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α-Türevli Asal Halkaların Sağ İdealleri Üzerine Bir Genelleştirme

Barış ALBAYRAK[1*]

ÖZ: Bu makalede, genelleştirilmiş $\alpha -$ türevli R asal halkasının sağ ideallerinde çalışılarak , $\alpha -$ türevli halkalardaki önceki çalışmalar genelleştirilmiş ve asal halkanın farklı durumlarda değişmeli olduğu gösterilmiştir. Bunlara ek olarak, iki genelleştirilmiş $\alpha -$ türev D ve H alınarak asal halkanın değişmelilik koşulları incelenmiştir.

Anahtar Kelimeler: Asal halka, İdeal, Genelleşirilmiş $\alpha -$türev

A Generalization on Right Ideals of Prime Rings with α-Derivations

ABSTRACT: In this paper, we generalize previous studies on right ideals of prime ring R with generalized $\alpha -$ derivation D and show that R is commutative under different conditions. Also, we investigate commutative property of prime ring R for two generalized $\alpha -$ derivations D and H.

Keywords: Prime ring, Ideal, Generalized α- derivation

[1] Barış ALBAYRAK (**Orcid ID:** 0000-0002-8255-4706), Çanakkale Onsekiz Mart Üniversitesi, Biga Uygulamalı Bilimler Fakültesi, Bankacılık ve Finans Bölümü, Çanakkale

*Sorumlu Yazar / Corresponding Author: Barış ALBAYRAK, e-mail: e-mail: balbayrak77@gmail.com

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INTRODUCTION

Let $Z(R)$ be center of ring R. Suppose that $x_1Rx_2=(0)$ for any $x_1,x_2\in R$. If $x_1=0$ or $x_2=0$, then R is termed prime ring. It is used the $[x_1,x_2]$ notation for commutator $x_1x_2-x_2x_1$ and $x_1\circ x_2$ notation for anticommutator $x_1x_2+x_2x_1$ for $x_1,x_2\in R$.

Studying the commutativity of rings is a field of study that has been investigated and kept up to date for many years. After define an additive map d from R into R that provides $d(x_1x_2)=d(x_1)x_2+x_1d(x_2)$ for all $x_1,x_2\in R$ as a derivation, several authors have studied commutative property for prime rings with derivation. In a study published in 1957, Posner showed that the ring is commutative if there is a nonzero-centering derivation in the prime ring (Posner E. C., 1957). Herstein showed that if R is a prime ring of $char(R)\neq 2$ containing a derivation $d\neq 0$ such that $[d(x_1),d(x_2)]=0$ for all $x_1,x_2\in R$, then R is commutative (Herstein I.N, 1979). Bresar generalized definition of derivation as the following: D from R into R is termed generalized derivation with determined derivation d if $D(x_1x_2)=D(x_1)x_2+x_1d(x_2)$ for all $x_1,x_2\in R$ (Bresar, 1991). Definition of $\alpha-$derivation and generalized $\alpha-$derivation is given as follows: Let α be an automorphism of R. If $d(x_1x_2)=d(x_1)x_2+\alpha(x_1)d(x_2)$ holds for all $x_1,x_2\in R$, then d is termed $\alpha-$derivation (Argaç, 2004), (Chang, 2009). Similarly, if $D(x_1x_2)=D(x_1)x_2+\alpha(x_1)d(x_2)$ holds for all $x_1,x_2\in R$, then D is termed generalized $\alpha-$derivation with determined $\alpha-$derivation d.

In recent years, many authors have proved commutative theorems for prime rings with derivation, generalized derivation, $\alpha-$ derivation and generalized $\alpha-$ derivation. Also many researchers have generalized results to ideals and Lie ideals of ring. Asraf and Rehman proved that R prime ring with $(0)\neq I$ ideal must be commutative if it contains a derivation d providing either of the properties $d(xy)+xy\in Z$ or $d(xy)-xy\in Z$ for all $x,y\in R$ (Ashraf M, Rehman N, 2001). In (Ashraf M, Asma A, Shakir A, 2007), the authors showed that commutativity of R prime ring with generalized derivation f for many different conditions, i.e. (i) $f(xy)\mp xy\in Z$, (ii) $f(xy)\mp yx\in Z$, (iii) $f(x)f(y)\mp xy\in Z$. Later that, these conditions are investigated for lie ideals of prime ring R (Gölbaşı Ö, Koç E, 2009). In (Nawas and Al-Omary, 2018), Abu Nawas and Al-Omary showed that if I is a left ideal of R prime ring with generalized derivation, then R is commutative under several conditions for I.

