Classifications of $K$-Contact Semi-Riemannian Manifolds

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Abstract

The object of the present paper is to characterize a K-contact semi-Riemannian manifold satisfying certain curvature conditions. We study Ricci semi-symmetric K-contact semi-Riemannian manifolds and obtain an equivalent condition. Next we prove that a K-contact semi-Riemannian manifold is of harmonic conformal curvature tensor if and only if the manifold is an Einstein manifold. Also we study $\xi$-conformally flat K-contact semi-Riemannian manifolds. Finally, we characterize conformally semisymmetric Lorentzian $K$-contact manifolds.

Keywords: Conformally semisymmetric manifolds; Harmonic conformal curvature tensor; K-contact manifolds; Ricci symmetric manifolds; Ricci semi-symmetric manifolds.


1. Introduction

Let $M^{2n+1}$ be a contact manifold with contact form $\eta$, associated vector field $\xi$ and $(1,1)$ tensor field $\phi$ ([1],[2]). A semi-Riemannian metric $g$ is said to be an associated metric if it satisfies: $\eta(X) = \varepsilon g(X, \xi)$, where $\varepsilon = \pm 1$. Then $M$ is said to be a contact semi-Riemannian (or Pseudo-Riemannian) manifold and $(M, \phi, \xi, \eta, g)$ is called a semi-Riemannian (or Pseudo-Riemannian) structure. Takahashi [10] introduced the concept of contact semi-Riemannian manifolds. Semi-Riemannian contact metric manifolds have been studied by Calvaruso and Perrone ([3],[4]) and Perrone ([7],[8]).

A contact semi-Riemannian manifold is said to be $K$-contact if $\xi$ is a Killing vector field and Sasakian if it is normal. A semi-Riemannian manifold is said to be Ricci semi-symmetric if $R(X,Y)S = 0$, where $R(X,Y)$ is the curvature operator and $S$ denotes the Ricci tensor. In [9] Tanno studied Ricci symmetric ($\nabla S = 0$) $K$-contact Riemannian manifolds. Also Yildiz and Ata [11] studied $K$-contact Riemannian manifolds. The paper is organized as follows:

After preliminaries in section 3, we extend the Tanno’s result for semi-Riemannian case. In a recent paper [7], Perrone showed that a conformally flat $K$-contact semi-Riemannian manifold is Sasakian. In section 4 we generalize the result for a $K$-contact semi-Riemannian manifold of harmonic conformal curvature tensor. In section 5, we have shown that any $\xi$-conformally flat semi-Riemannian $K$-contact manifold is Sasakian. Finally, we consider conformally semisymmetric Lorentzian $K$-contact manifolds.

2. Preliminaries

A contact semi-Riemannian manifold is Sasakian if it is normal, that is,

$$\{\phi X, \phi X\} + 2d\eta(X,X)\xi = 0.$$  (2.1)

This condition is equivalent to

$$(\nabla_X \phi)Y = g(X,Y)\xi - \varepsilon \eta(Y)X.$$  (2.2)

In a contact semi-Riemannian manifold the following relations hold ([7],[8]):

$$\eta(\xi) = 1, \varepsilon g(X,\xi) = \eta(X),$$  (2.3)

$$\phi^2 X = -X + \eta(X)\xi.$$  (2.4)

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\[ g(\phi X, \phi Y) = g(X, Y) - \varepsilon \eta(X)\eta(Y), \tag{2.5} \]

\[ \nabla_X \xi = \varepsilon \phi X, \tag{2.6} \]

where \( \nabla \) is the Levi-Civita connection.

Also in a \( K \)-contact semi-Riemannian manifold \( \xi \) satisfies the following relation (see [6]):

\[ Q \xi = 2n \varepsilon \xi, \tag{2.7} \]

that is, \( \xi \) is an eigen vector of the Ricci operator \( Q \) defined by [7]

\[ S(\xi, X) = g(Q\xi, X) = 2n \varepsilon g(\xi, X) = 2n \eta(X), \tag{2.8} \]

where \( S \) denotes the Ricci tensor.

Also the curvature tensor \( R \) satisfies

\[ R(X, \xi)\xi = \phi^2 X. \tag{2.9} \]

Further, since \( \xi \) is Killing in a \( K \)-contact manifold, \( S \) and \( r \) remain invariant under it.

