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On the Generalization of Opial Type Inequality for Convex Function

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Abstract

In this article, by using new different approach method, we establish some generalization of Opial like inequality for convex mappings.

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1. Introduction

We recall the following interesting Opial type inequalities in [1]:

Theorem 1.1. *Let $x:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is an absolutely continuous function such that $x'\in L_2[a,b]$.*
i) If $x(a)=x(b)=0$, then

$$\int_a^b |x(t)x'(t)|\,dt \le \frac{b-a}{4}\int_a^b |x'(t)|^2\,dt \tag{1.1}$$

ii) If $x(a)=0\,(or\ x(b)=0)$, then

$$\int_a^b |x(t)x'(t)|\,dt \le \frac{b-a}{2}\int_a^b |x'(t)|^2\,dt. \tag{1.2}$$

Therefore, some very interesting generalizations are given by B. G. Pachpatte who works with several functions in Opial type inequalities. We give the following case

$$\int_a^b \left[|f'(t)|\,|g(t)|+|g'(t)|\,|f(t)|\right]dt \le \frac{(b-a)}{2}\left(\int_a^b |f'(t)|^2+|g'(t)|^2\,dt\right)$$

where $f,g\in C^1([a,b])$ with $f(a)=g(a)=0$ (see, [1], [10]).
Opial's inequality and its generalizations, extensions and discretizations, play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations. Over the last twenty years a large number of papers have been appeared in the literature which deals with the simple proofs, various generalizations and discrete analogues of Opial inequality and its generalizations, see [1]-[17]. Now, we give the following case that is one of them:

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2. Main Results

Theorem 2.1. *Let $f,g:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be differentiable function such that $f(a)=g(a)=0$ (or $f(b)=g(b)=0$). Suppose that ϕ be convex and increasing functions on $[0,\infty)$ with $\phi(0)=0$, then we have the following inequality*

$$\begin{aligned}&\int_a^b\left[\phi'(|f(t)|)\left|f'(t)\right|\phi(|g(t)|)+\phi'(|g(t)|)\left|g'(t)\right|\phi(|f(t)|)\right]dt \\ \leq\ & \phi\left(\int_a^b\left|f'(t)\right|dt\right)\phi\left(\int_a^b\left|g'(t)\right|dt\right).\end{aligned}\tag{2.1}$$

Proof. Consider the following functions, for $t\in[a,b]$ and $f(a)=g(a)=0$,

$$y(t)=\int_a^t\left|f'(s)\right|ds,\ \ z(t)=\int_a^t\left|g'(s)\right|ds$$

such that we also define the functions

$$F(t)=\phi(y(t))=\phi\left(\int_a^t\left|f'(s)\right|ds\right),\ G(t)=\phi(z(t))=\phi\left(\int_a^t\left|g'(s)\right|ds\right)$$

such that $y'(t)=|f'(t)|$, $z'(t)=|g'(t)|$, $y(t)\geq|f(t)|$ and $z(t)\geq|g(t)|$. Thus, by chain rule of differentiation and by using the convexity of ϕ, we get

$$F'(t)=\phi'(y(t))\left|f'(t)\right|\geq\phi'(|f(t)|)\left|f'(t)\right|\tag{2.2}$$

and

$$G'(t)=\phi'(z(t))\left|g'(t)\right|\geq\phi'(|g(t)|)\left|g'(t)\right|.\tag{2.3}$$

Multiplying both sides of (2.2) and (2.3) by $G(t)$ and $F(t)$, respectivly, and adding side by side, we get

$$F'(t)G(t)+F(t)G'(t)\geq\phi'(|f(t)|)\left|f'(t)\right|\phi(z(t))+\phi'(|g(t)|)\left|g'(t)\right|\phi(y(t)).\tag{2.4}$$

and then integrating both sides of this inequality (2.4) over $[a,b]$ with respect to t, we obtain that

$$\begin{aligned}&\int_a^b\left[F'(t)G(t)+F(t)G'(t)\right]dt\\ =\ & F(b)G(b)-F(a)G(a)\\ =\ & \phi(y(b))\phi(z(b))-\phi(y(a))\phi(z(a))\\ \geq\ & \int_a^b\left[\phi'(|f(t)|)\left|f'(t)\right|\phi(z(t))+\phi'(|g(t)|)\left|g'(t)\right|\phi(y(t))\right]dt.\end{aligned}$$