In this study, we generalize to study on right ideals with generalized derivation to generalized $\alpha-$ derivation. Throughout the paper, we take R is a prime ring, $0\neq I$ is a right ideal of R and $0\neq D:R\to R$ is a generalized $\alpha-$ derivation with determined $\alpha-$ derivation d such that $Z(R)\cap d\big(Z(R)\big)\neq(0)$. We study following conditions and prove that R is commutative ring. (i) $[D(x_1),x_1]\in Z(R)$ for all $x_1\in I$. (ii) $D(x_1)\circ x_1\in Z(R)$ for all $x_1\in I$. (iii) $[D(x_1),D(x_2)]-[x_1,x_2]\in Z(R)$ for all $x_1,x_2\in I$. (iv) $D(x_1)\circ D(x_2)-x_1\circ x_2\in Z(R)$ for all $x_1,x_2\in I$. (v) $[D(x_1),D(x_2)]-x_1\circ x_2\in Z(R)$ for all $x_1,x_2\in I$. (vi) $D(x_1)\circ D(x_2)-[x_1,x_2]\in Z(R)$ for all $x_1,x_2\in I$. (vii) $D[x_1,x_2]-x_1\circ x_2\in Z(R)$ for all $x_1,x_2\in I$. $(viii)$ $D(x_1\circ x_2)-[x_1,x_2]\in Z(R)$ for all $x_1,x_2\in I$. (ix) $[D(x_1),\alpha(x_2)]-[x_1,x_2]\in Z(R)$ for all $x_1,x_2\in I$. (x) $D(x_1)\circ\alpha(x_2)-x_1\circ x_2\in Z(R)$ for all $x_1,x_2\in I$.

In addition, we investigate commutative property of prime ring R for two generalized $\alpha-$ derivations $0\neq D,H:R\to R$ with determined $\alpha-$ derivations $d,h:R\to R$ respectively. We study following conditions and prove that R is commutative ring. (i) $[D(x_1),H(x_2)]-[x_1,x_2]\in Z(R)$ for all $x_1,x_2\in I$. (ii) $D[x_1,x_2]-[x_2,H(x_1)]\in Z(R)$ for all $x_1,x_2\in I$. (iii) $D(x_1\circ x_2)-x_2\circ H(x_1)\in Z(R)$ for all $x_1,x_2\in I$. (iv) $[D(x_1),x_1]-[x_1,H(x_1)]\in Z(R)$ for all $x_1\in I$. (v) $D(x_1)\circ x_1-x_1\circ H(x_1)\in Z(R)$ for all $x_1\in I$.

MATERIALS AND METHODS

In this section, the common definitions and some well-known informations used in the article are given.
Folowing identities is provided for commutator and anticommutator for all $x_1, x_2, x_3 \in R$.

- $[x_1x_2, x_3] = x_1[x_2, x_3] + [x_1, x_3]x_2$
- $[x_1, x_2x_3] = [x_1, x_2]x_3 + x_2[x_1, x_3]$
- $(x_1x_2) \circ x_3 = x_1(x_2 \circ x_3) - [x_1, x_3]x_2 = (x_1 \circ x_3)x_2 + x_1[x_2, x_3]$
- $x_1 \circ (x_2x_3) = (x_1 \circ x_2)x_3 - x_2[x_1, x_3] = x_2(x_1 \circ x_3) + [x_1, x_2]x_3$

Remark 1 *Let R be ring and α be an automorphism of R.*
i. Suppose that I is right ideal of R, then $\alpha(I)$ is right ideal of R.
ii. If $\alpha(x_1) \in Z(R)$ then $x_1 \in Z(R)$

Remark 2 *For an elements $x_1 \in Z(R)$ and $x_2 \in R$ prime ring, if $x_1x_2 \in Z(R)$ then $x_2 \in Z(R)$ or $x_1 =$* 0.

Remark 3 *If $Z(R) \cap d\big(Z(R)\big) \neq (0)$, then there is a fixed element $0 \neq u \in Z(R)$ which $0 \neq d(u) \in Z(R)$.*

Lemma 4 *(Abu Nawas M. K, Al-Omary R. M, 2018, Lemma 2.5) Let R be a prime ring and $(0) \neq I$ be a left ideal of R such that*
(a) $[x_1, x_2] \in Z(R)$ for all $x_1, x_2 \in I$, or
(b) $x_1 o x_2 \in Z(R)$ for all $x_1, x_2 \in I$.
Then R is commutative.

RESULTS AND DISCUSSION

Throughout the paper, we take R is a prime ring, $I \neq (0)$ is a right ideal of R, α is an automorphism of R and $0 \neq D, H: R \to R$ are generalized $\alpha -$ derivations determined with $\alpha -$ derivations d and h such that $Z(R) \cap d(Z(R)) \neq (0)$ and $Z(R) \cap h(Z(R)) \neq (0)$.

