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On The Mittag-Leffler Polynomials and Deformed Mittag-Leffler Polynomials

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Abstract

The present study deals with some new properties for the Mittag-Leffler polynomials and the deformed Mittag-Leffler polynomials. The results obtained here include various families of multilinear and multilateral generating functions, miscellaneous properties and also some special cases for these polynomials.

Keywords: *Mittag-Leffler polynomials, deformed Mittag-Leffler polynomials, generating function, multilinear and multilateral generating function.*
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1. Introduction

The classical Mittag-Leffler polynomials $g_n(y)$ were introduced by Mittag-Leffler in an investigation of analytic representation of the integrals and invariants of a linear homogeneous differential equation (see [2]). For the summary of basic properties of them, one may refer to the paper of Bateman (see [3], [4]). They are given by ordinary generating function as

$$\sum_{n=0}^{\infty} g_n(y)x^n = \left(\frac{1+x}{1-x}\right)^y, \tag{1.1}$$

with $|x| < 1$. The first few them are

$$\begin{aligned} g_0(y) &= 1, \\ g_1(y) &= 2y, \\ g_2(y) &= 2y^2, \\ g_3(y) &= \frac{4}{3}y^3 - \frac{2}{3}y, \\ g_4(y) &= \frac{2}{3}y^4 - \frac{4}{3}y^2, \\ g_5(y) &= \frac{4}{15}y^5 - \frac{4}{3}y^3 + \frac{2}{5}y. \end{aligned}$$

In fact, the Mittag-Leffler polynomials can be expressed in terms of the Gauss hypergeometric function ${}_2F_1$ as

$$g_n(y) = 2y\,{}_2F_1(1-n, 1-y; 2; 2),$$

for all $n \geq 1$. Following Roman, throughout this paper $M_n(x)$, also called Mittag-Leffler polynomials, are defined by the generating function as

$$\sum_{n=0}^{\infty} M_n(y)\frac{x^n}{n!} = \left(\frac{1+x}{1-x}\right)^y, \quad \text{(see, [6])}. \tag{1.2}$$

We note that the Mittag-Lefffler polynomials form the associated Sheffer sequence for

$$f(x) = \frac{e^t - 1}{e^t + 1}$$

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and have the following explicit expression

$$M_n(y)=\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)(n-1)_{n-k}2^k(y)_k,$$

where $(x)_n$ is the falling factorial given by $(x)_n=x(x-1)\cdots(x-n+1)$, for $n\geq 1$, and $(x)_0=1$. Although they are known for a long time, a few new papers considering them, appear recently (see, for example, [7] and [8]). They satisfy recurrence relations

$g_n(y+1)-g_{n-1}(y+1)=g_n(y)+g_{n-1}(y)$,

$(n+1)g_{n+1}(y)-2yg_n(y)+(n-1)g_{n-1}(y)=0$.

In [5], it was defined deformed exponential function

$e_h(x,y)=(1+hx)^{y/h},\quad (x\in\mathbb{C}-\{-1/h\},y\in\mathbb{R})$.

Function $e_h(x,y)$ keeps some of basic properties of exponential function. For $y\in R$, the following holds:

$$e_h(x,y)>0\ \ (x<-1/h\ \text{ for }\ h<0 \text{ or } x>-1/h \text{ for } h>0),\ e_h(0,y)=e_h(x,0)=1.$$

If h exchanges the sign, we have

$$e_{-h}(x,y)=e_h(-x,-y)\ (x\neq 1/h).$$

The additional property is kept only in regard to second variable:

$$e_h(x,y_1)e_h(x,y_2)=e_h(x,y_1+y_2).$$

Deformed exponential functions can be represented as expansions:

$$e_h(x,y)=\sum_{n=0}^{\infty}\frac{1}{n!}x^n y^{(n,h)}\ (|hx|<1),$$

$$e_{-h}(x,y)=\sum_{n=0}^{\infty}\frac{1}{n!}x^n y^{[n,h]}\ (|hx|<1).$$

Generating function of Mittag–Leffler polynomials can be recognized as

$$G(x,y)=(1+x)^y(1-x)^{-y}=e_1(x,y)e_{-1}(x,y).$$

For $h\in R\backslash\{0\}$ we can define deformed Mittag-Leffler polynomials as coefficients in expansion [5],

$$G_h(x,y)=e_h(x,y)e_{-h}(x,y)=\sum_{n=0}^{\infty}g_n^{(h)}(y)x^n. \tag{1.3}$$

Hence,

$$g_n^{(h)}(y)=\frac{1}{n!}\sum_{m=0}^{n}\binom{n}{m}y^{(m,h)}y^{[n-m,h]}.$$

Theorem 1.1. *(see, [5]) The successive members of sequence $\{g_n^{(h)}(y)\}_{n\in N_0}$ satisfy the three-term recurrence relation*