Since $\phi(0)=0$ and ϕ is an increasing function, we have

$$\begin{aligned}&\int_a^b\left[\phi'(|f(t)|)\left|f'(t)\right|\phi(|g(t)|)+\phi'(|g(t)|)\left|g'(t)\right|\phi(|f(t)|)\right]dt\\ \leq\ & \phi\left(\int_a^b\left|f'(t)\right|dt\right)\phi\left(\int_a^b\left|g'(t)\right|dt\right)\end{aligned}$$

which completes the proof. Similar to the above proof, choosing the following functions for $t\in[a,b]$ and $f(b)=g(b)=0$

$$y_1(t)=\int_t^b\left|f'(s)\right|ds,\ \ z_1(t)=\int_t^b\left|g'(s)\right|ds$$

such that then we have

$$F_1(t)=\phi(y_1(t))=\phi\left(\int_t^b\left|f'(s)\right|ds\right),\ \ G_1(t)=\phi(z_1(t))=\phi\left(\int_t^b\left|g'(s)\right|ds\right)$$

such that $y_1'(t)=-|f'(t)|$, $z_1'(t)=-|g'(t)|$, $y_1(t)\geq|f(t)|$ and $z_1(t)\geq|g(t)|$. It follows that

$$-\left[F_1'(t)G_1(t)+F_1(t)G_1'(t)\right]\geq\phi'(|f(t)|)\left|f'(t)\right|\phi(z_1(t))+\phi'(|g(t)|)\left|g'(t)\right|\phi(y_1(t)).\tag{2.5}$$

This completes the proof of the inequality (2.1). ☐

Remark 2.2. *If we choose $f(t)=g(t)$ in Theorem 2.1, we have*

$$\int_a^b \phi'(|f(t)|)\,|f'(t)|\,\phi(|f(t)|)\,dt \le \frac{1}{2}\left[\phi\left(\int_a^b |f'(t)|\,dt\right)\right]^2. \tag{2.6}$$

If we take $\phi(t)=t$ in the inequality (2.6), then we have the following inequality

$$\int_a^b |f'(t)|\,|f(t)|\,dt \le \frac{1}{2}\left(\int_a^b |f'(t)|\,dt\right)^2.$$

By using Cauchy-Schwarz inequality, it follows that

$$\int_a^b |f'(t)|\,|f(t)|\,dt \le \frac{(b-a)}{2}\int_a^b |f'(t)|^2\,dt$$

which is the inequality (1.2).

Remark 2.3. *If we take $\phi(t)=t$ in the inequality (2.1), then we have the following inequality*

$$\int_a^b \left[|f'(t)|\,|g(t)|+|g'(t)|\,|f(t)|\right]dt \le \left(\int_a^b |f'(t)|\,dt\right)\left(\int_a^b |g'(t)|\,dt\right). \tag{2.7}$$

By using Cauchy-Schwarz inequality in the right hand sides of inequality (2.7), it follows that

$$\int_a^b \left[|f'(t)|\,|g(t)|+|g'(t)|\,|f(t)|\right]dt \le (b-a)\sqrt{\left(\int_a^b |f'(t)|^2\,dt\right)\left(\int_a^b |g'(t)|^2\,dt\right)}.$$

By using AGM inequality, we get

$$\int_a^b \left[|f'(t)|\,|g(t)|+|g'(t)|\,|f(t)|\right]dt \le \frac{(b-a)}{2}\left(\int_a^b |f'(t)|^2+|g'(t)|^2\,dt\right)$$

which is proved by Pacpatte in [10].