Lemma 3.1 *If (i) or (ii) is provided for all $x_1, x_2 \in I$, then R is commutative.*

(i) $[\alpha(x_2), x_1] \in Z(R)$
(ii) $\alpha(x_2) \circ x_1 \in Z(R)$

Proof. i. Let $[\alpha(x_2), x_1] \in Z(R)$ for all $x_1, x_2 \in I$. Then $\big[[\alpha(x_2), x_1], r\big] = 0$ for all $x_1, x_2 \in I$, $r \in R$. Replacing x_1 by $x_1\alpha(x_2)$, we have $0 = \big[[\alpha(x_2), x_1\alpha(x_2)], r\big]$. Using hypothesis in this equation, we get $[\alpha(x_2), x_1][\alpha(x_2), r] = 0$. Replacing r by rx_1, we find $[\alpha(x_2), x_1]r[\alpha(x_2), x_1] = 0$ for all $x_1, x_2 \in I$, $r \in R$. Since R is a prime ring, we obtain

$$[\alpha(x_2), x_1] = 0 \text{ for all } x_1, x_2 \in I.$$

Replacing x_1 by x_1r and using above relation, we get $x_1[\alpha(x_2), r] = 0$ for all $x_1, x_2 \in I, r \in R$. Hence, $I[\alpha(x_2), r] = 0$ for all $x_2 \in I$, $r \in R$. Since $0 \neq I$ is a right ideal and R is a prime ring, we have $[\alpha(I), r] = 0$ for all $r \in R$. Hence, we obtain $\alpha(I) \subset Z(R)$. From the Remark 1 and (Mayne, Lemma 3), R is commutative.

ii. Let $\alpha(x_2) \circ x_1 \in Z(R)$ for all $x_1, x_2 \in I$. Then $[\alpha(x_2) \circ x_1, r] = 0$ for all $x_1, x_2 \in I, r \in R$. Replacing x_1 by $x_1\alpha(x_2)$, we have $0 = [\alpha(x_2) \circ x_1\alpha(x_2), r]$. Using hypothesis in this equation, we get

$(\alpha(x_2) \circ x_1)[\alpha(x_2), r] = 0$. Replacing r by rt for any $t \in R$, we find $(\alpha(x_2) \circ x_1)t[\alpha(x_2), r] = 0$ for all $x_1, x_2 \in I, r, t \in R$. Since R is a prime ring, we obtain

$$\alpha(x_2) \circ x_1 = 0 \text{ or } [\alpha(x_2), r] = 0 \text{ for all } x_1, x_2 \in I, r \in R.$$

Let $C = \{x_2 \in I | \alpha(x_2) \circ x_1 = 0 \text{ for all } x_1 \in I\}$ and $D = \{x_2 \in I | [\alpha(x_2), r] = 0 \text{ for all } r \in R\}$. C and D are subgroups of additive group I whose $I = C \cup D$, but I can not be written as a union of its two proper subgroups. So, $I = C$ or $I = D$. If $I = C$, then $\alpha(x_2) \circ x_1 = 0$ for all $x_1, x_2 \in I$. Replacing x_1 by $x_1 r$, we get $x_1[\alpha(x_2), r] = 0$ for all $x_1, x_2 \in I$, $r \in R$. Hence, $I[\alpha(x_2), r] = 0$ for all $x_2 \in I$, $r \in R$. Since $0 \neq I$ is a right ideal and R is a prime ring, we have $[\alpha(x_2), r] = 0$ for all $x_2 \in I, r \in R$. So, $[\alpha(x_2), r] = 0$ for all $x_2 \in I$, $r \in R$ in both cases. Hence, we obtain $\alpha(I) \subset Z(R)$. From the Remark 1 and the (Mayne, Lemma 3), R is commutative.

Theorem 3.2 *If (i) or (ii) is provided for all $x_1 \in I$, then R is commutative.*
(i) $[D(x_1), x_1] \in Z(R)$
(ii)$D(x_1) \circ x_1 \in Z(R)$

Proof. i. Let $[D(x_1), x_1] \in Z(R)$ for all $x_1 \in I$. Replacing x_1 by $x_1 + x_2$, we get

$$[D(x_1), x_2] + [D(x_2), x_1] \in Z(R) \text{ for all } x_1, x_2 \in I. \quad (3.1)$$

From $Z(R) \cap d(Z(R)) \neq (0)$, we take fixed element $0 \neq u \in Z(R)$ which $0 \neq d(u) \in Z(R)$. Replacing x_2 by $x_2 u$ in Equation (3.1), we get $[D(x_1), x_2]u + x_2[D(x_1), u] + D(x_2)[u, x_1] + [D(x_2), x_1]u + \alpha(x_2)[d(u), x_1] + [\alpha(x_2), x_1]d(u) \in Z(R)$.

