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A Study on Lorentzian $\alpha-$Sasakian Manifolds

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Abstract

The object of the present paper is to study the geometric properties of Concircular curvature tensor on Lorentzian $\alpha-$Sasakian manifold admitting a type of quarter-symmetric metric connection. In the last, we provide an example of 3-dimensional Lorentzian $\alpha-$Sasakian manifold endowed with the quarter-symmetric metric connection which is under consideration is an $\eta-$Einstein manifold with respect to the quarter-symmetric metric connection.

Keywords: *Concircular curvature tensor; $\eta-$Einstein manifold; Lorentzian α-Sasakian manifold; Quarter-symmetric metric connection.*
2010 Mathematics Subject Classification: *53C15; 53C25; 53C40.*

1. Introduction

In 1975, Golab [5] defined and studied quarter-symmetric connection in differentiable manifolds. A linear connection $\bar{\nabla}$ on an n-dimensional Riemannian manifold (M,g) is said to be a quarter-symmetric connection [5] if its torsion tensor T defined by

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] \tag{1.1}$$

satisfies

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.2}$$

where ϕ is a $(1,1)$ tensor field, η is a 1-form and X,Y are vector fields on $\Gamma(TM)$, $\Gamma(TM)$ is the set of all differentiable vector fields on M. In particular, if $\phi X = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection [4].
Thus the notion of the quarter-symmetric connection generalizes the notion of the semi-symmetric connection. If moreover, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z) = 0, \tag{1.3}$$

for all X,Y,Z on $\Gamma(TM)$, then $\bar{\nabla}$ is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter symmetric non-metric connection. Recently quarter-symmetric metric connection have been studied by several authors ([8], [9], [12]).
A differentiable manifold M is said to be a Lorentzian manifold, if M has a Lorentzian metric g, which is a symmetric non-degenerate $(0,2)$ tensor field of index 1. Since the Lorentzian metric g is of index 1 therefore Lorentzian manifold M has not only spacelike vector fields but also lightlike and timelike vector fields. On a Lorentzian manifold this difference with Riemannian case gives interesting results. In 1989, K. Matsumoto used a structure vector field $-\xi$ instead of ξ in an almost para contact manifold and associated a Lorentzian metric with this resulting structure, called it as Lorentzian almost para contact manifold.

Yildiz and Murathan studied [15] Lorentzian $\alpha-$Sasakian manifolds in 2005 and obtained results for conformally flat and quasi-conformally flat Lorentzian $\alpha-$Sasakian manifolds. In 2009, Yildiz et al. ([16, 17]), further studied on three dimensional Lorentzian $\alpha-$Sasakian manifolds and a class of Lorentzian $\alpha-$Sasakian manifolds and obtained some important results. In 2013, U.C. De and K. De ([3]) studied on Lorentzian Trans-Sasakian manifolds, which is a generalization of Lorentzian $\alpha-$Sasakian manifolds.
A concircular transformation ([7], [13]) on an n-dimensional Riemannian manifold M is a transformation under which every geodesic circle of M transforms into a geodesic circle. Every concircular transformation is always a conformal transformation [7]. Thus the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle

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preserving diffeomorphism (see also [2]). An interesting invariant of a concircular transformation is the concircular curvature tensor $\bar{C}$. It is defined by ([13], [14])

$$\bar{C}(X,Y)Z = \bar{R}(X,Y)Z - \frac{\bar{r}}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y]. \tag{1.4}$$

for all vector fields $X,Y,Z \in \Gamma(TM)$, where $\bar{R}$ and $\bar{r}$ be the curvature tensor and scalar curvature with respect to the quarter-symmetric metric connection $\bar{\nabla}$ respectively.
Using (1.4), we obtain

$$`\bar{C}(X,Y,Z,W) = `\bar{R}(X,Y,Z,W) - \frac{\bar{r}}{2n(2n+1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \tag{1.5}$$

where $`\bar{C}(X,Y,Z,W) = g(\bar{C}(X,Y)Z,W)$, $`\bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W)$, where $X,Y,Z,W \in \Gamma(TM)$ and $\bar{C}$ is the concircular curvature tensor and $\bar{r}$ is the scalar curvature with respect to the quarter-symmetric metric connection respectively. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