Remark 2.4. *If we take $\phi(t)=\frac{t^p}{p}$ for $1\le p<\infty$ in the inequality (2.1), then we have the following inequality*

$$\begin{aligned}&\int_a^b \left[|f(t)|^{p-1}\,|f'(t)|\,|g(t)|^p+|g(t)|^{p-1}\,|g'(t)|\,|f(t)|^p\right]dt\\ \le\;& \frac{1}{p}\left(\int_a^b |f'(t)|\,dt\right)^p\left(\int_a^b |g'(t)|\,dt\right)^p.\end{aligned}$$

It follows from the Hölder's inequality with indices p and $\frac{p}{p-1}$, in the right hand sides of above inequality, and by using AGM inequality we get

$$\begin{aligned}&\int_a^b \left[|f(t)|^{p-1}\,|f'(t)|\,|g(t)|^p+|g(t)|^{p-1}\,|g'(t)|\,|f(t)|^p\right]dt\\ \le\;& \frac{(b-a)^{2p-2}}{p}\left(\int_a^b |f'(t)|^p\,dt\right)\left(\int_a^b |g'(t)|^p\,dt\right)\\ \le\;& \frac{(b-a)^{2p-2}}{p}\left(\int_a^b \left[|f'(t)|^p+|g'(t)|^p\right]dt\right)^2.\end{aligned}$$

Theorem 2.5. *Let $f,g:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be differentiable function such that $f(a)=f(b)=0$ and $g(a)=g(b)=0$. If ϕ is an convex and an increasing function on $[0,\infty)$ with $\phi(0)=0$, then we have the inequality*

$$\begin{aligned}&\int_a^b \left[\phi'(|f(t)|)\,|f'(t)|\,\phi(|g(t)|)+\phi'(|g(t)|)\,|g'(t)|\,\phi(|f(t)|)\right]dt\\ \le\;& \phi\left(\int_a^{\frac{a+b}{2}} |f'(t)|\,dt\right)\phi\left(\int_a^{\frac{a+b}{2}} |g'(t)|\,dt\right)\\ &+\phi\left(\int_{\frac{a+b}{2}}^b |f'(t)|\,dt\right)\phi\left(\int_{\frac{a+b}{2}}^b |g'(t)|\,dt\right).\end{aligned} \tag{2.8}$$

Proof. Consider the following functions, for $t \in [a,b]$ and $f(a)=f(b)=0$ and $g(a)=g(b)=0$,

$$y(t)=\int_a^t |f'(s)|\,ds,\ \ z(t)=\int_a^t |g'(s)|\,ds$$

$$y_1(t)=\int_t^b |f'(s)|\,ds,\ \ z_1(t)=\int_t^b |g'(s)|\,ds,$$

$$F(t)=\phi(y(t))=\phi\left(\int_a^t |f'(s)|\,ds\right),\ G(t)=\phi(z(t))=\phi\left(\int_a^t |g'(s)|\,ds\right)$$

and

$$F_1(t)=\phi(y_1(t))=\phi\left(\int_t^b |f'(s)|\,ds\right),\ G_1(t)=\phi(z_1(t))=\phi\left(\int_t^b |g'(s)|\,ds\right)$$

such that

$$y'(t)=|f'(t)|,\ z'(t)=|g'(t)|,\ y(t)\geq |f(t)| \text{ and } z(t)\geq |g(t)|$$

and

$$y_1'(t)=-|f'(t)|,\ z_1'(t)=-|g'(t)|,\ y_1(t)\geq |f(t)| \text{ and } z_1(t)\geq |g(t)|.$$

If we write the inequality (2.4) and (2.5) on the intervals $\left[a,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},b\right]$, respectively we have

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} \left[F'(t)G(t)+F(t)G'(t)\right]dt\\
=\;& F\left(\frac{a+b}{2}\right)G\left(\frac{a+b}{2}\right)-F(a)G(a)\\
=\;& \phi\left(y(\frac{a+b}{2})\right)\phi\left(z(\frac{a+b}{2})\right)-\phi(y(a))\phi(z(a))\\
\geq\;& \int_a^{\frac{a+b}{2}} \left[\phi'(|f(t)|)|f'(t)|\phi(z(t))+\phi'(|g(t)|)|g'(t)|\phi(y(t))\right]dt.
\end{aligned}$$

and

$$\begin{aligned}
&-\int_{\frac{a+b}{2}}^{b} \left[F_1'(t)G_1(t)+F_1(t)G_1'(t)\right]dt\\
=\;& -F_1(b)G_1(b)+F_1\left(\frac{a+b}{2}\right)G_1\left(\frac{a+b}{2}\right)\\
=\;& -\phi(y_1(b))\phi(z_1(b))+\phi\left(y_1\left(\frac{a+b}{2}\right)\right)\phi\left(z_1\left(\frac{a+b}{2}\right)\right)\\
\geq\;& \int_{\frac{a+b}{2}}^{b} \left[\phi'(|f(t)|)|f'(t)|\phi(z_1(t))+\phi'(|g(t)|)|g'(t)|\phi(y_1(t))\right]dt.
\end{aligned}$$