In this expression, using $u, d(u) \in Z(R)$ and Equation (3.1), we have

$$[\alpha(x_2), x_1]d(u) \in Z(R) \text{ for all } x_1, x_2 \in I.$$

Hence, using $0 \neq d(u) \in Z(R)$ and Remark 2, we obtain

$$[\alpha(x_2), x_1] \in Z(R) \text{ for all } x_1, x_2 \in I.$$

From the Lemma 3.1, R is commutative.
ii. Let $D(x_1) \circ x_1 \in Z(R)$ for all $x_1 \in I$. Replacing x_1 by $x_1 + x_2$, we have

$$D(x_1) \circ x_2 + D(x_2) \circ x_1 \in Z(R) \text{ for all } x_1, x_2 \in I. \quad (3.2)$$

Since $Z(R) \cap d(Z(R)) \neq (0)$, we take fixed element $0 \neq u \in Z(R)$ which $0 \neq d(u) \in Z(R)$. Replacing x_2 by $x_2 u$ in Equation (3.2), we obtain $(D(x_1) \circ x_2)u - x_2[D(x_1), u] + (D(x_2) \circ x_1)u + D(x_2)[u, x_1] + (\alpha(x_2) \circ x_1)d(u) + \alpha(x_2)[d(u), x_1] \in Z(R)$.

In this expression, using $u, d(u) \in Z(R)$ and Equation (3.2), we get

$$(\alpha(x_2) \circ x_1)d(u) \in Z(R) \text{ for all } x_1, x_2 \in I.$$

Hence, using $0 \neq d(u) \in Z(R)$ and Remark 2, we have

$$\alpha(x_2) \circ x_1 \in Z(R) \text{ for all } x_1, x_2 \in I.$$

From the Lemma 3.1, R is commutative.

Lemma 3.3 *If (i) or (ii) is provided for all $x_1, x_2 \in I$, then R is commutative.*
(i) $[D(x_1), \alpha(x_2)] \in Z(R)$
(ii) $D(x_1) \circ \alpha(x_2) \in Z(R)$

Proof. i. Let

$$[D(x_1),\alpha(x_2)]\in Z(R) \text{ for all } x_1,x_2\in I. \tag{3.3}$$

From $Z(R)\cap d(Z(R))\neq(0)$, we take fixed element $0\neq u\in Z(R)$ which $0\neq d(u)\in Z(R)$. Replacing x_1 by x_1u in Equation (3.3), we have
$[D(x_1),\alpha(x_2)]u+D(x_1)[u,\alpha(x_2)]+\alpha(x_1)[d(u),\alpha(x_2)]+[\alpha(x_1),\alpha(x_2)]d(u)\in Z(R)$

In this expression, using $u,d(u)\in Z(R)$ and Equation (3.3), we get

$$[\alpha(x_1),\alpha(x_2)]d(u)\in Z(R) \text{ for all } x_1,x_2\in I.$$

Hence, using $0\neq d(u)\in Z(R)$ and Remark 2, we obtain

$$[\alpha(x_1),\alpha(x_2)]\in Z(R) \text{ for all } x_1,x_2\in I.$$

Using the fact that α is automorphism, we get $\alpha([x_1,x_2])\in Z(R)$. So, from the Remark 1, $[x_1,x_2]\in Z(R)$ for all $x_1,x_2\in I$. From the Lemma 4, we obtain R is commutative.
ii. Let

$$D(x_1)\circ\alpha(x_2)\in Z(R) \text{ for all } x_1,x_2\in I. \tag{3.4}$$

From $Z(R)\cap d(Z(R))\neq(0)$, we take fixed element $0\neq u\in Z(R)$ which $0\neq d(u)\in Z(R)$. Replacing x_1 by x_1u in Equation (3.4), we have $(D(x_1)\circ\alpha(x_2))u+D(x_1)[u,\alpha(x_2)]+\alpha(x_1)[d(u),\alpha(x_2)]+(\alpha(x_1)\circ\alpha(x_2))d(u)\in Z(R)$.

In this expression, using $u,d(u)\in Z(R)$ and Equation (3.4), we have

$$(\alpha(x_1)\circ\alpha(x_2))d(u)\in Z(R) \text{ for all } x_1,x_2\in I.$$

Hence, using $0\neq d(u)\in Z(R)$ and Remark 2, we obtain

$$\alpha(x_1)\circ\alpha(x_2)\in Z(R) \text{ for all } x_1,x_2\in I.$$

Using the fact that α is automorphism, we get $\alpha(x_1\circ x_2)\in Z(R)$. So, from the Remark 1, $x_1\circ x_2\in Z(R)$ for all $x_1,x_2\in I$. From the Lemma 4, we get R is commutative.