Consequently,

\[ \xi S = 0 \tag{2.10} \]

and

\[ \xi r = 0 \tag{2.11} \]

where \( \xi \) denotes the Lie derivation.

We state the following:

**Lemma 2.1.** ([7],[8]) Let \( (M, \eta, g, \xi, \phi) \) be a \( K \)-contact semi-Riemannian manifold. Then \( M \) is Sasakian if and only if the curvature tensor \( R \) satisfies

\[ R(X, Y)\xi = \eta(X)Y - \eta(Y)X. \tag{2.12} \]

**Definition 2.2.** A contact semi-Riemannian manifold is said to be \( \eta \)-Einstein if

\[ S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{2.13} \]

where \( a, b \) are smooth functions.

**Remark 2.3.** The basic difference between the Riemannian and semi-Riemannian geometry is the existence of a null vector, that is, a vector \( v \) satisfying \( g(v, v) = 0 \), where \( g \) is the metric tensor. The signature of the metric \( g \) of a Riemannian manifold is \( (+,+,+,...,+) \) and of a semi-Riemannian manifold is \( (-,-,-,...,+) \). There are differences of the basic results of \( K \)-contact Riemannian and semi-Riemannian manifolds due the presence of \( \varepsilon \).

### 3. Ricci semisymmetric \( K \)-contact semi-Riemannian manifolds

It is known that [9] a \( K \)-contact Riemannian metric manifold is an Einstien manifold if and only if it is Ricci-symmetric. In this section, we prove a similar result for the semi-Riemannian case.

**Theorem 3.1.** A \( K \)-contact semi-Riemannian manifold is Ricci semi-symmetric if and only if the manifold is an Einstein manifold.

**Proof.** Let \( M \) be a Ricci semi-symmetric \( K \)-contact semi-Riemannian manifold. Then

\[ Rs = 0, \]

which implies

\[ S(R(X, Y)U, V) + S(U, R(X, Y)V) = 0. \tag{3.1} \]

Putting \( Y = U = \xi \) in (3.1) we obtain

\[ S(R(X, \xi)\xi, V) + S(\xi, R(X, \xi)V) = 0. \tag{3.2} \]
Using (2.4) and (2.9) in (3.2) yields
\[ S(-X + \eta(X)\xi, V) + S(\xi, R(X, \xi)V) = 0, \]
which implies
\[ -S(X, V) + \eta(X)S(\xi, V) + S(\xi, R(X, \xi)V) = 0. \] (3.3)
Interchanging X and V in (3.3) we obtain
\[ -S(V, X) + \eta(V)S(\xi, X) + S(\xi, R(V, \xi)X) = 0. \] (3.4)
Subtracting (3.3) from (3.4) it follows that
\[ \eta(X)S(\xi, V) - \eta(V)S(\xi, X) + S(\xi, R(X, \xi)V) + S(R(V, \xi)\xi, X) = 0. \] (3.5)
Putting \( V = \xi \) in (3.5) we obtain
\[ \eta(X)S(\xi, \xi) - \eta(\xi)S(\xi, X) - S(\xi, R(\xi, X)\xi) = 0. \]
Using (2.3) and (2.9) in the above equation yields
\[ \eta(X)S(\xi, \xi) - S(\xi, X) - S(\xi, X) + \eta(X)S(\xi, \xi) = 0, \]
which implies
\[ S(X, \xi) = 2n\eta(X). \] (3.6)
Consequently from (3.2) we get
\[ -S(X, V) + 2n\eta(X)\eta(V) + 2n\bar{\varepsilon}g(R(X, \xi)V, \xi) = 0. \] (3.7)
It follows that
\[ S(X, V) + 2n\eta(X)\eta(V) - 2n\bar{\varepsilon}g(-X + \eta(X)\xi, V) = 0, \]
which in terms implies that
\[ S(X, V) = 2n\bar{\varepsilon}g(X, V), \quad \text{where} \quad \bar{\varepsilon} = g(X_i, X_i). \]
Converse part follows easily. \( \square \)

In view of Theorem 3.1 and the fact that Ricci symmetric \((\nabla S = 0)\) implies Ricci-semisymmetric \((R, S = 0)\), we may conclude the following:

**Theorem 3.2.** In a \( K \)-contact semi-Riemannian manifold \( M \) the following conditions are equivalent
(i) \( M \) is an Einstein manifold
(ii) \( M \) is Ricci-symmetric
(iii) \( M \) is Ricci semi-symmetric.