In this paper, we study a type of quarter-symmetric metric connection on Lorentzian α-Sasakian manifolds. The paper is organized as follows: After introduction section two gives some prerequisites of a Lorentzian α-Sasakian manifold. In section three, we obtain a relation between the quarter-symmetric metric connection and Levi-civita connection. In section four, curvature tensor and Ricci tensor of Lorentzian $\alpha-$Sasakian manifold with respect to quarter-symmetric metric connection are given. Section five is devoted to the study of ξ-concircularly flat Lorentzian α-Sasakian manifold with respect to the quarter-symmetric metric connection. Quasi-concircularly flat and ϕ-concircularly flat Lorentzian α-Sasakian manifolds with respect to the quarter-symmetric metric connection have been studied in section six and seven respectively. In the next section, we study a Lorentzian α-Sasakian manifold satisfying $\bar{C}\cdot\bar{S} = 0$ with respect to a quarter-symmetric metric connection. In the last, we construct an example of a 3-dimensional Lorentzian α-Sasakian manifold endowed with the quarter-symmetric metric connection.

2. Preliminaries

An $(2n+1)$-dimensional differentiable manifold M is said to be a Lorentzian $\alpha-$Sasakian manifold, if it admits a structure (ϕ,ξ,η,g) consisting of a $(1,1)$ tensor field ϕ, vector field ξ, 1-form η and a Lorentzian metric g satisfying

$$\phi^2 X = X + \eta(X)\xi, \tag{2.1}$$

$$\phi\circ\xi = 0, \quad \eta\circ\phi = 0, \quad \eta(\xi) = -1, \quad g(X,\xi) = \eta(X), \tag{2.2}$$

$$g(\phi X,\phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$(\nabla_X\phi)Y = \alpha\{g(X,Y)\xi + \eta(Y)X\} \tag{2.4}$$

for any vector field X, Y on M, where ∇ denotes the covariant differentiation with respect to Lorentzian metric g.
Also a Lorentzian $\alpha-$Sasakian manifold satisfies [16]

$$\nabla_X\xi = \alpha\phi X, \tag{2.5}$$

$$(\nabla_X\eta)Y = \alpha g(X,\phi Y) \tag{2.6}$$

for X, Y tangent to M.
Moreover, the curvature tensor R, the Ricci tensor S and the Ricci operator Q in a Lorentzian α-Sasakian manifold M with respect to the Levi-Civita connection ∇, satisfies following relations [16]

$$R(\xi,X)Y = \alpha^2\{g(X,Y)\xi - \eta(Y)X\}, \tag{2.7}$$

$$R(X,Y)\xi = \alpha^2\{\eta(Y)X - \eta(X)Y\}, \tag{2.8}$$

$$R(\xi,X)\xi = -R(X,\xi)\xi = \alpha^2\{X + \eta(X)\xi\}, \tag{2.9}$$

$$S(X,\xi) = 2n\alpha^2\eta(X), \tag{2.10}$$

$$S(\xi,\xi) = -2n\alpha^2, \quad Q\xi = 2n\alpha^2\xi, \tag{2.11}$$

$$S(\phi X,\phi Y) = S(X,Y) - 2n\alpha^2 g(X,Y), \tag{2.12}$$

for all vector fields $X,Y \in \Gamma(TM)$.

3. Relation Between the Quarter-Symmetric Metric Connection and Riemannian Connection

Let ∇ be a Riemannian connection and $\bar{\nabla}$ be a linear connection on Lorentzian α-Sasakian manifold M such that

$$\bar{\nabla}_X Y = \nabla_X Y + H(X,Y), \tag{3.1}$$

where H is a tensor of type $(1,2)$. Now if $\bar{\nabla}$ be a quarter-symmetric connection on M, then we have [5]

$$H(X,Y) = \frac{1}{2}[T(X,Y) + T'(X,Y) + T'(Y,X)], \tag{3.2}$$

where

$$g\left(T'(X,Y),Z\right) = g(T(Z,X),Y). \tag{3.3}$$

Using (1.2) in (3.3), we get

$$T'(X,Y) = \eta(X)\phi Y - g(\phi X,Y)\xi. \tag{3.4}$$

In view of (1.2) and (3.4), equation (3.2) gives

$$H(X,Y) = \eta(Y)\phi X - g(\phi X,Y)\xi. \tag{3.5}$$

Hence from (3.1), a quarter-symmetric connection $\bar{\nabla}$ on a Lorentzian $\alpha-$Sasakian manifold M is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X,Y)\xi. \tag{3.6}$$