Theorem 3.4 *If (i), (ii), (iii) or (iv) is provided for all $x_1,x_2\in I$, then R is commutative.*
(i) $[D(x_1),D(x_2)]-[x_1,x_2]\in Z(R)$
(ii) $D(x_1)\circ D(x_2)-x_1\circ x_2\in Z(R)$
(iii) $[D(x_1),D(x_2)]-x_1\circ x_2\in Z(R)$
(iv) $D(x_1)\circ D(x_2)-[x_1,x_2]\in Z(R)$

Proof. i. By assumption,

$$[D(x_1),D(x_2)]-[x_1,x_2]\in Z(R) \text{ for all } x_1,x_2\in I. \tag{3.5}$$

From $Z(R)\cap d(Z(R))\neq(0)$, we take fixed element $0\neq u\in Z(R)$ which $0\neq d(u)\in Z(R)$. Replacing x_2 by x_2u in Equation (3.5) and using $u,d(u)\in Z(R)$, we have

$$[D(x_1),D(x_2)]u+[D(x_1),\alpha(x_2)]d(u)-[x_1,x_2]u\in Z(R).$$

Using Equation (3.5), we obtain

$$[D(x_1),\alpha(x_2)]d(u)\in Z(R) \text{ for all } x_1,x_2\in I.$$

From Remark 2, we get

$$[D(x_1), \alpha(x_2)] \in Z(R) \text{ for all } x_1, x_2 \in I.$$

So, R is commutative from Lemma 3.3.

ii. By assumption,

$$D(x_1) \circ D(x_2) - x_1 \circ x_2 \in Z(R) \text{ for all } x_1, x_2 \in I. \quad (3.6)$$

From $Z(R) \cap d\big(Z(R)\big) \neq (0)$, we choose fixed element $0 \neq u \in Z(R)$ which $0 \neq d(u) \in Z(R)$. Replacing x_2 by $x_2 u$ in Equation (3.6) and using $u, d(u) \in Z(R)$, we obtain
$\big(D(x_1) \circ D(x_2)\big)u + \big(D(x_1) \circ \alpha(x_2)\big)d(u) - (x_1 \circ x_2)u \in Z(R)$.

Using Equation (3.6), we get

$$\big(D(x_1) \circ \alpha(x_2)\big)d(u) \in Z(R) \text{ for all } x_1, x_2 \in I.$$

From Remark 2, we have

$$D(x_1) \circ \alpha(x_2) \in Z(R) \text{ for all } x_1, x_2 \in I.$$

So, R is commutative from Lemma 3.3.

iii. By assumption,

$$[D(x_1), D(x_2)] - x_1 \circ x_2 \in Z(R) \text{ for all } x_1, x_2 \in I. \quad (3.7)$$

From $Z(R) \cap d\big(Z(R)\big) \neq (0)$, we take fixed element $0 \neq u \in Z(R)$ which $0 \neq d(u) \in Z(R)$. Replacing x_2 by $x_2 u$ in Equation (3.7) and using $u, d(u) \in Z(R)$, we have $[D(x_1), D(x_2)]u + [D(x_1), \alpha(x_2)]d(u) - (x_1 \circ x_2)u \in Z(R)$.

Using Equation (3.7), we obtain

$$[D(x_1), \alpha(x_2)]d(u) \in Z(R) \text{ for all } x_1, x_2 \in I.$$

From Remark 2, we get

$$[D(x_1), \alpha(x_2)] \in Z(R) \text{ for all } x_1, x_2 \in I.$$

So, R is commutative from Lemma 3.3.

iv. By assumption,

$$D(x_1) \circ D(x_2) - [x_1, x_2] \in Z(R) \text{ for all } x_1, x_2 \in I. \quad (3.8)$$

From $Z(R) \cap d\big(Z(R)\big) \neq (0)$, we take fixed element $0 \neq u \in Z(R)$ which $0 \neq d(u) \in Z(R)$. Replacing x_2 by $x_2 u$ in Equation (3.8) and using $u, d(u) \in Z(R)$, we obtain $\big(D(x_1) \circ D(x_2)\big)u + \big(D(x_1) \circ \alpha(x_2)\big)d(u) - [x_1, x_2]u \in Z(R)$.

Using Equation (3.8), we get

$$\big(D(x_1) \circ \alpha(x_2)\big)d(u) \in Z(R) \text{ for all } x_1, x_2 \in I.$$

From Remark 2, we have

$$D(x_1) \circ \alpha(x_2) \in Z(R) \text{ for all } x_1, x_2 \in I.$$

So, R is commutative from Lemma 3.3.