Since $\phi(0)=0$ and ϕ is an increasing function, we have

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} \left[\phi'(|f(t)|)|f'(t)|\phi(|g(t)|)+\phi'(|g(t)|)|g'(t)|\phi(|f(t)|)\right]dt\\
\leq\;& \phi\left(\int_a^{\frac{a+b}{2}}|f'(t)|\,dt\right)\phi\left(\int_a^{\frac{a+b}{2}}|g'(t)|\,dt\right)
\end{aligned}\tag{2.9}$$

and

$$\begin{aligned}
&\int_{\frac{a+b}{2}}^{b} \left[\phi'(|f(t)|)|f'(t)|\phi(|g(t)|)+\phi'(|g(t)|)|g'(t)|\phi(|f(t)|)\right]dt\\
\leq\;& \phi\left(\int_{\frac{a+b}{2}}^{b}|f'(t)|\,dt\right)\phi\left(\int_{\frac{a+b}{2}}^{b}|g'(t)|\,dt\right).
\end{aligned}\tag{2.10}$$

Adding to inequalities (2.9) and (2.10), we obtain the required inequality (2.8). □

Remark 2.6. *If we choose $f(t)=g(t)$ in Theorem 2.5, we have*

$$2\int_a^b \phi'(|f(t)|)\left|f'(t)\right|\phi(|f(t)|)\,dt \le \left[\phi\left(\int_a^{\frac{a+b}{2}}\left|f'(t)\right|dt\right)\right]^2+\left[\phi\left(\int_{\frac{a+b}{2}}^b\left|f'(t)\right|dt\right)\right]^2. \tag{2.11}$$

If we take $\phi(t)=t$ in the inequality (2.11), then we have the following inequality

$$2\int_a^b \left|f'(t)\right||f(t)|\,dt \le \left(\int_a^{\frac{a+b}{2}}\left|f'(t)\right|dt\right)^2+\left(\int_{\frac{a+b}{2}}^b\left|f'(t)\right|dt\right)^2.$$

By using Cauchy-Schwarz inequality, it follows that

$$\int_a^b \left|f'(t)\right||f(t)|\,dt \le \frac{(b-a)}{4}\int_a^b \left|f'(t)\right|^2 dt$$

which is the inequality (1.1).

Remark 2.7. *If we take $\phi(t)=t$ in the inequality (2.8), then we have the following inequality*

$$\begin{aligned} &\int_a^b \left[\left|f'(t)\right||g(t)|+\left|g'(t)\right||f(t)|\right]dt \\ \le\ & \left(\int_a^{\frac{a+b}{2}}\left|f'(t)\right|dt\right)\left(\int_a^{\frac{a+b}{2}}\left|g'(t)\right|dt\right)+\left(\int_{\frac{a+b}{2}}^b\left|f'(t)\right|dt\right)\left(\int_{\frac{a+b}{2}}^b\left|g'(t)\right|dt\right). \end{aligned} \tag{2.12}$$

By using Cauchy-Schwarz inequality in the right hand sides of inequality (2.12) and using AGM inequality, it follows that

$$\begin{aligned} &\int_a^b \left[\left|f'(t)\right||g(t)|+\left|g'(t)\right||f(t)|\right]dt \\ \le\ & \frac{(b-a)}{2}\left[\sqrt{\left(\int_a^{\frac{a+b}{2}}\left|f'(t)\right|^2 dt\right)\left(\int_a^{\frac{a+b}{2}}\left|g'(t)\right|^2 dt\right)}+\sqrt{\left(\int_{\frac{a+b}{2}}^b\left|f'(t)\right|^2 dt\right)\left(\int_{\frac{a+b}{2}}^b\left|g'(t)\right|^2 dt\right)}\right] \\ \le\ & \frac{(b-a)}{4}\left(\int_a^b \left|f'(t)\right|^2+\left|g'(t)\right|^2 dt\right) \end{aligned}$$

which is provided by Pacpatte in(for $m=0$ in Theorem 4, [10]).