Also we have

$$\left(\bar{\nabla}_X g\right)(Y,Z) = Xg(Y,Z) - g\left(\bar{\nabla}_X Y,Z\right) - g\left(Y,\bar{\nabla}_X Z\right) \tag{3.7}$$

With the help of (3.6), after simplification (3.7) gives

$$\left(\bar{\nabla}_X g\right)(Y,Z) = 0, \quad \forall\, Y,Z \in \Gamma(TM). \tag{3.8}$$

By virtue of (3.6) and (3.8), we conclude that $\bar{\nabla}$ is a quarter-symmetric metric connection. Therefore (3.6) is the relation between Riemannian connection and quarter-symmetric metric connection on a Lorentzian α-Sasakian manifold.

4. Curvature Tensor and Ricci Tensor of Lorentzian $\alpha-$Sasakian Manifold with respect to the Quarter-Symmetric Metric Connection

Let $R(X,Y)Z$ and $\bar{R}(X,Y)Z$ be the curvature tensors of a Lorentzian α-Sasakian manifold M with respect to the Riemannian connection ∇ and quarter-symmetric metric connection $\bar{\nabla}$ respectively, then relation between $R(X,Y)Z$ and $\bar{R}(X,Y)Z$ is given by

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z + \alpha\eta(Z)[\eta(Y)X - \eta(X)Y] \\ &\quad + (2\alpha-1)[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X] \\ &\quad - \alpha[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi. \end{aligned} \tag{4.1}$$

From (4.1), we have

$$\bar{R}(\xi,X)Y = (\alpha^2-\alpha)[g(X,Y)\xi - \eta(Y)X], \tag{4.2}$$

$$\bar{R}(X,Y)\xi = \left(\alpha^2-\alpha\right)[\eta(Y)X - \eta(X)Y], \tag{4.3}$$

$$\bar{R}(\xi,Y)\xi = \left(\alpha^2-\alpha\right)[Y + \eta(Y)\xi]. \tag{4.4}$$

Let $\{e_1, e_2, ..., e_{2n}, e_{2n+1} = \xi\}$ be a local orthonormal basis of vector fields in M. Since on a semi-Riemannian manifold, we have [10]

$$\sum_{i=1}^{2n+1} \varepsilon_i g(R(e_i,Y)Z,e_i) = S(Y,Z),$$

$$\sum_{i=1}^{2n+1} \varepsilon_i S(e_i,Y)g(e_i,Z) = S(Y,Z),$$

$$\sum_{i=1}^{2n+1} \varepsilon_i g(e_i,Y)g(e_i,Z) = g(Y,Z),$$

and

$$\sum_{i=1}^{2n+1} \varepsilon_i g(\phi e_i,e_i) = trace(\phi),$$

where $\varepsilon_i = g(e_i,e_i)$, $i = 1,2,...,2n+1$. Using above results on a Lorentzian α-Sasakian manifold, it can be easily verify that

$$\sum_{i=1}^{2n} g(R(e_i,Y)Z,e_i) = S(Y,Z) - \alpha^2 g(\phi Y,\phi Z), \tag{4.5}$$

$$\sum_{i=1}^{2n} S(e_i,Y)g(e_i,Z) = S(Y,Z) + 2n\alpha^2\eta(Y)\eta(Z), \tag{4.6}$$

$$\sum_{i=1}^{2n} g(e_i,e_i) = 2n, \tag{4.7}$$

$$\sum_{i=1}^{2n} g(e_i,Y)g(e_i,Z) = g(\phi Y,\phi Z), \tag{4.8}$$

$$\sum_{i=1}^{2n} g(\phi e_i,e_i) = trace(\phi) \tag{4.9}$$

and

$$\sum_{i=1}^{2n} g(\bar{R}(e_i,Y)Z,e_i) = \bar{S}(Y,Z) - \left(\alpha^2-\alpha\right) g(\phi Y,\phi Z). \tag{4.10}$$