$(n+1)g_{n+1}^{(h)}(y)-2yg_n^{(h)}(y)-h^2(n-1)g_{n-1}^{(h)}(y)=0\ \ (n\geq 2)$,

$g_0^{(h)}(y)=1,\ g_1^{(h)}(y)=2y.$

The first members of the sequence $\{g_n^{(h)}(y)\}_{n\in N_0}$ are [5]:

$$\begin{aligned}
g_0^{(h)}(y) &= 1,\\
g_1^{(h)}(y) &= 2y,\\
g_2^{(h)}(y) &= 2y^2,\\
g_3^{(h)}(y) &= \frac{2}{3}y(2y^2+h^2),\\
g_4^{(h)}(y) &= \frac{2}{3}y^2(y^2+2h^2),\\
g_5^{(h)}(y) &= \frac{2}{15}y(2y^4+10h^2y^2+3h^4).
\end{aligned}$$

Theorem 1.2. *(see, [5]) The polynomial $g_n^{(h)}(y)$ can be represented over hypergeometric function as*

$$g_n^{(h)}(y)=2yh^{n-1}{}_2F_1\left(1-n,1-\frac{y}{h};2;2\right),\ \ n\geq 1. \tag{1.4}$$

2. Generating Functions For Deformed Mittag-Leffler Polynomials

Now, we give some special generating functions for the deformed Mittag-Leffler polynomials $g_n^{(h)}(y)$ given by (1.4).

Theorem 2.1. *The following generating function for deformed Mittag-Leffler polynomials $g_n^{(h)}(y)$ holds true:*

$$\sum_{n=0}^{\infty} g_n^{(h)}(y)x^n = \frac{2y}{h}(1-xh)^{-1}F_1\left[1-\frac{y}{h},1,1;2;2,\frac{2xh}{xh-1}\right],$$

where F_1 is the Lauricella's hypergeometric functions of three variables.

Proof. The polynomials ${}_2F_1[\rho-n,\alpha;\gamma;z]$ admit the following generating functions

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2F_1[\rho-n,\alpha;\gamma;z]t^n = (1-t)^{-\lambda}\ F_1\left[\alpha,\rho,\lambda;\gamma;z,\frac{zt}{t-1}\right],\quad |t|<1 \tag{2.1}$$

where we have used a special case of Lauricella function (see, for example, [1], p. 150). If we put $\lambda=1,\ \rho=1,\ \alpha=1-\frac{y}{h},\ \gamma=2,\ z=2$ in the generating function (2.1) and using (1.4) we get

$$\begin{aligned}\sum_{n=0}^{\infty} g_n^{(h)}(y)x^n &= \sum_{n=0}^{\infty} 2yh^{n-1}\,{}_2F_1(1-n,1-\frac{y}{h};2;2)x^n\\ &= \frac{2y}{h}\sum_{n=0}^{\infty}{}_2F_1(1-n,1-\frac{y}{h};2;2)(xh)^n\\ &= \frac{2y}{h}(1-xh)^{-1}F_1\left[1-\frac{y}{h},1,1;2;2,\frac{2xh}{xh-1}\right],\quad |xh|<1,\end{aligned}$$

which completes the proof. □

Theorem 2.2. *The following generating function for deformed Mittag-Leffler polynomials $g_n^{(h)}(y)$ holds true:*

$$\sum_{n=0}^{\infty} g_n^{(h)}(y)x^n = \frac{2y}{h}(1-xh)^{-1}\,{}_2F_1\left[1-\frac{y}{h},1;2;\frac{2}{1-xh}\right],$$

where ${}_2F_1$ is Gaussian hypergeometric function.

