On $\mathcal{I}_2$-Convergence and $\mathcal{I}_2^*$-Convergence of Double Sequences in Fuzzy Normed Spaces

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Abstract

In this paper first, we investigate some properties of $\mathcal{I}_2$-convergence in fuzzy normed spaces. After, we study some relationships between $\mathcal{I}_2$-convergence and $\mathcal{I}_2^*$-convergence of double sequences in fuzzy normed spaces.

Keywords: Double sequences, $\mathcal{I}_2$-convergence, $\mathcal{I}_2$-Cauchy, Fuzzy normed space.

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1. Introduction and Background

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [11] and Schoenberg [30]. A lot of developments have been made in this area after the various studies of researchers [21, 25]. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [15] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers $\mathbb{N}$. Das et al. [3] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. A lot of developments have been made in this area after the works of [4, 16, 27, 31].

The concept of ordinary convergence of a sequence of fuzzy numbers was firstly introduced by Matloka [18] and proved some basic theorems for sequences of fuzzy numbers. Nanda [23] studied the sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers are a complete metric space. Şençimen and Pehlivan [29] introduced the notions of statistically convergent sequence and statistically Cauchy sequence in a fuzzy normed linear space. Hazarika [13] studied the concepts of $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence and $\mathcal{I}$-Cauchy sequence in a fuzzy normed linear space. Dündar and Talo [8, 9] introduced the concepts of $\mathcal{I}_2$-convergence and $\mathcal{I}_2$-Cauchy sequence for double sequences of fuzzy numbers and studied some properties and relations of them. Hazarika and Kumar [14] introduced the notion of $\mathcal{I}_2$-convergence and $\mathcal{I}_2$-Cauchy double sequences in a fuzzy normed linear space. A lot of developments have been made in this area after the various studies of researchers [17, 20, 32, 26].

Now, we recall the concept of ideal, convergence, statistical convergence, ideal convergence of sequence, double sequence and fuzzy normed and some basic definitions (see [1, 2, 5, 6, 7, 8, 10, 11, 12, 19, 20, 21, 22, 24, 25, 28, 29]).

Fuzzy sets are considered with respect to a nonempty base set $X$ of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $\mu(x)$ taking values in $[0, 1]$. The function $\mu : X \to [0, 1]$ is called a fuzzy set.

A fuzzy set $\mu$ on $\mathbb{R}$ is called a fuzzy number if it has the following properties:

1. $\mu$ is normal, that is, there exists an $a_0 \in \mathbb{R}$ such that $\mu(a_0) = 1$;
2. $\mu$ is fuzzy convex, that is, for $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$, $\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x), \mu(y)\}$;
3. $\mu$ is upper semicontinuous;
4. $\operatorname{supp}(\mu) = \{x \in \mathbb{R} : \mu(x) > 0\}$, or denoted by $|\mu|$, is compact.

Let $L(\mathbb{R})$ be the set of all fuzzy numbers. If $u \in L(\mathbb{R})$ and $u(t) = 0$ for $t < 0$, then $u$ is called a non-negative fuzzy number. We write $L^+(\mathbb{R})$ by the set of all non-negative fuzzy numbers. We can say that $u \in L^+(\mathbb{R})$ if $u_{\alpha} \geq 0$ for each $\alpha \in [0, 1]$. Clearly we have $0 \in L(\mathbb{R})$. For
u ∈ L(ℝ), the α level set of u is defined by
\[ |u|_α^u = \begin{cases} \{ x ∈ ℝ : u(x) ≥ α \}, & \text{if } α ∈ (0, 1] \\ \{ x ∈ ℝ : u(x) ≥ α \}, & \text{if } α = 0. \end{cases} \]

A partial order ≤ on L(ℝ) is defined by u ≤ v if uα ≤ vα for all α ∈ [0, 1]. Arithmetic operation for t ∈ ℝ, ⊕, ⊗, and ⊙ on L(ℝ) are defined by
\[ (u ⊕ v)(t) = \sup_{s ∈ ℝ} \{ u(s) ∧ v(t − s) \}, \quad (u ⊗ v)(t) = \sup_{s ∈ ℝ} \{ u(s) ∧ v(s − t) \}, \quad (u ⊙ v)(t) = \sup_{s ∈ ℝ} \{ u(s) ∧ v(s) \} \]

for k ∈ ℝ+*, ku is defined as ku(t) = u(t/k) and 0u(t) = 0, t ∈ ℝ.