Then from (4.1), we obtain

$$\begin{aligned} \bar{S}(Y,Z) &= S(Y,Z) + \{(2n+1)\alpha - 1\}\eta(Y)\eta(Z) \\ &\quad + (\alpha-1)g(Y,Z) - (2\alpha-1)trace(\phi)\Phi(Y,Z), \end{aligned} \tag{4.11}$$

$$\bar{S}(Y,\xi) = 2n\left(\alpha^2-\alpha\right)\eta(Y), \tag{4.12}$$

$$\bar{S}(\xi,\xi) = -2n\left(\alpha^2-\alpha\right), \tag{4.13}$$

$$\bar{S}(\phi Y,\phi Z) = \bar{S}(Y,Z) - 2n\alpha^2 g(Y,Z) - 2n\alpha\eta(Y)\eta(Z). \tag{4.14}$$

where $\bar{S}$ and $\bar{r}$ be the Ricci tensor and scalar curvature with respect to the quarter-symmetric metric connection $\bar{\nabla}$ respectively.

5. ξ-Concircularly Flat Lorentzian α-Sasakian Manifold with Respect to the Quarter-Symmetric Metric Connection

Definition 5.1. *A Lorentzian α-Sasakian manifold is said to be $\xi-$concircularly flat [1] with respect to the quarter-symmetric metric connection if $\bar{C}(X,Y)\xi = 0$, where X, $Y \in \Gamma(TM)$.*

Theorem 5.2. *A Lorentzian α-Sasakian manifold admitting a quarter-symmetric metric connection $\bar{\nabla}$ is $\xi-$concircularly flat if and only if the scalar curvature $\bar{r}$ with respect to the quarter-symmetric metric connection is equal to $2n(2n+1)\left(\alpha^2-\alpha\right)$.*

Proof. From (1.4), we have

$$\bar{C}(X,Y)\xi = \bar{R}(X,Y)\xi - \frac{\bar{r}}{2n(2n+1)}[\eta(Y)X - \eta(X)Y]. \tag{5.1}$$

Using (4.3) in (5.1), we have

$$\begin{aligned} \bar{C}(X,Y)\xi &= \left(\alpha^2-\alpha\right)[\eta(Y)X - \eta(X)Y] \\ &\quad -\frac{\bar{r}}{2n(2n+1)}[\eta(Y)X - \eta(X)Y]. \end{aligned} \tag{5.2}$$

From (5.2), we have

$$\bar{C}(X,Y)\xi = [\left(\alpha^2-\alpha\right) - \frac{\bar{r}}{2n(2n+1)}][\eta(Y)X - \eta(X)Y]. \tag{5.3}$$

Thus from (5.3), if $\bar{C}(X,Y)\xi = 0$, then $\bar{r} = 2n(2n+1)\left(\alpha^2-\alpha\right)$ or $\eta(Y)X - \eta(X)Y = 0$, implies that $\eta(X) = 0$ which is not possible. Conversely, if $\bar{r} = 2n(2n+1)\left(\alpha^2-\alpha\right)$, then from (5.3), it follows that $\bar{C}(X,Y)\xi = 0$.
This completes the proof of the theorem. □

6. Quasi-Concircularly Flat Lorentzian α-Sasakian Manifold with Respect to the Quarter-Symmetric Metric Connection

Definition 6.1. *A Lorentzian α-Sasakian manifold is said to be quasi−concircularly flat with respect to the quarter-symmetric metric connection if*

$$`\bar{C}(\phi X,Y,Z,\phi W) = 0 \tag{6.1}$$

where X, Y, Z, $W \in \Gamma(TM)$.

Definition 6.2. *A Lorentzian α-Sasakian manifold is said to be an $\eta-$Einstein manifold [17] if its Ricci tensor S of the Levi-Civita connection is of the form*

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{6.2}$$

where a and b are smooth functions on the manifold.

