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Research Article

New generalizations of modular spaces

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Abstract

In the present paper, we introduce the concept of $\mathcal{F}$-modular, which is a generalization of the modular notion. Moreover, we introduce a K_p-modular and K-modular, and then compare these concepts together. Finally, we give a characterization of $\mathcal{F}$-modulars.

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1. Introduction

A modular on a space $\mathcal{X}$ is a mapping $\rho : \mathcal{X} \to [0, \infty]$ satisfying the following properties:

(i) $\rho(x) = 0$ if and only if $x = 0$,

(ii) $\rho(\alpha x) = \rho(x)$ for every scaler α with $|\alpha| = 1$,

(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for every $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$.

A modular ρ defines a corresponding modular space, i.e., the vector space $\mathcal{X}_\rho$ given by

$$\mathcal{X}_\rho = \{x \in \mathcal{X} : \quad \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

The theory of modular spaces was founded by Nakano [15] and was intensively developed by Luxemburg [10], Koshi and Shimogaki [8] and Yamamuro [18] and their collaborators. In the present time the theory of modulars and modular spaces is extensively applied, in particular, in the study of various Orlicz spaces [16] and interpolation theory [9, 12], which in their turn have broad applications [13]. Shateri [17] introduced the notion of a C^*-valued modular, and investigated some fixed point theroems in such spaces.
Recently, many interesting extentions of the metric space appeared. The notion of a b-metric space introduced by Czerwik [2]. Fagin, et al. [3] introduced s-relaxed$_p$ metric spaces. Gähler [4] defined the notion of a 2-metric on the product set $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$. The reader can see more generalizations of the notion of a metric space in [1, 5, 7, 12, 14]. Jleli and Samet [6] introduced the $\mathcal{F}$-metric concept. They defined a natural topology in such spaces, and studied their topological properties.

In this paper, by using some ideas of [6] we introduce the $\mathcal{F}$-modular concept, which is a generalization of the modular space notion. We prove that any modular is an $\mathcal{F}$-modular, but the converse is not true in general, which shows that our concept is more general than the standard modular concept.

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Moreover, we introduce a K_p-modular and K-modular, and then compare these concepts together. Finally, we introduce the notion of $\mathcal{F}$-modulars boundedness, which is used to provide a characterization of $\mathcal{F}$-modular spaces.

2. $\mathcal{F}$-modulars

We start by introducing the following set which plays an important role in our topic. Let $\mathcal{F}$ be the set of all functions $f:(0,+\infty)\to\mathbb{R}$ which satisfy in the following conditions

($\mathcal{F}_1$) f is non-decreasing,
($\mathcal{F}_2$) For every sequence $\{t_n\}$ in $(0,+\infty)$, $\lim_{n\to+\infty} t_n = 0$ if and only if $\lim_{n\to+\infty} f(t_n) = -\infty$.

Now we define a new concept of a modular space.

Definition 2.1. Let $\mathcal{X}$ be a linear space, and let $\delta:\mathcal{X}\to[0,+\infty)$ be a given mapping. Suppose there exists $(f,\gamma)\in\mathcal{F}\times[0,+\infty)$ such that

(δ_1) $\delta(x)=0$ if and only if $x=0$,
(δ_2) $\delta(\alpha x)=\delta(x)$ for every scaler α with $|\alpha|=1$,
(δ_3) For each $x,y\in\mathcal{X}$, for each $2\le n\in\mathbb{N}$, and for every $\{u_i\}_{i=1}^n$ in $\mathcal{X}$ with $u_1=x$ and $u_n=y$, if $\delta(\alpha x+\beta y)>0$ for $\alpha,\beta>0$ in which $\alpha+\beta=1$,
then

$$f(\delta(\alpha x+\beta y))\le f\Big(\sum_{i=1}^n \delta(u_i)\Big)+\gamma.$$

Then δ is called an $\mathcal{F}$-modular on $\mathcal{X}$, and the pair $(\mathcal{X},\delta)$ is said to be an $\mathcal{F}$-modular space.