Some arithmetic operations for α-level sets are defined as follows:
\[ |u|_α^u = [u_α^u, u_α^u], \quad |v|_α^v = [v_α^v, v_α^v], \quad |u ⊕ v|_α^v = [u_α^v + v_α^v, u_α^v + v_α^v], \quad |u ⊗ v|_α^v = [u_α^v − v_α^v, u_α^v − v_α^v], \quad |u ⊙ v|_α^v = [u_α^v, v_α^v] \]

and \[ 1 ⊓ u|_α^u = [1/α, 1], \quad u_α^u > 0. \]

For u, v ∈ L(ℝ), the supremum metric on L(ℝ) is defined as
\[ D(u, v) = \sup_{0 ≤ α ≤ 1} \max \{ |u|_α^u − |v|_α^v, |v|_α^v − |u|_α^u \} \]

It is known that D is a metric on L(ℝ) and (L(ℝ), D) is a complete metric space.

A sequence \( x = (x_0) \) of fuzzy numbers is said to be convergent to the fuzzy number \( x_0 \), if for every \( ε > 0 \) there exists a positive integer \( k_0 \) such that \( D(x_k, x_0) < ε \) for \( k > k_0 \) and a sequence \( x = (x_0) \) of fuzzy numbers convergent to levelwise to \( x_0 \) if and only if \( \lim_{k→∞} |x_k|_α^x = |x_0|_α^x \) and \( \lim |x_k|_α^x = |x_0|_α^x \), where \( |x_k|_α^x = (\{ x_k(α) \}_α^x, \{ x_k(α) \}_α^x) \) and \( |x_0|_α^x = (\{ x_0(α) \}_α^x, \{ x_0(α) \}_α^x) \), for every \( α ∈ (0, 1) \).

Let X be a vector space over ℝ, \( || \cdot ||_X : X → L^*(ℝ) \) and the mappings \( L, R \) respectively, left norm and right norm) : \([0, 1] × [0, 1] → [0, 1] \) be symmetric, nondecreasing in both arguments and satisfy \( L(0, 0) = 0 \) and \( R(1, 1) = 1 \).

The quadruple \( (X, || \cdot ||_X, L, R) \) is called fuzzy normed linear space (briefly FNS) and || \·|| a fuzzy norm if the following axioms are satisfied

1. \( ||x|| = \bar{0} \) if \( x = 0 \).
2. \( ||x + y|| ≤ ||x|| + ||y|| \) and \( ||ax|| = |a| ||x|| \) for \( x ∈ X \), \( r ∈ ℝ \).
3. For all \( x, y ∈ X \)
   \( (a) \) \( ||x + y|| + \delta (s + t) ≤ ||x|| + ||y|| + \delta (s + t) \), whenever \( s ≤ ||x||, t ≤ ||y|| \) and \( s + t ≤ ||x + y|| \),
   \( (b) \) \( ||x + y|| + \delta (s + t) ≤ \psi (s + t) \), whenever \( s ≥ ||x||, t ≥ ||y|| \) and \( s + t ≥ ||x + y|| \).