Theorem 6.3. *If a Lorentzian α-Sasakian manifold admitting a quarter-symmetric metric connection is quasi-concircularly flat, then the manifold with respect to the quarter-symmetric metric connection is an $\eta-$Einstein manifold.*

Proof. From (1.4), we have

$$\begin{aligned} `\bar{C}(X,Y,Z,W) &= `\bar{R}(X,Y,Z,W) - \frac{\bar{r}}{2n(2n+1)}[g(Y,Z)g(X,W) \\ &\quad -g(X,Z)g(Y,W)]. \end{aligned} \tag{6.3}$$

where $`\bar{C}(X,Y,Z,W) = g(\bar{C}(X,Y)Z,W)$ and $`\bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W)$.
Now putting $X = \phi X$ and $W = \phi W$ in (6.3), we get

$$\begin{aligned} `\bar{C}(\phi X,Y,Z,\phi W) &= `\bar{R}(\phi X,Y,Z,\phi W) - \frac{\bar{r}}{2n(2n+1)}[g(Y,Z)g(\phi X,\phi W) \\ &\quad -g(\phi X,Z)g(Y,\phi W)]. \end{aligned} \tag{6.4}$$

Using (6.1) in (6.4), we get

$$`\bar{R}(\phi X,Y,Z,\phi W) = \frac{\bar{r}}{2n(2n+1)}[g(Y,Z)g(\phi X,\phi W) - g(\phi X,Z)g(Y,\phi W)]. \tag{6.5}$$

Let $\{e_1, e_2, ..., e_{2n}, \xi\}$ be a local orthonormal basis of vector fields in M, then $\{\phi e_1, \phi e_2, ..., \phi e_{2n}, \xi\}$ is also a local orthonormal basis. Putting $X = W = e_i$ in (6.5) and summing over $i = 1$ to $2n$, we obtain

$$\sum_{i=1}^{2n} `\bar{R}(\phi e_i,Y,Z,\phi e_i) = \frac{\bar{r}}{2n(2n+1)}\sum_{i=1}^{2n}[g(Y,Z)g(\phi e_i,\phi e_i) - g(\phi e_i,Z)g(Y,\phi e_i)], \tag{6.6}$$

So by virtue of (2.3), (4.7), (4.8) and (4.10), the equation (6.6) takes the form

$$\bar{S}(Y,Z) = \left[\frac{\bar{r}(2n-1)}{2n(2n+1)} + \left(\alpha^2-\alpha\right)\right]g(Y,Z) - \left[\frac{\bar{r}}{2n(2n+1)} - \left(\alpha^2-\alpha\right)\right]\eta(Y)\eta(Z). \tag{6.7}$$

or

$$\bar{S}(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z),$$

where $a = \left[\frac{\bar{r}(2n-1)}{2n(2n+1)} + \left(\alpha^2-\alpha\right)\right]$ and $b = -\left[\frac{\bar{r}}{2n(2n+1)} - \left(\alpha^2-\alpha\right)\right]$.

From which it follows that the manifold is an $\eta-$Einstein manifold with respect to the quarter-symmetric metric connection.
This completes the proof of the theorem. □

7. ϕ-Concircularly Flat Lorentzian α-Sasakian Manifold with Respect to the Quarter-Symmetric Metric Connection

Definition 7.1. *A Lorentzian α-Sasakian manifold is said to be $\phi-$concircularly flat [11] with respect to the quarter-symmetric metric connection if*

$$`\bar{C}(\phi X,\phi Y,\phi Z,\phi W)=0, \tag{7.1}$$

where X, Y, Z, $W \in \Gamma(TM)$.

Theorem 7.2. *If a Lorentzian α-Sasakian manifold admitting a quarter-symmetric metric connection is ϕ-concircularly flat, then the manifold with respect to the quarter-symmetric metric connection is an $\eta-$Einstein manifold.*

Proof. From (1.4), we have

$$`\bar{C}(X,Y,Z,W) = `\bar{R}(X,Y,Z,W)-\frac{\bar{r}}{2n(2n+1)}[g(Y,Z)g(X,W) -g(X,Z)g(Y,W)]. \tag{7.2}$$

where $`\bar{C}(X,Y,Z,W)=g(\bar{C}(X,Y)Z,W)$ and $`\bar{R}(X,Y,Z,W)=g(\bar{R}(X,Y)Z,W)$.
Now putting $X=\phi X$, $Y=\phi Y$, $Z=\phi Z$, $W=\phi W$ in (7.2), we get

$$`\bar{C}(\phi X,\phi Y,\phi Z,\phi W) = `\bar{R}(\phi X,\phi Y,\phi Z,\phi W)-\frac{\bar{r}}{2n(2n+1)}[g(\phi Y,\phi Z)g(\phi X,\phi W) -g(\phi X,\phi Z)g(\phi Y,\phi W)]. \tag{7.3}$$