Theorem 3.5 *If (i), (ii), (iii) or (iv) is provided for all $x_1, x_2 \in I$, then R is commutative.*

(i) $D([x_1, x_2]) - x_1 \circ x_2 \in Z(R)$
(ii) $D(x_1 \circ x_2) - [x_1, x_2] \in Z(R)$
(iii) $[D(x_1), \alpha(x_2)] - [x_1, x_2] \in Z(R)$
(iv) $D(x_1) \circ \alpha(x_2) - x_1 \circ x_2 \in Z(R)$

Proof. i. Let

$$D([x_1, x_2]) - x_1 \circ x_2 \in Z(R) \text{ for all } x_1, x_2 \in I. \tag{3.9}$$

From $Z(R) \cap d\big(Z(R)\big) \neq (0)$, we take fixed element $0 \neq u \in Z(R)$ which $0 \neq d(u) \in Z(R)$. Replacing x_2 by $x_2 u$ in Equation (3.9), we have $D([x_1, x_2])u + \alpha([x_1, x_2])d(u) - (x_1 \circ x_2)u + x_2[x_1, u] \in Z(R)$.

In this expression, using $u \in Z(R)$ and Equation (3.9), we have

$$\alpha([x_1, x_2])d(u) \in Z(R) \text{ for all } x_1, x_2 \in I.$$

Hence, using $0 \neq d(u) \in Z(R)$ and Remark 2, we get

$$\alpha([x_1, x_2]) \in Z(R) \text{ for all } x_1, x_2 \in I.$$

So, from the Remark 1, $[x_1, x_2] \in Z(R)$ for all $x_1, x_2 \in I$. From the Lemma 4, we obtain R is commutative.

ii. Let

$$D(x_1 \circ x_2) - [x_1, x_2] \in Z(R) \text{ for all } x_1, x_2 \in I. \tag{3.10}$$

From $Z(R) \cap d\big(Z(R)\big) \neq (0)$, we take fixed element $0 \neq u \in Z(R)$ which $0 \neq d(u) \in Z(R)$. Replacing x_2 by $x_2 u$ in Equation (3.10) and using $u, d(u) \in Z(R)$, we have $D(x_1 \circ x_2)u + \alpha(x_1 \circ x_2)d(u) - [x_1, x_2]u \in Z(R)$.

Using Equation (3.4), we have

$$\alpha(x_1 \circ x_2)d(u) \in Z(R) \text{ for all } x_1, x_2 \in I.$$

Hence, using $0 \neq d(u) \in Z(R)$ and Remark 2, we obtain

$$\alpha(x_1 \circ x_2) \in Z(R) \text{ for all } x_1, x_2 \in I.$$

So, from the Remark 1, $x_1 \circ x_2 \in Z(R)$ for all $x_1, x_2 \in I$. From the Lemma 4, we have R is commutative.

iii. Let

$$[D(x_1), \alpha(x_2)] - [x_1, x_2] \in Z(R) \text{ for all } x_1, x_2 \in I. \tag{3.11}$$

From $Z(R) \cap d\big(Z(R)\big) \neq (0)$, we take fixed element $0 \neq u \in Z(R)$ which $0 \neq d(u) \in Z(R)$. Replacing x_1 by $x_1 u$ in Equation (3.11) and using u, $d(u) \in Z(R)$, we get $[D(x_1), \alpha(x_2)]u + [\alpha(x_1), \alpha(x_2)]d(u) - [x_1, x_2]u \in Z(R)$.

From Equation (3.11), we obtain

$$[\alpha(x_1), \alpha(x_2)]d(u) \in Z(R) \text{ for all } x_1, x_2 \in I.$$

Hence, using $0 \neq d(u) \in Z(R)$ and Remark 2, we get

$$[\alpha(x_1), \alpha(x_2)] \in Z(R) \text{ for all } x_1, x_2 \in I.$$

Using the fact that α is automorphism, we get $\alpha([x_1, x_2]) \in Z(R)$.So, from the Remark 1, $[x_1, x_2] \in Z(R)$ for all $x_1, x_2 \in I$. From the Lemma 4, we get R is commutative.

iv. Let

$$D(x_1)\circ\alpha(x_2)-x_1\circ x_2\in Z(R) \text{ for all } x_1,x_2\in I. \tag{3.12}$$

From $Z(R)\cap d(Z(R))\neq(0)$, we choose fixed element $0\neq u\in Z(R)$ which $0\neq d(u)\in Z(R)$. Replacing x_1 by x_1u in Equation (3.12) and using $u,d(u)\in Z(R)$, we have $\big(D(x_1)\circ\alpha(x_2)\big)u+\big(\alpha(x_1)\circ\alpha(x_2)\big)d(u)-(x_1\circ x_2)u\in Z(R)$