Let \( (X, || \cdot ||_C) \) be an ordinary normed linear space. Then, a fuzzy norm \( || \cdot || \) on X can be obtained by
\[ ||x|| = \begin{cases} 0, & \text{if } t ≤ a ||x||_C \text{ or } t ≥ b ||x||_C \\ \frac{t}{(t-a)||x||_C} - \frac{a}{t}, & \text{if } a ||x||_C ≤ t ≤ ||x||_C \\ \frac{b}{(t-b)||x||_C}, & \text{if } ||x||_C ≤ t ≤ b ||x||_C \end{cases} \]

where ||x||_C is the ordinary norm of x (≠ 0), 0 < a < 1 and 1 < b < ∞. For x = 0, define ||x|| = 0. Hence, \( (X, || \cdot ||) \) is a fuzzy normed linear space.

Let us consider the topological structure of an FNS \( (X, || \cdot ||) \). For every \( ε > 0, α ∈ (0, 1] \) and \( x ∈ X \), the \( (ε, α) \) neighborhood of x is the set \( N(ε, α) = \{ y ∈ X : ||x - y||_α^x ≤ ε \} \).

Let \( (X, || \cdot ||) \) be an FNS. A sequence \( (x_n)_{n=0}^{∞} \) in X is convergent to \( x ∈ X \) with respect to the fuzzy norm on X and we denote by \( x_n^{FN} x \), provided that \( D(0) = \lim_{n→∞} ||x_n - x||_α^x = 0 \); i.e., for every \( ε > 0 \) there is an \( N(ε) ∈ N \) such that \( D(0) = ||x_n - x||_α^x < ε \) for all \( n > N(ε) \). This means that for every \( ε > 0 \) there is an \( N(ε) ∈ N \) such that for all \( n > N(ε) \), \( \sup_{α ∈ [0, 1]} ||x_n - x||_α^x = ||x_n - x||_0^x < ε \).

Let \( (X, || \cdot ||) \) be an FNS. Then a double sequence \( (x_{jk}) \) is said to be convergent to \( x ∈ X \) with respect to the fuzzy norm on X if for every \( ε > 0 \) there exist a number \( N = N(ε) \) such that \( D(0) = ||x_{jk} - x||_α^x < ε \), for all \( j, k > N \).

In this case, we write \( x_{jk}^{FN} x \). This means that, for every \( ε > 0 \) there exist a number \( N = N(ε) \) such that \( \sup_{α ∈ [0, 1]} ||x_{jk} - x||_α^x = ||x_{jk} - x||_0^x < ε \), for all \( j, k > N \). In terms of neighborhoods, we have \( x_{jk}^{FN} x \) provided that for every \( ε > 0 \), there exist a number \( N = N(ε) \) such that \( x_{jk} ∈ N(ε, α) \) whenever, \( j, k > N \).

Let \( X ≠ Θ \). A class \( J \) of subsets of X is said to be an ideal in X provided:
(i) \( Θ ∈ J \), (ii) \( A, B ∈ J \) implies \( A ∪ B ∈ J \), (iii) \( A ∈ J, B ⊆ A \) implies \( B ∈ J \).

\( J \) is called a nontrivial ideal if \( x ∉ J \). A nontrivial ideal \( J \) in X is called admissible if \( \{ x \} ∈ J \) for each \( x ∈ X \).

A nontrivial ideal \( J_2 \) of \( N × N \) is called strongly admissible if \( \{ i \} × N \) and \( N × \{ i \} \) belong to \( J_2 \) for each \( i ∈ N \). It is evident that a strongly admissible ideal is also admissible. Throughout the paper we take \( J_2 \) as a strongly admissible ideal in \( N × N \).

Let \( J_2 = \{ A ⊆ N × N : \sup \{ m(A), i, j \} = m(A) \} \). Then \( J_2 \) is a strongly admissible ideal and clearly an ideal \( J_2 \) is strongly admissible if and only if \( J_2^0 \) is a subfilter.

Let \( X ≠ Θ \). A nonempty class \( F \) of subsets of X is said to be a filter in X provided:
(i) \( Θ ∉ F \), (ii) \( A, B ∈ F \) implies \( A ∩ B ∈ F \), (iii) \( A ∈ F, A ⊆ B \) implies \( B ∈ F \).