Proof. The polynomials ${}_2F_1[\rho-n,\alpha;\lambda+\rho;z]$ admit the following generating functions

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2F_1[\rho-n,\alpha;\lambda+\rho;z]t^n \\ &= (1-z)^{-\alpha}(1-t)^{-\lambda}\,{}_2F_1\left[\alpha,\lambda;\lambda+\rho;\frac{z}{(1-z)(t-1)}\right],\end{aligned} \tag{2.2}$$

where we have used a special case of hypergeometric function (see, for example, [1], p. 151). If we put $\lambda=1,\rho=1,\alpha=1-\frac{y}{h},\ z=2$ in the generating function (2.2) and using (1.4), we get

$$\begin{aligned}\sum_{n=0}^{\infty} g_n^{(h)}(y)x^n &= \sum_{n=0}^{\infty} 2yh^{n-1}\,{}_2F_1(1-n,1-\frac{y}{h};2;2)x^n\\ &= \frac{2y}{h}\sum_{n=0}^{\infty}{}_2F_1(1-n,1-\frac{y}{h};2;2)(xh)^n\\ &= \frac{2y}{h}(1-xh)^{-1}\,{}_2F_1\left[1-\frac{y}{h},1;2;\frac{2}{1-xh}\right],\end{aligned}$$

which completes the proof. □

Theorem 2.3. *The following bilateral generating function for deformed Mittag-Leffler polynomials $g_n^{(h)}(y)$ holds true:*

$$\begin{aligned}&\sum_{n=0}^{\infty} g_n^{(h)}(y)F^{u+1:p;r}_{v:q;s}\left[\begin{array}{c}-n,(e_u):(a_p);(c_r);\\ (f_v):(b_q);(d_s);\end{array} z,t\right]x^n\\ &= \frac{2y}{h}(-1)^{-1+\frac{y}{h}}(1-xh)^{-1}\\ &\quad\times F^{(3)}\left[\begin{array}{c}1::-;(e_u);-:1-\frac{y}{h};(a_p);(c_r);\\ -::-;(f_v);-:2;(b_q);(d_s);\end{array}\frac{2}{1-xh},\frac{z(xh)}{xh-1},\frac{t(xh)}{xh-1}\right],\end{aligned}$$

where $F^{(3)}$ is the Srivastava's general triple hypergeometric seies.

Proof. The polynomials ${}_2F_1\left[\rho-n,\alpha;\lambda+\rho;\varpi\right]$ admit the following bilateral generating functions

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2F_1\left[\rho-n,\alpha;\lambda+\rho;\varpi\right]F^{u+1:p;r}_{v:q;s}\left[\begin{array}{c}-n,(e_u):(a_p);(c_r);\\(f_v):(b_q);(d_s);\end{array}z,t\right]x^n \qquad (2.3)\\
=\;&(1-\varpi)^{-\alpha}(1-x)^{-\lambda}\\
&\times F^{(3)}\left[\begin{array}{c}\lambda::-;(e_u);-:\quad\alpha;(a_p);(c_r);\\-::-;(f_v);-:\lambda+\rho;(b_q);(d_s);\end{array}\frac{\varpi}{(1-\varpi)(x-1)},\frac{zx}{x-1},\frac{tx}{x-1}\right],
\end{aligned}$$

where we have used a special case of hypergeometric function (see, for example, [1], p. 151). If we put $\lambda=1,\ \rho=1,\ \alpha=1-\frac{y}{h},\ \varpi=2$ in the generating function (2.3) and using (1.4), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}g_n^{(h)}(y)\,F^{u+1:p;r}_{v:q;s}\left[\begin{array}{c}-n,(e_u):(a_p);(c_r);\\(f_v):(b_q);(d_s);\end{array}z,t\right]x^n\\
=\;&\sum_{n=0}^{\infty}2yh^{n-1}\,{}_2F_1(1-n,1-\frac{y}{h};2;2)F^{u+1:p;r}_{v:q;s}\left[\begin{array}{c}-n,(e_u):(a_p);(c_r);\\(f_v):(b_q);(d_s);\end{array}z,t\right]x^n\\
=\;&\frac{2y}{h}\sum_{n=0}^{\infty}{}_2F_1(1-n,1-\frac{y}{h};2;2)F^{u+1:p;r}_{v:q;s}\left[\begin{array}{c}-n,(e_u):(a_p);(c_r);\\(f_v):(b_q);(d_s);\end{array}z,t\right](xh)^n\\
=\;&\frac{2y}{h}(-1)^{-1+\frac{y}{h}}(1-xh)^{-1}\\
&\times F^{(3)}\left[\begin{array}{c}1::-;\ (e_u);-:1-\frac{y}{h};(a_p);(c_r);\\-::-;\ (f_v)\ ;-:2;(b_q);(d_s);\end{array}\frac{2}{1-xh},\frac{z(xh)}{xh-1},\frac{t(xh)}{xh-1}\right],
\end{aligned}$$