Note that if ρ is a modular on $\mathcal{X}$, then it is an $\mathcal{F}$-modular with $f(t)=lnt$ and $\gamma=0$. Clearly (δ_1) and (δ_2) satisfied. On the other hand, for each $x,y\in\mathcal{X}$, for every $2\le n\in\mathbb{N}$, and for every $\{u_i\}_{i=1}^n$ in $\mathcal{X}$ with $u_1=x$ and $u_n=y$, we have

$$\ln(\rho(\alpha x+\beta y))\le\ln(\rho(x)+\rho(y))\le\ln\Big(\sum_{i=1}^n\rho(u_i)\Big),$$

for $\alpha,\beta>0$ in which $\alpha+\beta=1$.

In the following example we give an $\mathcal{F}$-modular space which is not a modular space.

Example 2.2. Let $\mathcal{X}=[1,\infty)$, define the mapping $\delta:\mathcal{X}\to[0,+\infty)$ as follows

$$\delta(x)=\begin{cases}x^2 & x\in[1,2),\\ x & x\ge 2,\end{cases}$$

for all $x\in\mathcal{X}$. Then δ is not a modular. Indeed, δ does not satisfy (iii), because for $x=1$, $y=2$, $\alpha=\frac{1}{5}$ and $\beta=\frac{4}{5}$, we get

$$\delta(\alpha x+\beta y)=\delta(\frac{1}{5}+\frac{8}{5})=\delta(\frac{9}{5})=\frac{81}{25}>\delta(x)+\delta(y)=3.$$

Now, we show that δ is an $\mathcal{F}$-modular. Fix $x,y\in\mathcal{X}$, and let $\{u_i\}_{i=1}^n$ in $\mathcal{X}$ with $u_1=x$ and $u_n=y$. Put $I=\{i=1,\dots,n;\ u_i\in[1,2)\}$ and $J=\{1,2,\dots\}-I$, then we have

$$\sum_{i=1}^n\delta(u_i)=\sum_{i\in I}\delta(u_i)+\sum_{j\in J}\delta(u_j)=\sum_{i\in I}u_i^2+\sum_{j\in J}u_j.$$

Now we have two cases.
Case 1: If $\alpha x+\beta y\notin[1,2)$, then

$$\begin{aligned}\delta(\alpha x+\beta y)&=\alpha x+\beta y\\&\leq x+y\leq\sum_{i=1}^{n}u_i=\sum_{i\in I}u_i+\sum_{j\in J}u_j\\&\leq\sum_{i\in I}u_i^2+\sum_{j\in J}u_j\\&=\sum_{i=1}^{n}\delta(u_i).\end{aligned}$$

Case 2: If $\alpha x+\beta y\in[1,2)$, then we have

$$\begin{aligned}\delta(\alpha x+\beta y)&=(\alpha x+\beta y)^2\\&\leq 2(\alpha x+\beta y)\\&\leq 2(x+y)\\&\leq 2\Big(\sum_{i\in I}u_i+\sum_{j\in J}u_j\Big)\\&\leq 2\Big(\sum_{i\in I}u_i^2+\sum_{j\in J}u_j\Big)\\&=2\sum_{i=1}^{n}\delta(u_i).\end{aligned}$$

The above cases show that δ satisfies (δ_3) with $f(t)=\ln t$, $t>0$ and $\gamma=\ln 2$. Therefore δ is an $\mathcal{F}$-modular.

Now,we define a K_p-modular on a space $\mathcal{X}$, also we provide an example of an $\mathcal{F}$-modular space that cannot be an K_p-modular space, which confirms that the class of $\mathcal{F}$-modular spaces is more large than the class of K_p-modular spaces.