\( F \) is a nontrivial ideal in X, \( X ≠ Θ \), then the class \( F(\mathcal{F}) = \{ M ⊆ X : (\exists A ∈ F)(M = X \setminus A) \} \) is a filter on X, called the filter associated with \( F \).
Let \((X, \rho)\) be a linear metric space and \(\mathcal{I} \subseteq 2^{\mathbb{N} \times \mathbb{N}}\) be a strongly admissible ideal. A double sequence \(x = (x_{mn})\) in \(X\) is said to be \(\mathcal{I}\)-convergent to \(L \in X\), if for any \(\varepsilon > 0\) we have \(A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}\) and we write \(\lim_{m,n \to \infty} x_{mn} = L\).

If \(\mathcal{I} \subseteq 2^{\mathbb{N} \times \mathbb{N}}\) is a strongly admissible ideal, then usual convergence implies \(\mathcal{I}\)-convergence.

Let \((X, \|\cdot\|)\) be a fuzzy normed space. A double sequence \(x = (x_{mn})\) is said to be \(\mathcal{I}\)-convergent to \(L \in X\) with respect to fuzzy norm on \(X\), if for each \(\varepsilon > 0\), the set \(A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - L\|_{\varepsilon} \geq \varepsilon\} \in \mathcal{I}\). In this case, we write \(x_{mn} \xrightarrow{\mathcal{I}} L_1\) or \(x_{mn} \rightarrow L_1\) if for every \(\varepsilon > 0\) there exists a \(K(\varepsilon) \in \mathbb{N}\) such that \(\|x_{mn} - L_1\|_{\varepsilon} < \varepsilon\) holds for all \(m, n \geq K(\varepsilon)\). Note that \(x_{mn} \xrightarrow{\mathcal{I}} L_1\) if and only if \(\lim_{m,n \to \infty} x_{mn} = L_1\).

A useful interpretation of the above definition is the following:

\[x_{mn} \xrightarrow{\mathcal{I}} L_1 \iff \forall \varepsilon > 0, \exists K(\varepsilon) \in \mathbb{N} : \|x_{mn} - L_1\|_{\varepsilon} < \varepsilon, \forall m, n \geq K(\varepsilon)\]

Let \((X, \|\cdot\|)\) be a fuzzy normed space. A double sequence \(x = (x_{mn})\) is said to be \(\mathcal{I}\)-convergent to \(L \in X\) with respect to the fuzzy norm on \(X\) if there exists a set \(M \in \mathcal{I}(\mathcal{I})\), \(M = \{m_1 < \cdots < m_k < \cdots : n_1 < \cdots < n_j < \cdots \} \subseteq \mathbb{N} \times \mathbb{N}\) such that \(\lim_{m,n \to \infty} x_{mn} = L_1\) in \(X\). In terms of neighborhoods, we have \(x_{mn} \xrightarrow{\mathcal{I}} L_1\) if and only if \(\lim_{m,n \to \infty} x_{mn} = L_1\).

2. Main Results

In this section, we investigate some properties of \(\mathcal{I}\)-convergence in fuzzy normed spaces. After, we study some relationships between \(\mathcal{I}\)-convergence and \(\mathcal{I}\)-convergence of double sequences in fuzzy normed spaces.

Theorem 2.1. Let \((X, \|\cdot\|)\) be a fuzzy normed space. If a double sequence \((x_{mn})\) in \(X\) is \(\mathcal{I}\)-convergent to \(L_1\), then \(L_1\) determined uniquely.