Using (7.1) in (7.3), we get

$$`\bar{R}(\phi X,\phi Y,\phi Z,\phi W)=\frac{\bar{r}}{2n(2n+1)}[g(\phi Y,\phi Z)g(\phi X,\phi W)-g(\phi X,\phi Z)g(\phi Y,\phi W)]. \tag{7.4}$$

Let $\{e_1,e_2,....,e_{2n},\xi\}$ be a local orthonormal basis of vector fields in M, then $\{\phi e_1,\phi e_2,....,\phi e_{2n},\xi\}$ is also a local orthonormal basis. Putting $X=W=e_i$ in (7.4) and summing over $i=1$ to $2n$, we obtain

$$\sum_{i=1}^{2n}`\bar{R}(\phi e_i,\phi Y,\phi Z,\phi e_i)=\frac{\bar{r}}{2n(2n+1)}\sum_{i=1}^{2n}[g(\phi Y,\phi Z)g(\phi e_i,\phi e_i)-g(\phi e_i,\phi Z)g(\phi Y,\phi e_i)], \tag{7.5}$$

So by virtue of (4.7), (4.8) and (4.10), the equation (7.5) takes the form

$$\bar{S}(\phi Y,\phi Z)=\left[\frac{\bar{r}(2n-1)}{2n(2n+1)}+\left(\alpha^2-\alpha\right)\right]g(\phi Y,\phi Z). \tag{7.6}$$

By making use of (2.3) and (4.14) in equation (7.6), we obtain

$$\bar{S}(Y,Z) = \left[\frac{\bar{r}(2n-1)}{2n(2n+1)}+\left(\alpha^2-\alpha\right)+2n\alpha^2\right]g(Y,Z) +\left[\frac{\bar{r}(2n-1)}{2n(2n+1)}+\left(\alpha^2-\alpha\right)+2n\alpha\right]\eta(Y)\eta(Z), \tag{7.7}$$

or

$$\bar{S}(Y,Z)=ag(Y,Z)+b\eta(Y)\eta(Z),$$

where $a=\left[\frac{\bar{r}(2n-1)}{2n(2n+1)}+\left(\alpha^2-\alpha\right)+2n\alpha^2\right]$ and $b=\left[\frac{\bar{r}(2n-1)}{2n(2n+1)}+\left(\alpha^2-\alpha\right)+2n\alpha\right]$.

From which it follows that the manifold is an $\eta-$Einstein manifold with respect to the quarter-symmetric metric connection.
This completes the proof of the theorem. □

8. Lorentzian α-Sasakian Manifold Satisfying $\bar{C}\cdot\bar{S}=0$ with Respect to the Quarter-Symmetric Metric Connection

Definition 8.1. *A Lorentzian α-Sasakian manifold is said to be an Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form*

$$S(X,Y)=ag(X,Y), \tag{8.1}$$

where a is a constant on the manifold.

Theorem 8.2. *If Lorentzian α-Sasakian manifold satisfying $\bar{C}\cdot\bar{S}=0$ with respect to a quarter-symmetric metric connection, then the manifold is an Einstein manifold with respect to the quarter-symmetric metric connection.*

Proof. We consider Lorentzian α-Sasakian manifolds with respect to a quarter-symmetric metric connection $\bar{\nabla}$ satisfying the curvature condition $\bar{C}\cdot\bar{S}=0$. Then

$$\left(\bar{C}(X,Y)\cdot\bar{S}\right)(Z,W)=0. \tag{8.2}$$

So,

$$\bar{S}(\bar{C}(X,Y)Z,W)+\bar{S}\left(Z,\bar{C}(X,Y)W\right)=0. \tag{8.3}$$

Putting $X=\xi$ in (8.3), we get

$$\bar{S}(\bar{C}(\xi,Y)Z,W)+\bar{S}\left(Z,\bar{C}(\xi,Y)W\right)=0. \tag{8.4}$$

From equation (1.4), we have

$$\bar{C}(\xi,Y)Z=\bar{R}(\xi,Y)Z-\frac{\bar{r}}{2n(2n+1)}[g(Y,Z)\xi-\eta(Z)Y]. \tag{8.5}$$

Using (4.2) in the equation (8.5), we obtain

$$\bar{C}(\xi,Y)Z=\{\alpha^2-\alpha-\frac{\bar{r}}{2n(2n+1)}\}[g(Y,Z)\xi-\eta(Z)Y]. \tag{8.6}$$