which completes the proof. □

Theorem 2.4. *The following bilinear generating function for deformed Mittag–Leffler polynomials $g_n^{(h)}(y)$ holds true:*

$$\begin{aligned}
&\sum_{n=0}^{\infty}g_n^{(h)}(y)\,g_{n+1}^{(h)}(z)\,x^n\\
=\;&\frac{4yz}{h}(-1)^{-1+\frac{y}{h}}\left(1-xh^2\right)^{-1}\\
&\times F_2\left[1,1-\frac{y}{h},1-\frac{z}{h};2,2;\frac{2}{1-xh^2},\frac{2xh^2}{xh^2-1}\right],
\end{aligned}$$

where F_2 is the Appell's hypergeometric functions of two variables.

Proof. The polynomials ${}_2F_1(\rho-n,\alpha;\beta;x)$ admit the following bilinear generating functions

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(\beta-\rho)_n}{n!}\,{}_2F_1(\rho-n,\alpha;\beta;x)\,{}_2F_1(-n,\gamma;\delta;y)t^n \qquad (2.4)\\
=\;&(1-x)^{-\alpha}(1-t)^{\rho-\beta}F_2\left[\beta-\rho,\alpha,\gamma;\beta,\delta;\frac{x}{(1-x)(t-1)},\frac{yt}{t-1}\right],
\end{aligned}$$

where we have used a special case of hypergeometric function (see, for example, [1], p. 295). If we put $x=2,\quad \rho=1,\ \beta=2,\ \alpha=1-\frac{y}{h}$, $\gamma=1-\frac{z}{h},\quad \delta=2,\quad y=2,\quad t=xh^2$ in the generating function (2.4) and using (1.4), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}g_n^{(h)}(y)\,g_{n+1}^{(h)}(z)\,x^n\\
=\;&\sum_{n=0}^{\infty}2yh^{n-1}\,{}_2F_1(1-n,1-\frac{y}{h};2;2)2zh^n\,{}_2F_1(-n,1-\frac{z}{h};2;2)x^n\\
=\;&\frac{4yz}{h}\sum_{n=0}^{\infty}{}_2F_1(1-n,1-\frac{y}{h};2;2)\,{}_2F_1(-n,1-\frac{z}{h};2;2)\left(xh^2\right)^n\\
=\;&\frac{4yz}{h}(-1)^{-1+\frac{y}{h}}\left(1-xh^2\right)^{-1}\\
&\times F_2\left[1,1-\frac{y}{h},1-\frac{z}{h};2,2;\frac{2}{1-xh^2},\frac{2xh^2}{xh^2-1}\right],
\end{aligned}$$

which completes the proof. □

Theorem 2.5. *The following generating function for deformed Mittag–Leffler polynomials $g_n^{(h)}(y)$ holds true:*

$$\sum_{n=0}^{\infty}g_n^{(h)}(y)\,z^n=\frac{2y}{h}(1-zh)^{-1}F^{1:1;1}_{1:0;0}\left[\begin{array}{c}1-\frac{y}{h},1:1;\\2:-\quad;\quad-;\end{array}2,\frac{2zh}{zh-1}\right],$$

where $F^{1:1;1}_{1:0;0}$ is generalized (Kampé de Fériet's) hypergeometric function of two variables.

Proof. The generalized hypergeometric polynomials ${}_{p+1}F_q\left[-m-n,\left(a_p\right);\left(b_q\right);x\right]$ admit the following generating functions