Proof. Let \((x_{mn})\) be any double sequence and suppose that

\[F \mathcal{I} - \lim_{m,n \to \infty} x_{mn} = L_1\quad \text{and} \quad F \mathcal{I} - \lim_{m,n \to \infty} x_{mn} = L_2,\]

where \(L_1 \neq L_2\). Since \(L_1 \neq L_2\), we may suppose that \(L_1 > L_2\). Select \(\varepsilon = \frac{L_1 - L_2}{2}\), so that the neighborhoods \((L_1 - \varepsilon, L_1 + \varepsilon)\) and \((L_2 - \varepsilon, L_2 + \varepsilon)\) of \(L_1\) and \(L_2\) respectively are disjoint. Since \(L_1\) and \(L_2\) both are \(\mathcal{I}\)-limit of the sequence \((x_{mn})\). Therefore, both the sets

\[A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - L_1\|_{\varepsilon} \geq \varepsilon\}\quad \text{and} \quad B(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - L_2\|_{\varepsilon} \geq \varepsilon\}\]

belongs to \(\mathcal{I}\). This implies that the sets

\[A'(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - L_1\|_{\varepsilon} < \varepsilon\}\quad \text{and} \quad B'(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - L_2\|_{\varepsilon} < \varepsilon\}\]

belongs to \(\mathcal{I}(\mathcal{I})\). Since \(\mathcal{I}(\mathcal{I})\) is a filter on \(\mathbb{N} \times \mathbb{N}\) therefore \(A'(\varepsilon) \cap B'(\varepsilon)\) is a non-empty set in \(\mathcal{I}(\mathcal{I})\). In this way we obtain a contradiction to the fact that the neighborhoods \((L_1 - \varepsilon, L_1 + \varepsilon)\) and \((L_2 - \varepsilon, L_2 + \varepsilon)\) of \(L_1\) and \(L_2\), respectively, are disjoint. Hence, we have \(L_1 = L_2\).

Theorem 2.2. Let \((X, \|\cdot\|)\) be a fuzzy normed space. \((x_{mn})\) be a double sequence in \(X\) and \(L_1 \in X\). Then, \(F \mathcal{I} - \lim_{m,n \to \infty} x_{mn} = L_1 \iff F \mathcal{I} - \lim_{m,n \to \infty} x_{mn} = L_1\).

Proof. Let \(F \mathcal{I} - \lim_{m,n \to \infty} x_{mn} = L_1\). For \(\varepsilon > 0\) there exists a positive integer \(k_0 = k_0(\varepsilon)\) such that \(\|x_{mn} - L_1\|_\varepsilon < \varepsilon\), whenever \(m, n > k_0\). This implies that the set

\[A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - L_1\|_{\varepsilon} \geq \varepsilon\} \subseteq (\mathbb{N} \times \{1,2,\ldots,k_0\}) \cup \{(1,2,\ldots,k_0) \times \mathbb{N}\}.

Since \(\mathcal{I}\) is a admissible ideal, then

\[(\mathbb{N} \times \{1,2,\ldots,k_0\}) \cup \{(1,2,\ldots,k_0) \times \mathbb{N}\} \subseteq \mathcal{I}\]

and so \(A(\varepsilon) \subseteq \mathcal{I}\). Hence, we have

\[F \mathcal{I} - \lim_{m,n \to \infty} x_{mn} = L_1.\]
Theorem 2.3. Let \((X, \|\|)\) be a fuzzy normed space.

(i) If \(X\) has no accumulation point, then \(F_{\mathcal{F}_2}\)-convergence and \(F_{\mathcal{F}_2^\ast}\)-convergence coincide for each strongly admissible ideal \(\mathcal{F}_2\).

(ii) If \(X\) has an accumulation point \(L\), then there exists a strongly admissible ideal \(\mathcal{F}_2\) and a double sequence \((x_{mn})\) for which \(F_{\mathcal{F}_2^\ast}\lim_{m,n \to \infty} x_{mn} = L\) but \(F_{\mathcal{F}_2}\lim_{m,n \to \infty} x_{mn}\) does not exist.