Using (8.6) and putting $Z=\xi$ in (8.4) and using the equations (2.2), (4.12), we obtain

$$\{\alpha^2-\alpha-\frac{\bar{r}}{2n(2n+1)}\}[\bar{S}(Y,W)-2n(\alpha^2-\alpha)g(Y,W)]=0. \tag{8.7}$$

Therefore,

$$\bar{S}(Y,W)=2n(\alpha^2-\alpha)g(Y,W)$$

provided $\bar{r}\neq 2n(2n+1)(\alpha^2-\alpha)$.
This means that the manifold is an Einstein manifold with respect to the quarter-symmetric metric connection.
This completes the proof. □

9. Example

In this section we construct an example on Lorentzian α-Sasakian manifold endowed with the quarter-symmetric metric connection. We consider the 3-dimensional manifold $M^3=\{(x,y,z):x,y,z\in\mathbb{R}\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. We choose the vector fields

$$e_1=e^z\frac{\partial}{\partial y},\ e_2=e^z(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}),\ e_3=\alpha\frac{\partial}{\partial z}=\xi,$$

which are linearly independent at each point of M^3.
Let g be a Lorentzian metric defined by

$$g(e_1,e_1)=1,\ g(e_2,e_2)=1,\ g(e_3,e_3)=-1,$$

and $g(e_i,e_j)=0$ if $i\neq j$.
Let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1=-e_1,\ \phi e_2=-e_2,\ \phi e_3=0.$$

and η be a $1-$form defined by $\eta(X)=g(X,e_3)$ for any $X\in\Gamma(TM^3)$
Now using the linearity of ϕ and g, we obtain

$$\phi^2X=X+\eta(X)\xi,$$

$$\eta(\xi)=-1,$$

and

$$g(\phi X,\phi Y)=g(X,Y)+\eta(X)\eta(Y),$$

for any vector fields $X,Y\in\Gamma(TM^3)$. Thus for $e_3=\xi$, the structure (ϕ,ξ,η,g) defines a Lorentzian para-contact metric structure on M^3.
Now, we have

$$[e_1,e_2]=0,\ [e_2,e_3]=-\alpha e_2,\ [e_1,e_3]=-\alpha e_1,$$

Let ∇ be the Levi-Civita connection of the Lorentzian metric g which is given by Koszul's formula defined by

$$2g(\nabla_XY,Z)=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)-g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y]).$$

Using Koszul's formula, we obtain the following:

$$\begin{aligned} &\nabla_{e_1}e_1 = -\alpha e_3,\ \nabla_{e_1}e_2 = 0,\ \nabla_{e_1}e_3 = -\alpha e_1, \\ &\nabla_{e_2}e_1 = 0,\ \nabla_{e_2}e_2 = -\alpha e_3,\ \nabla_{e_2}e_3 = -\alpha e_2, \\ &\nabla_{e_3}e_1 = 0,\ \nabla_{e_3}e_2 = 0,\ \nabla_{e_3}e_3 = 0, \end{aligned} \tag{9.1}$$

In view of the above results, we see that

$$(\nabla_X \eta)Y = \alpha g(\phi X, Y)\xi,$$

$$\nabla_X \xi = \alpha \phi X,$$

for all $X, Y \in \Gamma(TM^3)$ and $\xi = e_3$. Therefore the manifold is a Lorentzian α-Sasakian manifold with the structure (ϕ, ξ, η, g).
It is known that

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$$

Now using (9.1), we can easily obtain the non-zero components of the curvature tensor R as follows:

$$\begin{aligned} &R(e_1,e_2)e_1 = -\alpha^2 e_2,\ R(e_1,e_2)e_2 = \alpha^2 e_1, \\ &R(e_1,e_3)e_1 = -\alpha^2 e_3,\ R(e_1,e_3)e_3 = -\alpha^2 e_1 \\ &R(e_2,e_3)e_2 = -\alpha^2 e_3,\ R(e_2,e_3)e_3 = -\alpha^2 e_2, \end{aligned} \tag{9.2}$$

