u+1}}, y \dotplus \frac{5}{2^{v+1}}\right)\right| dx\,dy\end{aligned}$$

and in general we have

$$4|\hat{f}(m,n)| \leq \int\int_{\overline{\mathbb{I}}^2} |\Delta f_{jk}(x,y)|\,dx\,dy, \tag{4.1}$$

where

$$\Delta f_{jk}(x,y) = f\left(x \dotplus \frac{2j}{2^{u+1}}, y \dotplus \frac{2k}{2^{v+1}}\right) - f\left(x \dotplus \frac{2j}{2^{u+1}}, y \dotplus \frac{(2k+1)}{2^{v+1}}\right) - f\left(x \dotplus \frac{(2j+1)}{2^{u+1}}, y \dotplus \frac{2k}{2^{v+1}}\right) + f\left(x \dotplus \frac{(2j+1)}{2^{u+1}}, y \dotplus \frac{(2k+1)}{2^{v+1}}\right),$$

for all $j = 1, ..., 2^u$ and for all $k = 1, ..., 2^v$.

Dividing both sides of the above inequality by $\lambda_j^1\ \lambda_k^2$ and then summing over $j = 1$ to 2^u and $k = 1$ to 2^v, we get

$$4|\hat{f}(2^u,2^v)|\left(\sum_{j=1}^{2^u}\sum_{k=1}^{2^v}\frac{1}{\lambda_j^1\ \lambda_k^2}\right) \le \int\int_{\bar{\mathbb{I}}^2}\left(\sum_{j=1}^{2^u}\sum_{k=1}^{2^v}\frac{|\Delta f_{jk}(x,y)|}{\left(\lambda_j^1\ \lambda_k^2\right)^{\frac{1}{p(\tau(2^{u+v}))}+\frac{1}{q(\tau(2^{u+v}))}}}\right) dx\, dy,$$

where $q(\tau(2^{u+v}))$ is the index conjugate to $p(\tau(2^{u+v}))$.

Applying Hölder's inequality on the right side of the above inequality, we get

$$4|\hat{f}(2^u,2^v)|\left(\sum_{j=1}^{2^u}\sum_{k=1}^{2^v}\frac{1}{\lambda_j^1\ \lambda_k^2}\right) \le \int\int_{\bar{\mathbb{I}}^2}\left(\sum_{j=1}^{2^u}\sum_{k=1}^{2^v}\frac{|\Delta f_{jk}(x,y)|^{p(\tau(2^{u+v}))}}{\lambda_j^1\ \lambda_k^2}\right)^{\frac{1}{p(\tau(2^{u+v}))}}\left(\sum_{j=1}^{2^u}\sum_{k=1}^{2^v}\frac{1}{\lambda_j^1\ \lambda_k^2}\right)^{\frac{1}{q(\tau(2^{u+v}))}} dx\, dy.$$

Hence,

$$4|\hat{f}(2^u,2^v)|\left(\sum_{j=1}^{2^u}\sum_{k=1}^{2^v}\frac{1}{\lambda_j^1\ \lambda_k^2}\right)^{\frac{1}{p(\tau(2^{u+v}))}} \le \int\int_{\bar{\mathbb{I}}^2}\left(\sum_{j=1}^{2^u}\sum_{k=1}^{2^v}\frac{|\Delta f_{jk}(x,y)|^{p(\tau(2^{u+v}))}}{\lambda_j^1\ \lambda_k^2}\right)^{\frac{1}{p(\tau(2^{u+v}))}}. \tag{4.2}$$

For any $x, y \in \mathbb{R}$, all these points $x \dotplus 2jh_1,\ x \dotplus (2j+1)h_1,\ for\ j = 1, ..., 2^u$, and $y \dotplus 2kh_2,\ y \dotplus (2k+1)h_2,\ for\ k = 1, ..., 2^v$, lie in the interval of length 1. Thus, $f \in \bigwedge BV(p(n) \uparrow \infty, \varphi, \bar{\mathbb{I}}^2)$ implies

$$\left(\sum_{j=1}^{2^u}\sum_{k=1}^{2^v}\frac{|\Delta f_{jk}(x,y)|^{p(\tau(2^{u+v}))}}{\lambda_j^1\ \lambda_k^2}\right)^{\frac{1}{p(\tau(2^{u+v}))}} = O(1).$$

This together with above inequality (4.2) imply that

$$|\hat{f}(2^u,2^v)| = O\left(\frac{1}{\left(\sum_{j=1}^{2^u}\sum_{k=1}^{2^v}\frac{1}{\lambda_j^1\ \lambda_k^2}\right)^{\frac{1}{p(\tau(2^{u+v}))}}}\right).$$

This completes the proof of the theorem.

Proof of Corollary 3.2. Since $f \in \bigwedge^* BV(p(n) \uparrow \infty, \varphi, \bar{\mathbb{I}}^2)$ is bounded [6, p. 436] and $\bigwedge^* BV(p(n) \uparrow \infty, \varphi, \bar{\mathbb{I}}^2) \subset \bigwedge BV(p(n) \uparrow \infty, \varphi, \bar{\mathbb{I}}^2)$, the Corollary 3.2 follows from Theorem 3.1.

One can extend these results for functions of $N-$variables $(N > 2)$ analogously to the above-mentioned results for functions of two variables.

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