Proof. (i) Let \(x = (x_{mn})\) be a double sequence in \(X\) and \(L \in X\). By Lemma 1.1, \(x_{mn} \xrightarrow{F_{\mathcal{F}_2}} L_1\) implies \(x_{mn} \xrightarrow{F_{\mathcal{F}_2^\ast}} L_1\). Assume that \(F_{\mathcal{F}_2}\lim_{m,n \to \infty} x_{mn} = L\). Since \(X\) has no accumulation point, so there exists \(\varepsilon > 0\) such that

\[B_L(\varepsilon, 0) = \{x \in X : \|x - L\|_0^+ < \varepsilon\} = \{L\}.
\]

Since \(F_{\mathcal{F}_2^\ast}\lim_{m,n \to \infty} x_{mn} = L\), so

\[
\{(m,n) \in N \times N : \|x_{mn} - L\|_0^+ \geq \varepsilon\} \subseteq \mathcal{F}_2.
\]

Hence, we have

\[
\{(m,n) \in N \times N : \|x_{mn} - L\|_0^+ \geq \varepsilon\} \subseteq (\{m,n) \in N \times N : \|x_{mn} - L\|_0^+ = 0\} \subseteq \mathcal{F}(\mathcal{F}_2).
\]

Therefore, \(F_{\mathcal{F}_2^\ast}\lim_{m,n \to \infty} x_{mn} = L\).

(ii) Since \(L\) is an accumulation point of \(X\), so there exists a sequence \((t_i)_{i \in \mathbb{N}}\) of distinct points all different from \(L\) in \(X\) which is convergent to \(L\) such that the sequence \(\{\|t_i - L\|_0^\delta\}_{i \in \mathbb{N}}\) is decreasing to 0. Let \((T_i)_{i \in \mathbb{N}}\) be a decomposition of \(\mathbb{N}\) onto infinite sets and put \(\Delta_i = \{(m,n) : \min\{m,n\} \in T_i\}\). Then, \(\Delta_i \subseteq \mathbb{N}\) is a decomposition of \(\mathbb{N} \times \mathbb{N}\) and the ideal

\[\mathcal{F}_2 = \{A : A \text{ is included in a finite union of } \Delta_i\}\]

is a strongly admissible ideal. Put \(x_{mn} = t_i\) if and only if \((m,n) \in \Delta_i\). Put \(s_n = \{\|t_n - L\|_0^\delta\}_{i \in \mathbb{N}}\), for \(n \in \mathbb{N}\). Let \(\delta > 0\) be given. Select \(\gamma \in \mathbb{N}\) such that \(s_\gamma < 4\). Then,

\[A(\delta) = \{(m,n) \in N \times N : \|x_{mn} - L\|_0^+ \geq \delta\} \subseteq \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_\gamma.
\]

Hence, \(A(\delta) \subseteq \mathcal{F}_2\) and \(F_{\mathcal{F}_2^\ast}\lim_{m,n \to \infty} x_{mn} = L\).

Now suppose that \(F_{\mathcal{F}_2^\ast}\lim_{m,n \to \infty} x_{mn} = L\). Then, there exists \(H \in \mathcal{F}_2\) such that for \(M = N \times N \setminus H\) we have \(FP - \lim_{m,n \to \infty} x_{mn} = L\), for \((m,n) \in M\).

By definition of the ideal \(\mathcal{F}_2\), there exists \(k \in \mathbb{N}\) such that

\[H \subseteq \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_k.
\]

But then, \(\Delta_{k+1} \subseteq N \times N \setminus H = M\). Then, from the construction of \(\Delta_{k+1}\) it follows that for any \(n_0 \in \mathbb{N}\),

\[\|x_{mn} - L\|_0^+ = s_{k+1} > 0
\]

hold for infinitely many \((m,n)\)'s with \((m,n) \in M\) and \(m,n \geq n_0\). This contradicts that \(FP - \lim_{m,n \to \infty} x_{mn} = L\), for \((m,n) \in M\). Also the assumption \(F_{\mathcal{F}_2^\ast}\lim_{m,n \to \infty} x_{mn} = q\), for \(q \neq L\) leads to the contradiction. \(\square\)