Using Equation (3.12), we have

$$\big(\alpha(x_1)\circ\alpha(x_2)\big)d(u)\in Z(R) \text{ for all } x_1,x_2\in I.$$

Hence, using $0\neq d(u)\in Z(R)$ and Remark 2, we obtain

$$\alpha(x_1)\circ\alpha(x_2)\in Z(R) \text{ for all } x_1,x_2\in I.$$

Using the fact that α is automorphism, we get $\alpha(x_1\circ x_2)\in Z(R)$.So, from the Remark 1, $x_1\circ x_2\in Z(R)$ for all $x_1,x_2\in I$. From the Lemma 4, we have R is commutative.

Theorem 3.6 *If (i), (ii) or (iii) is provided for all $x_1,x_2\in I$, then R is commutative.*

(i) $[D(x_1),H(x_2)]-[x_1,x_2]\in Z(R)$
(ii) $D[x_1,x_2]-[x_2,H(x_1)]\in Z(R)$
(iii) $D(x_1\circ x_2)-x_2\circ H(x_1)\in Z(R)$

Proof. i) For all $x_1,x_2\in I$, let

$$[D(x_1),H(x_2)]-[x_1,x_2]\in Z(R). \tag{3.13}$$

By hypothesis, $Z(R)\cap h(Z(R))\neq(0)$. Then, we take fixed element $0\neq u\in Z(R)$ which $0\neq h(u)\in Z(R)$. Replacing x_2 by x_2u in Equation (3.13) and using $u,h(u)\in Z(R)$, we get $[D(x_1),H(x_2)]u+[D(x_1),\alpha(x_2)]h(u)-[x_1,x_2]u\in Z(R)$.

From Equation (3.13), we obtain

$$[D(x_1),\alpha(x_2)]h(u)\in Z(R) \text{ for all } x_1,x_2\in I.$$

In this expression, using $0\neq h(u)\in Z(R)$ and Remark 2, we have

$$[D(x_1),\alpha(x_2)]\in Z(R) \text{ for all } x_1,x_2\in I.$$

From the Lemma 3.3, R is commutative.
ii) For all $x_1,x_2\in I$, let

$$D[x_1,x_2]-[x_2,H(x_1)]\in Z(R). \tag{3.14}$$

By hypothesis, $Z(R)\cap d(Z(R))\neq(0)$. Then, we take fixed element $0\neq u\in Z(R)$ which $0\neq d(u)\in Z(R)$. Replacing x_2 by x_2u in Equation (3.14) and using u, $d(u)\in Z(R)$, we have $D([x_1,x_2])u+\alpha([x_1,x_2])d(u)-[x_2,H(x_1)]u\in Z(R)$.

From Equation (3.14), we get

$$\alpha([x_1,x_2])d(u)\in Z(R) \text{ for all } x_1,x_2\in I.$$

In this expression, using $0\neq d(u)\in Z(R)$ and Remark 2, we have

$$\alpha([x_1,x_2])\in Z(R) \text{ for all } x_1,x_2\in I.$$

So, from the Remark 1, $[x_1, x_2] \in Z(R)$ for all $x_1, x_2 \in I$. From the Lemma 4, we obtain R is commutative.

$ii i$) For all $x_1, x_2 \in I$, let

$$D(x_1 \circ x_2) - x_2 \circ H(x_1) \in Z(R). \tag{3.15}$$

By hypothesis, $Z(R) \cap d\big(Z(R)\big) \neq (0)$. Then, we choose fixed element $0 \neq u \in Z(R)$ which $0 \neq d(u) \in Z(R)$. Replacing x_2 by $x_2 u$ in Equation (3.15) and using $u, d(u) \in Z(R)$, we get $D(x_1 \circ x_2)u + \alpha(x_1 \circ x_2)d(u) - \big(x_2 \circ H(x_1)\big)u \in Z(R)$.

From Equation (3.15), we obtain

$$\alpha(x_1 \circ x_2)d(u) \in Z(R) \text{ for all } x_1, x_2 \in I.$$

In this expression, using $0 \neq d(u) \in Z(R)$ and Remark 2, we have

$$\alpha(x_1 \circ x_2) \in Z(R) \text{ for all } x_1, x_2 \in I.$$

So, from the Remark 1, $x_1 \circ x_2 \in Z(R)$ for all $x_1, x_2 \in I$. From the Lemma 4, we have R is commutative.