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\ {}_{p+1}F_q\left[-m-n,\left(a_p\right);\left(b_q\right);x\right]z^n \\ =\ &(1-z)^{-\lambda}F_{q:0;0}^{p:1;1}\left[\begin{array}{c}\left(a_p\right):-m;\lambda; \\ \left(b_q\right):-;-;\end{array}\ x,\frac{xz}{z-1}\right],\qquad |z|<1\end{aligned} \tag{2.5}$$

where we have used a special case of generalized hypergeometric function (see, for example, [1], p. 268). If we put $\lambda=1,\ p=1,\ q=1,$ $m=-1,\ \left(a_p\right)=1-\frac{y}{h},\ \left(b_q\right)=2,\ \ x=2$ in the generating function (2.5) and using (1.4), we get

$$\begin{aligned}\sum_{n=0}^{\infty}g_n^{(h)}(y)z^n &= \sum_{n=0}^{\infty}2yh^{n-1}\ {}_2F_1(1-n,1-\frac{y}{h};2;2)z^n \\ &= \frac{2y}{h}\sum_{n=0}^{\infty}{}_2F_1(1-n,1-\frac{y}{h};2;2)(zh)^n \\ &= \frac{2y}{h}(1-zh)^{-1}F_{1:0;0}^{1:1;1}\left[\begin{array}{c}1-\frac{y}{h},1:1; \\ 2:-\ \ ;\ \ -;\end{array}\ 2,\frac{2zh}{zh-1}\right],\end{aligned}$$

which completes the proof. □

3. Bilinear and Bilateral Generating Functions

In this section, we derive several bilinear and bilateral generating fuctions for the Mittag–Leffler polynomials $g_n(y)$ and deformed Mittag–Leffler polynomials $g_n^{(h)}(y)$ which generated by (1.1) and given explicitly by (1.3) using the similar method considered in (see [9], [10],[11]).

Theorem 3.1. *Let*

$$\theta_{\mu,\varphi}(y;s_1,\ldots,s_r;\zeta):\ =\sum_{k=0}^{\left[\frac{n}{p}\right]}a_k g_{n-pk}(y)\,\Omega_{\mu+\varphi k}(s_1,\ldots,s_r)\,\zeta^k. \tag{3.1}$$

If

$$\bigwedge\nolimits_{\mu,\varphi}(s_1,\ldots,s_r;\tau):\ =\sum_{k=0}^{\infty}a_k\Omega_{\mu+\varphi k}(s_1,\ldots,s_r)\,\tau^k$$

then, for every nonnegative integer μ, *we have*

$$\sum_{n=0}^{\infty}\theta_{\mu,\varphi}\left(y;s_1,\ldots,s_r;\frac{\eta}{x^p}\right)x^n=\left(\frac{1+x}{1-x}\right)^y\bigwedge\nolimits_{\mu,\varphi}(s_1,\ldots,s_r;\eta). \tag{3.2}$$

Proof. If we denote the left-hand side of (3.2) by S and use (3.1),

$$S=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\left[\frac{n}{p}\right]}a_k g_{n-pk}(y)\,\Omega_{\mu+\varphi k}(s_1,\ldots,s_r)\frac{\eta^k}{x^{pk}}\right)x^n.$$

Replacing n by $n+pk$,

$$\begin{aligned}T&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k g_n(y)\,\Omega_{\mu+\varphi k}(s_1,\ldots,s_r)\frac{\eta^k}{x^{pk}}x^{n+pk} \\ &=\sum_{n=0}^{\infty}g_n(y)x^n\sum_{k=0}^{\infty}a_k\Omega_{\mu+\varphi k}(s_1,\ldots,s_r)\,\eta^k \\ &=\left(\frac{1+x}{1-x}\right)^y\bigwedge\nolimits_{\mu,\varphi}(s_1,\ldots,s_r;\eta),\end{aligned}$$

which completes the proof. □

Theorem 3.2. *Let*

$$\theta_{\mu,\varphi}^{h}(y;s_1,\ldots,s_r;\zeta):\ =\sum_{k=0}^{\left[\frac{n}{p}\right]}a_k g_{n-pk}^{(h)}(y)\,\Omega_{\mu+\varphi k}(s_1,\ldots,s_r)\,\zeta^k. \tag{3.3}$$

If

$$\bigwedge\nolimits_{\mu,\varphi}(s_1,\ldots,s_r;\tau):\ =\sum_{k=0}^{\infty}a_k\Omega_{\mu+\varphi k}(s_1,\ldots,s_r)\,\tau^k$$

then, for every nonnegative integer μ, *we have*

$$\sum_{n=0}^{\infty}\theta_{\mu,\varphi}^{h}\left(y;s_1,\ldots,s_r;\frac{\eta}{x^p}\right)x^n=G_h(x,y)\bigwedge\nolimits_{\mu,\varphi}(s_1,\ldots,s_r;\eta). \tag{3.4}$$