Definition 2.3. Let $\Delta:\mathcal{X}\to[0,+\infty)$ be a mapping satisfies $(\delta_1),(\delta_2)$, and
(Δ_3) There exists $K\geq 1$ such that for every $x,y\in\mathcal{X}$, for every $2\leq n\in\mathbb{N}$, for every $\{u_i\}_{i=1}^n$ in $\mathcal{X}$ with $u_1=x$ and $u_n=y$, we have

$$\Delta(\alpha x+\beta y)\leq K\Big(\sum_{i=1}^{n}\Delta(u_i)\Big),$$

for $\alpha,\beta>0$ in which $\alpha+\beta=1$. Then Δ is called a K_p-modular, and $(\mathcal{X},\Delta)$ is said to be a K_p-modular space.

It is clear that Δ satisfies (δ_3) with $f(t)=\ln t$, $t>0$ and $\gamma=\ln K$, and hence Δ is an $\mathcal{F}$-modular. Notice that the mapping δ in Example 2.2 satisfies in (Δ_3) with $K=2$.
The following example shows that the class of $\mathcal{F}$-modulars is more large than the class of K_p modulars.

Example 2.4. Let $\mathcal{X}=\mathbb{Z}$. Define the mapping $\delta:\mathcal{X}\to[0,+\infty)$ by

$$\delta(x)=\begin{cases}\frac{1}{e^{|x|}} & x\neq 0,\\ 0 & x=0,\end{cases}\tag{2.1}$$

for all $x\in\mathcal{X}$. Then δ is a $\mathcal{F}$-modular. It is clear that δ satisfies (δ_1) and (δ_2). In order to check (δ_3), let

$$f(t)=-\frac{1}{t},\quad (t>0).$$

It can be easily seen that $f \in \mathcal{F}$. Fix $x, y \in \mathcal{X}$ and $\alpha, \beta > 0$ in which $\alpha + \beta = 1$ with $\delta(\alpha x + \beta y) > 0$. For every $n \in \mathbb{N}$, and for every $\{u_i\}$ in $\mathcal{X}$ with $u_1 = x$ and $u_2 = y$, we have

$$\begin{aligned} &1 + f\Big(\sum_{i=1}^{n} \delta(u_i)\Big) - f(\delta(x) + \delta(y)) \\ &= 1 - \frac{1}{\sum_{i=1}^{n} \frac{1}{e^{|u_i|}}} + \frac{1}{\frac{1}{e^{|\alpha x + \beta y|}}} \\ &= 1 - \frac{1}{\sum_{i=1}^{n} \frac{1}{e^{|u_i|}}} + e^{|\alpha x + \beta y|} \\ &> 1 + 1 + e^{|\alpha x + \beta y|} \\ &\geq 0. \end{aligned}$$

Note that the first inequality holds because $\sum_{i=1}^{n} \frac{1}{e^{|u_i|}} > 0 > -1$ and so $-\frac{1}{\sum_{i=1}^{n} \frac{1}{e^{|u_i|}}} > 1$. Therefore we get

$$f(\delta(x) + \delta(y)) \leq f\Big(\sum_{i=1}^{n} \delta(u_i)\Big) + 1.$$

Consequently δ is an $\mathcal{F}$-modular.

Next, we shall prove δ is not a K_p-modular. Suppose that δ satifies (Δ_3) with a certain $K \geq 1$. Consider $u_1 = x = 4n, u_2 = y = 0$ and $\alpha = \beta = \frac{1}{2}$. Then we have

$$\delta(\alpha x + \beta y) = \delta(2n) \leq K(\delta(x) + \delta(y)) = K\delta(4n),$$

this implies that

$$e^{2n} \leq K.$$

Passing to the limit as $n \to +\infty$, we obtain a contradiction. Therefore, δ can not be a K_p-modular.

In following, we introduce another concept of a modular space which is more large than the class of $\mathcal{F}$-modular spaces and K_p-modular spaces.

Definition 2.5. Let $\mathcal{X}$ be a linear space, and let $\rho : \mathcal{X} \to [0, +\infty)$ be a mapping. Let there exists $K \geq 1$ such that

(ρ_1) $\rho(x) = 0$ if and only if $x = 0$,
(ρ_2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$,
(ρ_3) $\rho(\alpha x + \beta y) \leq K(\rho(x) + \rho(y))$, for $\alpha, \beta > 0$ in which $\alpha + \beta = 1$.
Then ρ is called a K-modular.