Theorem 2.4. Let \((X, \|\|)\) be a fuzzy normed space. If \(X\) has at least one accumulation point and for any arbitrary double sequence \((x_{mn})\) of elements of \(X\) and for each \(L \in X\), \(F_{\mathcal{F}_2^\ast}\lim_{m,n \to \infty} x_{mn} = L\) implies \(F_{\mathcal{F}_2^\ast}\lim_{m,n \to \infty} x_{mn} = L\), then \(\mathcal{F}_2\) has the property \((AP_2)\).

Proof. Assume that \(L \in X\) is an accumulation point of \(X\). There exists a sequence \((t_k)_{k \in \mathbb{N}}\) of distinct elements of \(X\) such that \(t_k \neq L\) for any \(k\), \(\lim_{k \to \infty} t_k = L\) and the sequence \(\{\|t_k - L\|_0^\delta\}_{k \in \mathbb{N}}\) is decreasing to 0. Put

\[s_k = \{\|t_k - L\|_0^\delta\}_{k \in \mathbb{N}}\]

for \(k \in \mathbb{N}\). Let \(\{A_i\}_{i \in \mathbb{N}}\) be a disjoint family of nonempty sets from \(\mathcal{F}_2\).

Define a sequence \((x_{mn})\) as following:

\[x_{mn} = \begin{cases} t_i, & \text{if } (m,n) \in A_i \\ L, & \text{if } (m,n) \notin A_i \end{cases}
\]

for any \(i\). Let \(\delta > 0\). Select \(k \in \mathbb{N}\) such that \(s_k < \delta\). Then,

\[A(\delta) = \{(m,n) \in N \times N : \|x_{mn} - L\|_0^+ \geq \delta\} \subseteq A_1 \cup A_2 \cup \cdots \cup A_k.
\]

Hence, \(A(\delta) \subseteq \mathcal{F}_2\) and so,

\[F_{\mathcal{F}_2}\lim_{m,n \to \infty} x_{mn} = L.
\]

By virtue of our assumption, we have

\[F_{\mathcal{F}_2^\ast}\lim_{m,n \to \infty} x_{mn} = L.
\]

So, there exists a set \(H \in \mathcal{F}_2\) such that \(M = N \times N \setminus H \in \mathcal{F}(\mathcal{F}_2)\) and

\[\lim_{m,n \to \infty} x_{mn} = L.\quad (2.1)
\]
Now, put \( H_i = A_i \cap H \), for \( i \in \mathbb{N} \). Then, \( H_i \in \mathcal{F}_2 \), for each \( i \in \mathbb{N} \). Also,

\[
\bigcup_{i=1}^{\infty} H_i = H \cap \bigcup_{i=1}^{\infty} A_i \subset H \quad \text{and so} \quad \bigcup_{i=1}^{\infty} H_i \in \mathcal{F}_2.
\]

Fix \( i \in \mathbb{N} \). If \( A_i \cap M \) is not included in the finite union of rows and columns in \( \mathbb{N} \times \mathbb{N} \), then \( M \) must contain an infinite sequence of elements \( \{(m_k, n_k)\} \), where both \( m_k, n_k \to \infty \) and \( x_{m_kn_k} = l_k \neq L \), for all \( k \in \mathbb{N} \) which contradicts (2.1). Hence, \( A_i \cap M \) must be contained in the finite union of rows and columns in \( \mathbb{N} \times \mathbb{N} \). Therefore,

\[
A_i \Delta H_i = A_i \setminus H_i = A_i \setminus H = A_i \cap M
\]

is also included in the finite union of rows and columns. Thus, \( \mathcal{F}_2 \) has the property (AP2).

\[ \square \]

References