Proof. If we denote the left-hand side of (3.4) by T and use (3.3),

$$T=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\left[\frac{n}{p}\right]}a_k g_{n-pk}^{(h)}(y)\,\Omega_{\mu+\varphi k}(s_1,\ldots,s_r)\,\frac{\eta^k}{x^{pk}}\right)x^n.$$

Replacing n by $n+pk$,

$$\begin{aligned}T &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k g_n^{(h)}(y)\,\Omega_{\mu+\varphi k}(s_1,\ldots,s_r)\,\frac{\eta^k}{x^{pk}}x^{n+pk}\\ &= \sum_{n=0}^{\infty}g_n^{(h)}(y)\,x^n\sum_{k=0}^{\infty}a_k\Omega_{\mu+\varphi k}(s_1,\ldots,s_r)\,\eta^k\\ &= G_h(x,y)\bigwedge\nolimits_{\mu,\varphi}(s_1,\ldots,s_r;\eta),\end{aligned}$$

which completes the proof. □

4. Special Cases and Miscellaneous Properties

It is possible to give many applications of our theorems with help of appropriate choices of the multivariable functions $\Omega_{\mu+\varphi k}(s_1,\ldots,s_r)$, $k\in\mathbb{N}_0,\ r\in\mathbb{N}$. We first set

$$\Omega_{\mu+\varphi k}(s_1,\ldots,s_r)=\Phi_{\mu+\varphi k}^{(\alpha)}(s_1,\ldots,s_r)$$

in Theorem 3.1, where the multivariable polynomials $\Phi_{\mu+\varphi k}^{(\alpha)}(x_1,\ldots,x_r)$ [9], generated by

$$\sum_{n=0}^{\infty}\Phi_n^{(\alpha)}(x_1,\ldots,x_r)z^n=(1-x_1 z)^{-\alpha}\,e^{(x_2+\ldots+x_r)z}, \tag{4.1}$$

where $|z|<|x_1|^{-1}$.
We are thus led to the following result which provides a class of bilateral generating functions for the Mittag–Leffler polynomials $g_n(y)$ and the family of multivariable polynomials given explicitly by (4.1).

Corollary 1. *If*

$$\begin{aligned}\bigwedge\nolimits_{\mu,\varphi}(s_1,\ldots,s_r;\tau) \quad : &= \sum_{k=0}^{\infty}a_k\Phi_{\mu+\varphi k}^{(\alpha)}(s_1,\ldots,s_r)\tau^k\\ &(a_k\neq 0,\ \mu,\varphi\in\mathbb{C})\end{aligned}$$

then, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\left[\frac{n}{p}\right]}a_k g_{n-pk}(y)\,\Phi_{\mu+\varphi k}^{(\alpha)}(s_1,\ldots,s_r)\frac{\eta^k}{x^{pk}}\right)x^n\\ =&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k g_n(y)\,\Phi_{\mu+\varphi k}^{(\alpha)}(s_1,\ldots,s_r)\frac{\eta^k}{x^{pk}}x^{n+pk}\\ =&\sum_{n=0}^{\infty}g_n(y)\,x^n\sum_{k=0}^{\infty}a_k\Phi_{\mu+\varphi k}^{(\alpha)}(s_1,\ldots,s_r)\eta^k\\ =&\left(\frac{1+x}{1-x}\right)^y\bigwedge\nolimits_{\mu,\varphi}(s_1,\ldots,s_r;\eta).\end{aligned}$$

Remark 4.1. *Using the generating relation (4.1) for the multivariable polynomials and getting $a_k=1,\ \mu=0,\ \varphi=1$ in Corollary 1, we get*

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\left[\frac{n}{p}\right]}g_{n-pk}(y)\,\Phi_k^{(\alpha)}(s_1,\ldots,s_r)\frac{\eta^k}{x^{pk}}\right)x^n\\ =&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}g_n(y)\,\Phi_k^{(\alpha)}(s_1,\ldots,s_r)\frac{\eta^k}{x^{pk}}x^{n+pk}\\ =&\sum_{n=0}^{\infty}g_n(y)\,x^n\sum_{k=0}^{\infty}\Phi_k^{(\alpha)}(s_1,\ldots,s_r)\eta^k\\ =&\left(\frac{1+x}{1-x}\right)^y(1-s_1\eta)^{-\alpha}e^{(s_2+\ldots+s_r)\eta},\\ &\left(|x|<1,\ |\eta|<\left\{|s_1|^{-1}\right\}\right).\end{aligned}$$