Remark 2.8. *If we take $\phi(t)=\frac{t^p}{p}$ for $1\le p<\infty$ in the inequality (2.8), then we have the following inequality*

$$\begin{aligned} &\int_a^b \left[|f(t)|^{p-1}\left|f'(t)\right||g(t)|^p+|g(t)|^{p-1}\left|g'(t)\right||f(t)|^p\right]dt \\ \le\ & \frac{1}{p}\left(\int_a^{\frac{a+b}{2}}\left|f'(t)\right|dt\right)^p\left(\int_a^{\frac{a+b}{2}}\left|g'(t)\right|dt\right)^p \\ & +\frac{1}{p}\left(\int_{\frac{a+b}{2}}^b\left|f'(t)\right|dt\right)^p\left(\int_{\frac{a+b}{2}}^b\left|g'(t)\right|dt\right)^p. \end{aligned}$$

It follows from the Hölder's inequality with indices p and $\frac{p}{p-1}$, in the right hand sides of above inequality, and by using AGM inequality we get

$$\begin{aligned} &\int_a^b \left[|f(t)|^{p-1}\left|f'(t)\right||g(t)|^p+|g(t)|^{p-1}\left|g'(t)\right||f(t)|^p\right]dt \\ \le\ & \frac{(b-a)^{2p-2}}{p2^{2p-2}}\left\{\left(\int_a^{\frac{a+b}{2}}\left|f'(t)\right|^p dt\right)\left(\int_a^{\frac{a+b}{2}}\left|g'(t)\right|^p dt\right)\right. \\ & \left.+\left(\int_{\frac{a+b}{2}}^b\left|f'(t)\right|^p dt\right)\left(\int_{\frac{a+b}{2}}^b\left|g'(t)\right|^p dt\right)\right\} \\ \le\ & \frac{(b-a)^{2p-2}}{p2^{2p-1}}\left\{\left(\int_a^{\frac{a+b}{2}}\left|f'(t)\right|^p dt\right)^2+\left(\int_a^{\frac{a+b}{2}}\left|g'(t)\right|^p dt\right)^2\right. \\ & \left.+\left(\int_{\frac{a+b}{2}}^b\left|f'(t)\right|^p dt\right)^2+\left(\int_{\frac{a+b}{2}}^b\left|g'(t)\right|^p dt\right)^2\right\}. \end{aligned}$$

By using Cauchy-Schwarz inequality

$$\begin{aligned} &\int_a^b \left[|f(t)|^{p-1} \left|f'(t)\right| |g(t)|^p + |g(t)|^{p-1} \left|g'(t)\right| |f(t)|^p \right] dt \\ \leq \quad & \frac{(b-a)^{2p-1}}{p2^{2p}} \int_a^b \left[\left|f'(t)\right|^{2p} + \left|g'(t)\right|^{2p} \right] dt \end{aligned} \tag{2.13}$$

If we take $p=1$ *in the inequality (2.13), then we have the following inequality*

$$\int_a^b \left[\left|f'(t)\right| |g(t)| + \left|g'(t)\right| |f(t)| \right] dt \leq \frac{(b-a)}{4} \int_a^b \left[\left|f'(t)\right|^2 + \left|g'(t)\right|^2 \right] dt$$

which is presented by Pacpatte in(for $m=0$ *in Theorem 4, [10]). If we take* $p=2$ *in the inequality (2.13), then we have the following inequality*

$$\int_a^b \left[|f(t)| \left|f'(t)\right| |g(t)|^2 + |g(t)| \left|g'(t)\right| |f(t)|^2 \right] dt \leq \frac{(b-a)^3}{32} \int_a^b \left[\left|f'(t)\right|^4 + \left|g'(t)\right|^4 \right] dt.$$

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