### 4. \( K \)-Contact Semi-Riemannian Manifolds With Harmonic Conformal Curvature Tensor

This section deals with a \( K \)-contact Semi-Riemannian manifold with \( divC = 0 \).

**Definition 4.1.** The conformal curvature tensor is given by
\[ C(X, Y)Z = R(X, Y)Z - \frac{1}{2n-1}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] + \frac{r}{2n(2n-1)}[g(Y, Z)X - g(X, Z)Y], \]
where \( Q \) is the Ricci operator and \( r \) is the scalar curvature.

It is known that([5],[6])
\[ (divC)(X, Y)Z = \frac{2(n-1)}{2n-1}\left\{(\nabla_XS)(Y, Z) - (\nabla_YS)(X, Z) - \frac{1}{4n}[g(Y, Z)dr(X) - g(X, Z)dr(Y)]\right\}, \]
where ‘\( div \)’ denotes divergence.

**Definition 4.2.** A Riemannian or a semi-Riemannian manifold is said to be of harmonic conformal curvature tensor if
\[ (divC)(X, Y)Z = 0, \]
where ‘\( div \)’ denotes divergence.

We prove the following:

**Theorem 4.3.** A \( K \)-contact semi-Riemannian manifold is of harmonic conformal curvature if and only if the manifold is an Einstein manifold.
Proof. Let $M$ be a semi-Riemannian $K$-contact manifold satisfying $\text{div} \, C = 0$. It is known that $\text{div} \, C = 0$ implies

$$(\nabla_Y S)(Y, Z) - (\nabla_Y S)(X, Z) = \frac{1}{4n}[g(Y, Z)dr(X) - g(X, Z)dr(Y)].$$  \hfill (4.1)

From (2.10) and (2.11) it follows that

$$(\nabla_Z S)(Y, Z) = -S(\nabla_Y Z, Z) - S(Y, \nabla_Z Z).$$  \hfill (4.2)

$$dr(\xi) = 0.$$  \hfill (4.3)

Consequently putting $X = \xi$ in (4.1) we get

$$(\nabla_Z S)(Y, Z) - (\nabla_Z S)(\xi, Z) = \frac{1}{4n}[g(Y, Z)dr(\xi) - g(\xi, Z)dr(Y)].$$  \hfill (4.4)

Thus from (4.2) and (4.3) it follows that

$$-S(\nabla_Y \xi, Z) - S(Y, \nabla_Z \xi) - (\nabla_Y S)(\xi, Z) = -\frac{1}{4n}g(\xi, Z)dr(Y).$$  \hfill (4.5)

That is,

$$-S(Y, \nabla_Z \xi) - \nabla_Y S(\xi, Z) + S(\xi, \nabla_Y Z) = -\frac{1}{4n}g(\xi, Z)dr(Y).$$  \hfill (4.6)

Using (2.6) and (2.8) in (4.6) we get,

$$\varepsilon S(Y, \phi Z) + 2n\nabla_Y \eta(Z) - 2n\eta(\nabla_Y Z) = \frac{\varepsilon}{4n}g(\xi, Z)dr(Y),$$  \hfill (4.7)

which implies

$$\varepsilon S(Y, \phi Z) + 2n(\nabla_Y \eta)[Z] = \frac{\varepsilon}{4n}g(\xi, Z)dr(Y).$$  \hfill (4.8)

Hence

$$S(Y, \phi Z) + 2n\varepsilon(\nabla_Y \eta)(Z) = \frac{1}{4n}g(\xi, Z)dr(Y).$$  \hfill (4.9)

We note that

$$(\nabla_Y \eta)(Z) = \nabla_Y \eta(Z) - \eta(\nabla_Y Z)$$  \hfill (4.10)

$$= \varepsilon \nabla_Y g(Z, \xi) - \varepsilon g(\nabla_Y Z, \xi)$$

$$= \varepsilon g(Z, \nabla_Y \xi)$$

$$= \varepsilon g(Z, \phi Y)$$

$$= g(Z, \phi Y).$$

So from (4.10) it follows that

$$S(Y, \phi Z) + 2n\varepsilon g(Z, \phi Y) = \frac{1}{4n}g(\xi, Z)dr(Y).$$  \hfill (4.11)