If we set

$\Omega_{\mu+\varphi k}(s_1,...,s_r) = h^{(\beta_1,...,\beta_s)}_{\mu+\varphi k}(s_1,...,s_r),$

in Theorem 3.2. Recall that, by $h_n^{(\beta_1,...,\beta_r)}(z_1,...,z_r)$ we denote the multivariable Lagrange-Hermite polynomials (see, e.g. [11]) generated by

$$\sum_{n=0}^{\infty} h_n^{(\beta_1,...,\beta_r)}(z_1,...,z_r)t^n = \prod_{j=1}^{r}\left\{\left(1-z_jt^j\right)^{-\beta_j}\right\}, \tag{4.2}$$

where $|t| < \min\left\{|z_1|^{-1}, |z_2|^{-1/2},...,|z_r|^{-1/r}\right\}$.

We are thus led to the following result which provides a class of bilateral generating functions for the deformed Mittag–Leffler polynomials $g_n^{(h)}(y)$ and the family of multivariable polynomials given explicitly by (4.2).

Corollary 2. *If*

$$\bigwedge\nolimits_{\mu,\varphi}(s_1,...,s_r;\tau) := \sum_{k=0}^{\infty} a_k h^{(\beta_1,...,\beta_s)}_{\mu+\varphi k}(s_1,...,s_r)\,\tau^k$$

$(a_k \neq 0,\ k \in \mathbb{N}_0)$

and

$$\theta^h_{\mu,\varphi}(y;s_1,...,s_r;\zeta) := \sum_{k=0}^{\left[\frac{n}{p}\right]} a_k g^{(h)}_{n-pk}(y)\, h^{(\beta_1,...,\beta_s)}_{\mu+\varphi k}(s_1,...,s_r)\,\zeta^k$$

where $p,q \in \mathbb{N}$ *and, we have*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \theta^h_{\mu,\varphi}\left(y;s_1,...,s_r;\frac{\eta}{x^p}\right)x^n \\
=\ &\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{p}\right]} a_k g^{(h)}_{n-pk}(y)\, h^{(\beta_1,...,\beta_s)}_{\mu+\varphi k}(s_1,...,s_r)\,\frac{\eta^k}{x^{pk}}x^n \\
=\ &\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k g^{(h)}_{n}(y)\, h^{(\beta_1,...,\beta_s)}_{\mu+\varphi k}(s_1,...,s_r)\,\frac{\eta^k}{x^{pk}}x^{n+pk} \\
=\ &\sum_{n=0}^{\infty} g^{(h)}_{n}(y)\,x^n \sum_{k=0}^{\infty} a_k h^{(\beta_1,...,\beta_s)}_{\mu+\varphi k}(s_1,...,s_r)\,\eta^k \\
=\ &G_h(x,y)\bigwedge\nolimits_{\mu,\varphi}(s_1,...,s_r;\eta).
\end{aligned}$$

Remark 4.2. *Using the relation (4.2) for the deformed Mittag–Leffler polynomials* $g_n^{(h)}(y)$ *and* $a_k = 1,\ \mu = 0,\ \varphi = 1$ *in Corollary 2, we receive*

$$\begin{aligned}
&\sum_{n=0}^{\infty} g^{(h)}_{n}(y)\,x^n \sum_{k=0}^{\infty} h^{(\beta_1,...,\beta_s)}_{k}(s_1,...,s_r)\,\eta^k \\
=\ &G_h(x,y)\prod_{j=1}^{r}\left\{\left(1-s_j\eta^j\right)^{-\beta_j}\right\}, \\
|\eta| <\ &\min\left\{|s_1|^{-1},|s_2|^{-1/2},...,|s_r|^{-1/r}\right\},\ j=1,2,...,r.
\end{aligned}$$

If we set

$r=1,\ s_1 = s$ and $\Omega_{\mu+\varphi k}(s) = g^{(h)}_{\mu+\varphi k}(s),$

in Theorem 3.2. We are thus led to the following result which provides a class of bilinear generating functions for the deformed Mittag-Leffler polynomials $g_n^{(h)}(y)$.