Theorem 3.7 *Suppose that* $\{u \in Z(R) | 0 \neq d(u), 0 \neq h(u) \in Z(R), d(u) \neq \mp h(u)\} \neq \emptyset$. *If (i) or (ii) is provided for all* $x_1 \in I$, *then* R *is commutative.*

(i) $[D(x_1), x_1] - [x_1, H(x_1)] \in Z(R)$
(ii)$D(x_1) \circ x_1 - x_1 \circ H(x_1) \in Z(R)$

Proof. i. Let $[D(x_1), x_1] - [x_1, H(x_1)] \in Z(R)$ for all $x_1 \in I$. Replacing x_1 by $x_1 + x_2$ for any $x_2 \in I$, we get

$$[D(x_1), x_2] + [D(x_2), x_1] - [x_1, H(x_2)] - [x_2, H(x_1)] \in Z(R). \tag{3.16}$$

From the definition of the set $\{u \in Z(R) | 0 \neq d(u), 0 \neq h(u) \in Z(R), d(u) \neq \mp h(u)\} \neq \emptyset$, we conclude that there is a fixed element $u \in Z(R)$ which $0 \neq d(u), 0 \neq h(u) \in Z(R)$ and $d(u) \neq \mp h(u)$. If u was zero element, $d(u)$ would be also zero element. So, $0 \neq u$, since $0 \neq d(u)$. Replacing x_2 by $x_2 u$ in Equation (3.16) and using $u, d(u), h(u) \in Z(R)$, we get $[D(x_1), x_2]u + [D(x_2), x_1]u + [\alpha(x_2), x_1]d(u) - [x_1, H(x_2)]u - [x_1, \alpha(x_2)]h(u) - [x_2, H(x_1)]u \in Z(R)$.

From Equation (3.16), we have

$$[\alpha(x_2), x_1]d(u) - [x_1, \alpha(x_2)]h(u) \in Z(R) \text{ for all } x_1, x_2 \in I.$$

Using equation $-[x_1, \alpha(x_2)] = [\alpha(x_2), x_1]$, we get

$$[\alpha(x_2), x_1]\big(d(u) + h(u)\big) \in Z(R) \text{ for all } x_1, x_2 \in I.$$

Hence, using $0 \neq d(u)$, $0 \neq h(u) \in Z(R)$, $d(u) \neq -h(u)$ and Remark 2, we obtain

$$[\alpha(x_2), x_1] \in Z(R) \text{ for all } x_1, x_2 \in I.$$

From the Lemma 3.1, R is commutative.
ii. Let $D(x_1) \circ x_1 - x_1 \circ H(x_1) \in Z(R)$ for all $x_1 \in I$. Replacing x_1 by $x_1 + x_2$ for any $x_2 \in I$, we get

$$D(x_1) \circ x_2 + D(x_2) \circ x_1 - x_1 \circ H(x_2) - x_2 \circ H(x_1) \in Z(R)\,. \tag{3.17}$$

From the definition of the set $\{u \in Z(R) | 0 \neq d(u), 0 \neq h(u) \in Z(R), d(u) \neq \mp h(u)\} \neq \emptyset$, we conclude that there is a fixed element $u \in Z(R)$ which $0 \neq d(u), 0 \neq h(u) \in Z(R)$ and $d(u) \neq \mp h(u)$. If u was zero element, $d(u)$ would be also zero element. So, $0 \neq u$, since $0 \neq d(u)$. Replacing x_2 by

x_2u in Equation (3.17) and using $u, d(u), h(u) \in Z(R)$, we get $(D(x_1) \circ x_2)u + (D(x_2) \circ x_1)u + (\alpha(x_2) \circ x_1)d(u) - (x_1 \circ H(x_2))u - (x_1 \circ \alpha(x_2))h(u) - (x_2 \circ H(x_1))u \in Z(R)$.

From Equation (3.17), we have

$$(\alpha(x_2) \circ x_1)d(u) - (x_1 \circ \alpha(x_2))h(u) \in Z(R).$$

Using equation $x_1 \circ \alpha(x_2) = \alpha(x_2) \circ x_1$, we get

$$(\alpha(x_2) \circ x_1)(d(u) - h(u)) \in Z(R) \text{ for all } x_1, x_2 \in I.$$

Hence, using $0 \neq d(u)$, $0 \neq h(u) \in Z(R)$, $d(u) \neq h(u)$ and Remark 2, we obtain

$$\alpha(x_2) \circ x_1 \in Z(R) \text{ for all } x_1, x_2 \in I.$$

From the Lemma 3.1, R is commutative.

CONCLUSION

Prime rings with generalized $\alpha-$ derivations are commutative under different conditions. In this study showed that, previous studies are also provided for right ideals of prime rings with generalized $\alpha-$ derivations. Also, prime rings with two generalized $\alpha-$ derivations are also commutative for different conditions.

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