Putting $Z = \phi Z$ in (4.11) we get

$$S(Y, \phi^2 Z) + 2n\varepsilon g(\phi Z, \phi Y) = \frac{1}{4n}g(\phi Z)dr(Y).$$  \hfill (4.12)

Again using (2.4) and (2.5) in (4.12) we get

$$S(Y, -Z + \eta[Z, \xi]) + 2n\varepsilon g(Z, Y) - 2n\eta(Z)\eta(Y) = 0.$$  \hfill (4.13)

That is,

$$-S(Y, Z) + \eta(Z)S(Y, \xi) + 2n\varepsilon g(Z, Y) - 2n\eta(Z)\eta(Y) = 0.$$  \hfill (4.14)

Using (2.8) in (4.14), we obtain

$$S(Y, Z) = 2n\varepsilon g(Y, Z), \quad \text{where} \quad \varepsilon = g(X_i, X_i).$$  \hfill (4.15)

Conversely, if the $K$-contact manifold is an Einstein manifold, then it can be easily seen that the conformal curvature tensor is harmonic. This completes the proof.

Since $\nabla C = 0$ (conformally symmetric) implies $\text{div} \, C = 0$, therefore we can state the following:

**Corollary 4.4.** A conformally symmetric $K$-contact semi-Riemannian manifold is an Einstein manifold.

Note: A conformally symmetric $K$-contact Riemannian manifold has been studied by Zhen [13].
5. \(\xi\)-Conformally Flatness

\(\xi\)-conformally flat \(K\)-contact manifolds have been studied by Zhen et al. Since at each point \(p \in M^n\) the tangent space \(T_p(M^n)\) can be decomposed into the direct sum \(T_p(M^n) = \phi(T_p(M^n)) \oplus \{\xi_p\}\), where \(\{\xi_p\}\) is the one-dimensional linear subspace of \(T_p(M^n)\) generated by \(\xi_p\), the conformal curvature tensor \(C\) is a map \(C: T_p(M^n) \times T_p(M^n) \rightarrow T_p(M^n) \rightarrow \phi(T_p(M^n)) \oplus \{\xi_p\}\).

**Definition 5.1.** [12] A \(K\)-contact semi-Riemannian manifold is said to be \(\xi\) conformally flat if the projection of the image of \(C\) onto \(\{\xi_p\}\) is zero, that is, \(C(X, Y)\xi = 0\), where \(C\) is the conformal curvature tensor.

**Proposition 5.2.** A \(\xi\)-conformally flat \(K\)-contact semi-Riemannian manifold is an \(\eta\)-Einstein manifold.

**Proof.** Let \(M^{2n+1}\) be a \(\xi\)-conformally flat \(K\)-contact semi-Riemannian manifold. Then \(C(X, Y)\xi = 0\) implies that

\[
R(X, Y)\xi = \frac{1}{2n-1}[S(Y, \xi)X - S(X, \xi)Y + g(Y, \xi)QX - g(X, \xi)QY] + \frac{r}{2n(2n-1)}[g(Y, \xi)X - g(X, \xi)Y] = 0.
\]

Putting \(Y = \xi\) in (5.1) and using (2.3), (2.8) and (2.9) we obtain,

\[
-X + \eta(X)\xi = \frac{1}{2n-1}[2nX - 2n\eta(X)\xi] + \varepsilon QX - 2n\eta(X)\xi] - \frac{r\varepsilon}{2n(2n-1)}[X - \eta(X)\xi],
\]

which implies

\[
QX = \frac{1}{\varepsilon}(1 - 4n + \frac{r\varepsilon}{2n})X + \frac{1}{\varepsilon}(6n - 1 - \frac{r\varepsilon}{2n})\eta(X)\xi.
\]

That is,

\[
S(X, Y) = a(g(X, Y) + b\eta(X)\eta(Y),
\]

where

\[
a = \frac{1}{\varepsilon}(1 - 4n + \frac{r\varepsilon}{2n})
\]

and

\[
b = \frac{1}{\varepsilon}(6n - 1 - \frac{r\varepsilon}{2n}).
\]

Consequently \(M\) is \(\eta\)-Einstein.