Corollary 3. *If*

$$\bigwedge\nolimits_{\mu,\varphi}(s;\tau) := \sum_{k=0}^{\infty} a_k g^{(h)}_{\mu+\varphi k}(s)\,\tau^k$$

$(a_k \neq 0,\ k \in \mathbb{N}_0)$

and

$$\theta^h_{\mu,\varphi}(y;s;\zeta) := \sum_{k=0}^{\left[\frac{n}{p}\right]} a_k g^{(h)}_{n-pk}(y)\, g^{(h)}_{\mu+\varphi k}(s)\,\zeta^k$$

where $p,q\in\mathbb{N}$ *and, we have*

$$\begin{aligned}\sum_{n=0}^{\infty}\theta_{\mu,\varphi}^{h}\left(y;s;\frac{\eta}{x^{p}}\right)x^{n} &= \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{p}\right]}a_{k}g_{n-pk}^{(h)}(y)\,g_{\mu+\varphi k}^{(h)}(s)\frac{\eta^{k}}{x^{pk}}x^{n}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_{k}g_{n}^{(h)}(y)\,g_{\mu+\varphi k}^{(h)}(s)\frac{\eta^{k}}{x^{pk}}x^{n+pk}\\ &= \sum_{n=0}^{\infty}g_{n}^{(h)}(y)\,x^{n}\sum_{k=0}^{\infty}a_{k}g_{\mu+\varphi k}^{(h)}(s)\eta^{k}\\ &= G_{h}(x,y)\bigwedge\nolimits_{\mu,\varphi}(s;\eta).\end{aligned}$$

Remark 4.3. *For the deformed Mittag-Leffler polynomials* $g_n^{(h)}(y)$ *and* $a_k=1,\ \mu=0,\ \varphi=1$ *in Corollary 3, we receive*

$$\begin{aligned}&\sum_{n=0}^{\infty}g_{n}^{(h)}(y)\,x^{n}\sum_{k=0}^{\infty}g_{k}^{(h)}(s)\eta^{k}\\ =\ &G_{h}(x,y)\,G_{h}(\eta,s).\end{aligned}$$

Notice that, for every suitable choice of the coefficients a_k $(k\in N_0)$, if the multivariable functions $\Omega_{\mu+\psi k}(y_1,\ldots,y_r)$, $r\in N$, are expressed as an appropriate product of several simpler relatively functions, the assertions of Theorem 3.1 and 3.2, can be applied to yield many different families of multilinear and multilateral generating functions for the Mittag–Leffler polynomials $g_n(y)$ and the deformed Mittag–Leffler polynomials $g_n^{(h)}(y)$.

Theorem 4.4. *The Mittag-Leffler polynomials* $g_n(y)$ *have the following relationships [12]:*

a) $g_n(y)=\left(\frac{2y}{n!}\right)M_{n-1}(y-1;2;-1)$, where $M_n(x;\beta,c)$ is Meixner polynomials.
b) $M_n(y)=n!g_n(y)$, where $M_n(y)$ is given by (1.2).

Theorem 4.5. *The deformed Mittag–Leffler polynomials* $g_n^{(h)}(y)$ *have the following relationships:*

a) $g_n^{(h)}(y)=2y\left(\frac{1-y}{h}\right)^{1-n}F_1\left[1-n,1-\frac{y}{h};1+\frac{y}{h};2;2-y,y\right].$
b) $g_n^{(h)}(y)=2y\left(\frac{1-y}{h}\right)^{1-n}\left(\frac{y}{2-y}\right)^{1-\frac{y}{h}}F_2\left[2,1-\frac{y}{h},1-n;2,2;\frac{2-2y}{2-y},y\right].$

Proof. a) Using relations (1.4) and

$$_2F_1\left(a,b;b+b^{'};\frac{x-y}{1-y}\right)=\frac{1}{(1-y)^{-a}}F_1\left[a,b,b^{'};b+b^{'};x,y\right],\ \text{(see, [1], p.164)},\tag{4.3}$$

we arrived the result.
b) Using relations (1.4), (4.3) and

$$F_2\left(a,b,b^{'};a,c^{'};x,y\right)=(1-x)^{-b}F_1\left[b^{'},b,a-b;c^{'};\frac{y}{1-x},y\right],\ \text{(see, [1], p.163)},$$

we get the desired result. □

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