Notice that, each modular is a K-modular with $K = 1$. Also every K_p-modular is a K-modular. In following we give an example that shows the converse is not true in general.

Example 2.6. Let $\mathcal{X} = [0, 1]$, and let $\delta : \mathcal{X} \to [0, +\infty)$ be the mapping defined by

$$\delta(x) = \begin{cases} x^2 & x \in [0, 1), \\ 0 & x = 1. \end{cases}$$

It can be easily seen that δ is a K-modular with $K = 2$. Next, we prove that δ is not an $\mathcal{F}$-modular. Suppose that there exists (f, γ) such that δ satisfies (δ_3). Let $n \in \mathbb{N}$, and put

$$x = u_1 = 0, y = u_n = 1, u_i = \frac{1}{n}, \quad i = 2, \ldots, n-1.$$

Then for $\alpha = \beta = \frac{1}{2}$, (δ_3) implies that

$$f(\delta(\frac{x}{2} + \frac{y}{2})) \leq f(\delta(u_1) + \delta(u_2) + \cdots + \delta(u_{n-1}) + \delta(u_n)) + \gamma.$$

Hence

$$f(\frac{1}{2}) = \frac{1}{4} \leq f\Big(\frac{n-2}{n^2}\Big) + \gamma.$$

By $(\mathcal{F}_2)$, we have

$$\lim_{n\to+\infty} f\Big(\frac{n-2}{n^2}\Big) + \gamma = -\infty,$$

which is a contradiction. Consequently δ is not an $\mathcal{F}$-modular.
Moreover δ is not a K_p-modular. Infact if δ satisfies (Δ_3), and let

$$x = u_1 = 0, y = u_n = 1, u_i = \frac{1}{n}, \quad i = 2, \dots, n-1,$$

then for $\alpha = \beta = \frac{1}{2}$, by (Δ_3) we conclude that

$$\delta(\frac{x}{2} + \frac{y}{2}) \leq \delta(u_1) + \delta(u_2) + \cdots + \delta(u_{n-1}) + \delta(u_n) + \gamma.$$

Therefore

$$\frac{1}{2} = \frac{1}{4} \leq \frac{n-2}{n^2}.$$

By $(\mathcal{F}_2)$, we have

$$\lim_{n\to+\infty} \frac{n-2}{n^2} = 0,$$

which is a contradiction.

Remark 2.7. One can easily see that the mapping δ defined by (2.1) in Example 2.4, is not also a K-modular on $\mathcal{X}$.

3. $\mathcal{F}$-modular boundedness

In this section, we introduce the concept of $\mathcal{F}$-modular boundedness, which is used to provide a characterization of $\mathcal{F}$-modular spaces.

Definition 3.1. Let $\mathcal{X}$ be a linear space, and let $\delta : \mathcal{X} \to [0, +\infty)$ be a mapping satisfying (δ_1) and (δ_2). We call the pair $(\mathcal{X}, \delta)$ is $\mathcal{F}$-modular bounded with respect to $(f, \gamma) \in \mathcal{F} \times [0, +\infty)$, if there exists a modular ρ on $\mathcal{X}$ such that for every $x, y \in \mathcal{X}$, and for $\alpha, \beta > 0$ in which $\alpha + \beta = 1$, $\delta(\alpha x + \beta y) > 0$ implies that

$$f(\rho(\alpha x + \beta y)) \leq f(\delta(x) + \delta(y)) \ \text{ and } \ f(\delta(\alpha x + \beta y)) \leq f(\rho(\alpha x + \beta y)) + \gamma. \tag{3.1}$$

We can prove the following result.