Let X, Y and Z be any three vector fields given by

$$\begin{aligned} &X = X^1 e_1 + X^2 e_2 + X^3 e_3, \\ &Y = Y^1 e_1 + Y^2 e_2 + Y^3 e_{3,} \\ &Z = Z^1 e_1 + Z^2 e_2 + Z^3 e_3 \end{aligned} \tag{9.3}$$

where X^i, Y^i and Z^i, for all $i = 1,2,3$ are all non-zero real numbers. Then

$$R(X,Y)Z = R(X^1 e_1 + X^2 e_2 + X^3 e_3, Y^1 e_1 + Y^2 e_2 + Y^3 e)(Z^1 e_1 + Z^2 e_2 + Z^3 e_3). \tag{9.4}$$

Using equation (9.2) in (9.4), we get

$$R(X,Y)Z = \alpha^2 \{g(Y,Z)X - g(X,Z)Y\}. \tag{9.5}$$

Hence, the 3-dimensional Lorentzian α-Sasakian manifold is of constant curvature α^2. Also from (9.5) , we obtain

$$S(Y,Z) = 2\alpha^2 g(Y,Z) \tag{9.6}$$

which gives $S(e_1,e_1) = S(e_2,e_2) = 2\alpha^2$, $S(e_3,e_3) = -2\alpha^2$ and therefore the scalar curvature $r = 6\alpha^2$.
Now using (9.1) in (3.7) , we obtain the components of quarter$-$symmetric metric connection $\bar{\nabla}$ as follows:

$$\begin{aligned} &\bar{\nabla}_{e_1}e_1 = -(\alpha-1)e_3,\ \bar{\nabla}_{e_1}e_2 = 0,\ \bar{\nabla}_{e_1}e_3 = -(\alpha-1)e_1, \\ &\bar{\nabla}_{e_2}e_1 = 0,\ \bar{\nabla}_{e_2}e_2 = -(\alpha-1)e_3,\ \bar{\nabla}_{e_2}e_3 = -(\alpha-1)e_2, \\ &\bar{\nabla}_{e_3}e_1 = 0, \quad \bar{\nabla}_{e_3}e_2 = 0, \quad \bar{\nabla}_{e_3}e_3 = 0, \end{aligned} \tag{9.7}$$

Using above results, we can easily obtain the components of curvature tensor $\bar{R}$ with respect to quarter$-$symmetric metric connection $\bar{\nabla}$ as follows:

$$\begin{aligned} &\bar{R}(e_1,e_2)e_1 = -(\alpha-1)^2 e_2,\ \bar{R}(e_1,e_2)e_2 = (\alpha-1)^2 e_1,\ \bar{R}(e_1,e_2)e_3 = 0, \\ &\bar{R}(e_1,e_3)e_1 = -\alpha(\alpha-1)e_3,\ \bar{R}(e_1,e_3)e_2 = 0,\ \bar{R}(e_1,e_3)e_3 = -\alpha(\alpha-1)e_1 \\ &\bar{R}(e_2,e_3)e_1 = 0,\ \bar{R}(e_2,e_3)e_2 = -\alpha(\alpha-1)e_3,\ \bar{R}(e_2,e_3)e_3 = -\alpha(\alpha-1)e_2, \end{aligned} \tag{9.8}$$

With the help of (9.8) , we find the Ricci tensors $\bar{S}$ with respect to the quarter-symmetric metric connection as:

$$\bar{S}(e_1,e_1) = \bar{S}(e_2,e_2) = (2\alpha-1)(\alpha-1),\ \bar{S}(e_3,e_3) = -2\alpha(\alpha-1).$$

From above results, it follows that the scalar curvature tensor with respect to the quarter-symmetric metric connection is $\bar{r} = 2(3\alpha-1)(\alpha-1)$.

Using (4.11) and (9.6) in 3-dimensional Lorentzian $\alpha-$Sasakian manifold M^3, we have

$$\bar{S}(Y,Z) = (2\alpha-1)(\alpha-1)g(Y,Z) - (\alpha-1)\eta(Y)\eta(Z).$$

Thus the three dimensional Lorentzian $\alpha-$Sasakian manifold M^3 is an $\eta-$Einstein manifold with respect to the quarter-symmetric metric connection $\bar{\nabla}$.
If we take $\alpha = 1$ in this example, then 3-dimensional Lorentzian $\alpha-$Sasakian manifold M^3 becomes flat with respect to the quarter-symmetric metric connection $\bar{\nabla}$.

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