**Theorem 5.3.** A \(\xi\)-conformally flat \(K\)-contact manifold is Sasakian.

**Proof.** In view of (5.3) and (2.3) we obtain the following:

\[
QX = aX + b\eta(X)\xi,
\]

\[
S(X, \xi) = ag(X, \xi) + b\varepsilon\eta(X)
\]

and

\[
r = (2n+1)a + b.
\]

From (5.4) and (5.5), it follows that

\[
a + b = \frac{2n}{\varepsilon}
\]

Using (2.3),(5.6),(5.7) and (5.8) in (5.1) we have

\[
R(X, Y)\xi = [\frac{a}{2n\varepsilon} - \frac{b}{2n(2n-1)\varepsilon} + \frac{b\varepsilon}{2n-1}]\{\eta(Y)X - \eta(X)Y\}.
\]

Using (5.9) in (5.10) we obtain

\[
R(X, Y)\xi = \eta(X)Y - \eta(Y)X.
\]

Therefore from Lemma 2.1 we conclude the theorem.

Since conformally flatness implies \(\xi\)-conformally flat, therefore we obtain the following:

**Corollary 5.4.** A conformally flat \(K\)-contact semi-Riemannian manifold is Sasakian.

The above corollary has been proved by Perrone in his paper [7].
6. Conformally semisymmetric Lorentzian $K$-contact manifolds

The Ricci tensor $S_L$ of $\eta$-Einstein Lorentzian $K$-contact structure $(\eta, g_L)$ is given by [7]

\[
S_L = \left(\frac{r_L}{2n} + 1\right)g_L + \left(\frac{r_L}{2n} + 2n + 1\right)\eta \otimes \eta,
\]

where the scalar curvature $r_L = r + 4n$ is a constant when $n > 1$, and $g_L$ is Einstein if and only if $r_L = -2n(2n + 1)$. From (6.1)

\[
S_L(X, Y) = \left(\frac{r_L}{2n} + 1\right)g_L(X, Y) + \left(\frac{r_L}{2n} + 2n + 1\right)\eta(X)\eta(Y) = A g_L(X, Y) + B \eta(X)\eta(Y),
\]

where $A = \left(\frac{r_L}{2n} + 1\right)$, $B = \left(\frac{r_L}{2n} + 2n + 1\right)$ are constants.

The conformal curvature tensor $C$ is given by

\[
C_L(X, Y)Z = R_L(X, Y)Z - \frac{1}{2n-1}\left\{g_L(Y, Z)Q_L(X) - g_L(X, Z)Q_L(Y) + S_L(Y, Z)X - S_L(X, Z)Y\right\} + \frac{r_L}{2n(2n-1)}\left[g_L(Y, Z)X - g_L(X, Z)Y\right],
\]

where $Q_L$ is the Ricci operator defined by $g_L(QU, V) = S_L(QU, V)$, for all vector fields $X, Y, U$. Using (6.2) in (6.3) we get

\[
C_L(X, Y)Z = R_L(X, Y)Z - \frac{1}{2n-1}\left\{(\frac{r_L}{2n} + 1)X
\right.
\]

\[
+ \left(\frac{r_L}{2n} + 1 + 2n\right)\eta(X)\eta(Z)
\]

\[
- g_L(X, Z)\left\{(\frac{r_L}{2n} + 1)Y + (\frac{r_L}{2n} + 1 + 2n)\eta(Y)\eta(Z)\right\}
\]

\[
+ \left\{\left(\frac{r_L}{2n} + 1\right)g_L(Y, Z) + \left(\frac{r_L}{2n} + 1 + 2n\right)\eta(Y)\eta(Z)\right\}X
\]

\[
- \left\{\left(\frac{r_L}{2n} + 1\right)g_L(X, Z) + \left(\frac{r_L}{2n} + 1 + 2n\right)\eta(X)\eta(Z)\right\}Y
\]

\[
+ \frac{r_L}{2n(2n-1)}\left[g_L(Y, Z)X - g_L(X, Z)Y\right],
\]

from which it follows that

\[
C_L(X, Y)Z = R_L(X, Y)Z - \frac{1}{2n-1}\left\{2\left(\frac{r_L}{2n} + 1\right)g_L(Y, Z)X
\right.
\]

\[
- 2\left(\frac{r_L}{2n} + 1\right)g_L(X, Z)Y
\]

\[
+ \left(\frac{r_L}{2n} + 2n + 1\right)g_L(Y, Z)\eta(X)\xi - \left(\frac{r_L}{2n} + 1 + 2n\right)g_L(X, Z)\eta(Y)\xi
\]

\[
+ \frac{r_L}{2n + 1 + 2n}\eta(Y)\eta(Z)X - \left(\frac{r_L}{2n} + 1 + 2n\right)\eta(X)\eta(Z)Y
\]

\[
+ \frac{r_L}{2n(2n-1)}g_L(Y, Z)X - g_L(X, Z)Y.
\]