Theorem 3.2. *Let $\mathcal{X}$ be a space and let $\delta : \mathcal{X} \to [0, +\infty)$ be a mapping satisfying (δ_1) and (δ_2). Let $(f, \gamma) \in \mathcal{F} \times [0, +\infty)$ such that f is continuous from the right. Then the followings are equivalent:*

(i) *$(\mathcal{X}, \delta)$ is an $\mathcal{F}$-modular on $\mathcal{X}$ with (f, γ) given above.*
(ii) *$(\mathcal{X}, \delta)$ is an $\mathcal{F}$-modular bounded with respect to (f, γ).*

Proof. Suppose that $(\mathcal{X}, \delta)$ is an $\mathcal{F}$-modular on $\mathcal{X}$ with respect to (f, γ). We define the mapping $\rho : \mathcal{X} \to [0, +\infty)$ by

$$\rho(\alpha x + \beta y) = \inf\Big\{\sum_{i=1}^{n} \delta(u_i) : n \in \mathbb{N}, n \geq 2, (u_i)_{i=1}^{n} \subset \mathcal{X}, u_1 = x, u_n = y\Big\},$$

for all $x, y \in \mathcal{X}$ and for $\alpha, \beta > 0$ in which $\alpha + \beta = 1$. We show that ρ is a modular on $\mathcal{X}$. Note that

$$\rho(x) = \inf\Big\{\sum_{i=1}^{n} \delta(u_i) : n \in \mathbb{N}, n \geq 2, (u_i)_{i=1}^{n} \subset \mathcal{X}, u_1 = u_n = x\Big\},$$

so if $x = 0$, then $\rho(x) = 0$. Since $\delta(\alpha x) = \delta(x)$, for each α such that $|\alpha| = 1$, it follows from the definition of ρ that

$$\rho(\alpha x) = \rho(x), \quad x \in \mathcal{X}.$$

Now, let $x \in \mathcal{X}$ be such that $\rho(x) = 0$. Suppose that $x \neq 0$. Given $\varepsilon > 0$, then there exists $n \in \mathbb{N}$, $n \geq 2$, and $(u_i)_{i=1}^n \subset \mathcal{X}$ with $u_1 = u_n = x$ such that

$$\sum_{i=1}^n \delta(u_i) < \varepsilon.$$

By $(\mathcal{F}_1)$, we obtain

$$f\Big(\sum_{i=1}^n \delta(u_i)\Big) \leq f(\varepsilon). \tag{3.2}$$

More over, putting $y = x$ in (δ_3) deduce that

$$f(\delta(x)) \leq f\Big(\sum_{i=1}^n \delta(u_i)\Big) + \gamma. \tag{3.3}$$

Using (3.2) and (3.3), we get

$$f(\delta(x)) \leq f(\varepsilon) + \gamma.$$

By $(\mathcal{F}_2)$, we obtain

$$\lim_{\varepsilon \to 0^+} (f(\varepsilon) + \gamma) = -\infty,$$

which is a contradiction. Now, let $x, y \in \mathcal{X}$ and let $\alpha, \beta > 0$ be such that $\alpha + \beta = 1$. Suppose $\varepsilon > 0$ is arbitrary. Then by definition of ρ, there exist $\{u_i\}_{i=1}^n$ and $\{v_j\}_{j=1}^m$ in $\mathcal{X}$ such that $u_1 = u_n = x$, $v_1 = v_m = y$, and

$$\sum_{i=1}^n \delta(u_i) < \rho(x) + \varepsilon, \quad \sum_{j=1}^m \delta(v_j) < \rho(y) + \varepsilon.$$

Put $w_1 = u_1 = x$, and $w_i = u_i$ for every $2 \leq i \leq n$, $w_i = v_{n+m-i-1}$ for every $n+1 \leq i \leq n+m-1$, and $w_{n+m} = u_m = y$. Then we obtain

$$\begin{aligned} \rho(\alpha x + \beta y) &\leq \sum_{i=1}^{n+m} \delta(w_i) \\ &= \sum_{i=1}^n \delta(u_i) + \sum_{j=1}^m \delta(v_j) \\ &< \rho(x) + \rho(y) + 2\varepsilon. \end{aligned}$$

Passing to the limit as $\varepsilon \to 0^+$, we get

$$\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y).$$

Now, we shall prove that δ satisfies (3.1). For this, let $x, y \in \mathcal{X}$, and for $\alpha, \beta > 0$ in which $\alpha + \beta = 1$, $\delta(\alpha x + \beta y) > 0$. It is clear that

$$\rho(\alpha x + \beta y) \leq \delta(x) + \delta(y),$$

and $(\mathcal{F}_1)$ implies that

$$f(\rho(\alpha x + \beta y)) \leq f(\delta(x) + \delta(y)). \tag{3.4}$$

Let $\varepsilon > 0$. Then, there exist $n \in \mathbb{N}$, and $\{u_i\}_{i=1}^n \subset \mathcal{X}$ with $u_1 = x$ and $u_n = y$ such that

$$\sum_{i=1}^n \delta(u_i) < \rho(\alpha x + \beta y) + \varepsilon.$$

Hence

$$f\Big(\sum_{i=1}^{n}\delta(u_i)\Big) \le f(\rho(\alpha x+\beta y)+\varepsilon).$$

By (δ_3), we obtain

$$f(\delta(\alpha x+\beta y) \le f(\rho(\alpha x+\beta y)+\varepsilon)+\gamma.$$

Passing to the limit as $\varepsilon \to 0^+$, and using the right continuity of f, we get

$$f(\delta(\alpha x+\beta y) \le f(\rho(\alpha x+\beta y))+\gamma. \tag{3.5}$$

We deduce from (3.4) and (3.5) that

$$f(\rho(\alpha x+\beta y)) \le f(\delta(x)+\delta(y)) \le f(\rho(\alpha x+\beta y))+\gamma.$$

Therefore $(\mathcal{X},\delta)$ is $\mathcal{F}$-modular bounded with respect to (f,γ).
Now, let $(\mathcal{X},\delta)$ is $\mathcal{F}$-modular bounded with respect to (f,γ), that is, there exists a modular ρ on $\mathcal{X}$ such that (3.1) satisfied. Let $x,y\in\mathcal{X}$, and let $\alpha,\beta>0$ be such that $\alpha+\beta=1$, and $\delta(\alpha x+\beta y)>0$. Suppose $n\in\mathbb{N}$, and $\{u_i\}_{i=1}^n$ with $u_1=x, u_n=y$. Since ρ is a modular, we have

$$\rho(\alpha x+\beta y) \le \rho(x)+\rho(y) \le \sum_{i=1}^{n}\rho(u_i). \tag{3.6}$$

Using $(\mathcal{F}_1)$ and the fact that if $\delta(x)+\delta(y)>0$, and

$$f(\rho(\alpha x+\beta y)) \le f(\delta(x)+\delta(y)),$$

we deduce that

$$\rho(\alpha x+\beta y) \le \delta(x)+\delta(y). \tag{3.7}$$

By (3.6) and (3.7), we get

$$f(\rho(\alpha x+\beta y)) \le f(\delta(x)+\delta(y)) \;\text{ and }\; f(\delta(\alpha x+\beta y)) \le f(\rho(\alpha x+\beta y))+\gamma.$$

By $(\mathcal{F}_1)$ we deduce that

$$f(\rho(\alpha x+\beta y))+\gamma \le f\Big(\sum_{i=1}^{n}\delta(u_i)\Big)+\gamma.$$

On the other hand

$$f(\delta(\alpha x+\beta y)) \le f(\rho(\alpha x+\beta y))+\gamma,$$

we conclude that

$$f(\delta(\alpha x+\beta y)) \le f\Big(\sum_{i=1}^{n}\delta(u_i)\Big)+\gamma.$$

Therefore, δ is an $\mathcal{F}$-modular on $\mathcal{X}$. □

Theorem 3.2 gives a characterization of $\mathcal{F}$-modulars as follows.

Corollary 3.3. *An $\mathcal{F}$-modular on a space $\mathcal{X}$ is an $\mathcal{F}$-modular bounded mapping.*

Remark 3.4. Note that in the proof of Theorem 3.2, the right continuity assumption of f is used only to prove that $(i)\Rightarrow(ii)$. However, $(ii)\Rightarrow(i)$ holds for any $f\in\mathcal{F}$.

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