The above equation implies

\[
C_L(X, Y)Z = R_L(X, Y)Z - \left\{\frac{2}{2n-1} - \frac{r_L}{2n}\right\}g_L(Y, Z)X
\]

\[
- \frac{1}{2n-1}\left\{\frac{r_L}{2n} + 2n + 1\right\}g_L(Y, Z)\eta(X)\xi - g_L(X, Z)\eta(Y)\xi
\]

\[
+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y.
\]

Putting $X = Z = \xi$ in (6.6) we infer that

\[
C_L(\xi, Y)\xi = R_L(\xi, Y)\xi - \left\{\frac{2}{2n-1} - \frac{r_L}{2n}\right\}\eta(Y)\xi + Y - \frac{1}{2n-1}\left\{\frac{r_L}{2n} + 2n + 1\right\}\eta(Y)\xi - Y.
\]

As we are interested to study conformally semisymmetric Lorentzian $K$-contact manifolds, therefore

\[
(R_L(\xi, \xi), C_L)(U, V)W = 0.
\]

This implies

\[
R_L(X, \xi)C_L(\xi, Y)W - C_L(R_L(X, \xi)U, V)W - C_L(U, R_L(X, \xi)V)W - C_L(U, V)R_L(X, \xi)W = 0,
\]

for all vector fields $U, V, X, W$. Putting $U = W = \xi$ in (6.8) we get

\[
R_L(X, \xi)C_L(\xi, V)\xi - C_L(R_L(X, \xi)\xi, V)\xi - C_L(\xi, R_L(X, \xi)V)\xi - C_L(\xi, V)R_L(X, \xi)\xi = 0.
\]
From (6.7) we get
\[ C_L (\xi, V) \xi = a(V - \eta(V) \xi), \]
where \( a = \frac{2m + 2(2m + 1)}{2(2m - 1)} \) = constant. Hence
\[ R_L(X, \xi) C_L (\xi, V) \xi = (1 - a) \{ R_L(X, \xi) V + \eta(V) X - \eta(X) \eta(V) \xi \}. \]  

Similarly,
\[ C_L(R_L(X, \xi) \xi, V) \xi = -R_L(X, V) \xi + \{ (2a - 1) \{ \eta(V) X - \eta(X) V \} \} \]

Using (6.11), (6.12), (6.13), (6.14) in (6.9) we obtain
\[ R_L(X, V) \xi + R_L(\xi, V) X - a \{ 2(\eta(V) X - 3 \eta(X) V + 2 g_L(V, X) \xi - \eta(X) \eta(V) \xi) \} + \{ \eta(V) X - \eta(X) V + g_L(V, X) \xi - \eta(X) \eta(V) \xi \} = 0. \]

Interchanging \( X \) and \( V \) in (6.15) we get
\[ R_L(V, X) \xi + R_L(\xi, X) V - a \{ 2 \eta(X) V - 3 \eta(X) X + 2 g_L(V, X) \xi - \eta(X) \eta(V) \xi \} + \{ \eta(V) X - \eta(X) V + g_L(V, X) \xi - \eta(X) \eta(V) \xi \} = 0. \]

Subtraction (6.16) from (6.15) we get
\[ R_L(X, V) \xi = \frac{5a - 2}{3} \{ \eta(V) X - \eta(X) V \}. \]

It is clear that for \( a = 1 \), \( R_L(X, V) \xi = \eta(V) X - \eta(X) V \) and hence Weyl Conformally semi-symmetric Lorentzian \( \eta \)-Einstein \( K \)-contact manifold is a Sasakian manifold. Again \( a \) is equivalent to \( r_L = 4n \). Thus in view of the above we can state the following:

**Theorem 6.1.** A conformally semisymmetric Lorentzian \( K \)-contact \( \eta \)-Einstein manifold is a Sasakian manifold, provided \( r_L = 4n \).

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