Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (3) (2020), 1093 – 1106
DOI : 10.15672/hujms.647228

Research Article

Ladders and fan graphs are cycle-antimagic

Martin Bača[1], Pon Jeyanthi[2], Narayanaperumal Thillaiammal Muthuraja[3], Pothukutti Nadar Selvagopal[4], Andrea Semaničová-Feňovčíková*[1]

[1] *Department of Applied Mathematics and Informatics, Technical University, Letná 9, Košice, Slovakia*
[2] *Research Centre, Department of Mathematics, Govindammal Aditanar college for Women, Tiruchendur, Tamilnadu, India*
[3] *Department of Mathematics, Manonmaniam Sundaranar University PG Extension Centre, Nagercoil, Tamilnadu, India*
[4] *Al Musanna College of Technology, Sultanate of Oman*

Abstract

A simple graph $G = (V, E)$ admits an H-covering if every edge in E belongs to at least one subgraph of G isomorphic to a given graph H. The graph G admitting an H-covering is (a, d)-H-antimagic if there exists a bijection $f : V \cup E \to \{1, 2, \dots, |V| + |E|\}$ such that, for all subgraphs H' of G isomorphic to H, the H'-weights, $wt_f(H') = \sum_{v\in V(H')} f(v) + \sum_{e\in E(H')} f(e)$, form an arithmetic progression with the initial term a and the common difference d. Such a labeling is called *super* if the smallest possible labels appear on the vertices. In this paper we prove the existence of super (a, d)-H-antimagic labelings of fan graphs and ladders for H isomorphic to a cycle.

Mathematics Subject Classification (2010). 05C78, 05C70

Keywords. H-covering, (super) (a, d)-H-antimagic total labeling, cycle-antimagic labeling, ladder, fan graph

1. Introduction

Let $G = (V, E)$ be a finite simple graph. An *edge-covering* of G is a family of subgraphs $H_1, H_2, \dots, H_t$ such that each edge of E belongs to at least one of the subgraphs H_i, $i = 1, 2, \dots, t$. Then it is said that G admits an $(H_1, H_2, \dots, H_t)$-*(edge) covering*. If every subgraph H_i is isomorphic to a given graph H, then the graph G admits an H*-covering*. A bijective function $f : V \cup E \to \{1, 2, \dots, |V| + |E|\}$ is an (a, d)-H*-antimagic labeling* of a graph G admitting an H-covering whenever, for all subgraphs H' isomorphic to H, the H'-weights

$$wt_f(H') = \sum_{v\in V(H')} f(v) + \sum_{e\in E(H')} f(e)$$

form an arithmetic progression $a, a + d, a + 2d, \dots, a + (t - 1)d$, where $a > 0$ and $d \geq 0$ are two integers, and t is the number of all subgraphs of G isomorphic to H. Such a labeling

*Corresponding Author.
Email addresses: martin.baca@tuke.sk (M. Bača), jeyajeyanthi@rediffmail.com (P. Jeyanthi), bareeshraja@yahoo.com (N.T. Muthu Raja), ps_gopaal@yahoo.co.in (P. Selvagopal), andrea.fenovcikova@tuke.sk (A. Semaničová-Feňovčíková)
Received: 28.02.2018; Accepted: 08.08.2019

is called *super* if the smallest possible labels appear on the vertices. A graph that admits a (super) (a,d)-H-antimagic labeling is called *(super)* (a,d)-H*-antimagic*. For $d=0$ it is called H*-magic* and H*-supermagic*, respectively.

The H-supermagic graphs were first studied by Gutiérrez and Lladó [10] as an extension of the edge-magic and super edge-magic graphs introduced by Kotzig and Rosa [14] and Enomoto et al. [9], respectively. Lladó and Moragas [15] studied the cycle-(super)magic behavior of several classes of connected graphs. Further results on H-supermagic labelings are available in [18–21].

The (a,d)-H-antimagic labeling was introduced by Inayah et al. [11]. In [12] the super (a,d)-H-antimagic labeling for some shackles of a connected graph H was investigated. In [22] it was proved that wheels W_n, $n \geq 3$ are super (a,d)-C_k-antimagic for every $k = 3,4,\dots,n-1,n+1$ and $d = 0,1,2$.

The (super) (a,d)-H-antimagic labeling is related to a super d-antimagic labeling of type $(1,1,0)$ of a plane graph, that is the generalization of a face-magic labeling introduced by Lih [16]. A labeling of type $(1,1,0)$, that is a total labeling, of a plane graph is said to be d-antimagic if for every positive integer s the set of weights of all s-sided faces is $W_s=\{a_s, a_s+d, a_s+2d, \dots, a_s+(f_s-1)d\}$ for some integers a_s and $d \geq 0$, where f_s is the number of the s-sided faces. Note that, we allow different sets W_s for different s. The *weight* of a face under a labeling of type $(1,1,0)$ is the sum of labels of all the edges and vertices surrounding that face. If $d=0$ then Lih called such labeling as *magic* [16] and described magic (0-antimagic) labelings of type $(1,1,0)$ for wheels, friendship graphs and prisms. For further information on super d-antimagic labelings, we refer to [1, 3, 6].

For $H \cong K_2$, a (super) (a,d)-H-antimagic labeling is also called a (super) (a,d)-edge-antimagic total labeling, introduced in [23] and further studied in [5, 17]. The vertex version of these labelings for generalized pyramid graphs is given in [2].

The existence of super (a,d)-H-antimagic labelings for disconnected graphs is studied in [7]. In [4] it is shown that the disjoint union of multiple copies of a (super) $(a,1)$-tree-antimagic graph is also a (super) $(b,1)$-tree-antimagic. This leads to a natural question. Whether the similar result holds also for other differences and other H-antimagic graphs.

A *fan graph* F_n, $n \geq 2$, is a graph obtained by joining all the vertices of the path P_n to a further vertex, called the *centre*. The vertices on the path are called *path vertices*, the edges adjacent to the central vertex are called *spokes* and the remaining edges are called *path edges*. Thus F_n contains $n+1$ vertices, say, $v_1, v_2, \dots, v_{n+1}$, and $2n-1$ edges, say, $v_{n+1}v_i$, $1 \leq i \leq n$, and v_iv_{i+1}, $1 \leq i \leq n-1$.

A *ladder* $P_n \square P_2$, $n \geq 2$, is a Cartesian product of a path on n vertices and a path on 2 vertices. The ladder $P_n \square P_2$ contains $2n$ vertices, say, $v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n$, and $3n-2$ edges v_iv_{i+1}, $1 \leq i \leq n-1$, u_iu_{i+1}, $1 \leq i \leq n-1$ and v_iu_i, $1 \leq i \leq n$.

In this paper we investigate the super (a,d)-H-antimagic labelings of fan graphs and ladders for H isomorphic to a cycle. We also prove some results on the existence of super (a,d)-cycle-antimagic labelings of fan graphs for several values of parameter d.

In [8] Bača et al. proved the following:

Proposition 1.1 ([8]). *The fan* F_n*,* $n \geq 3$*, admits a super* (a,d)*-*C_k*-antimagic labeling for* $k = 3,4,\dots,\lfloor \frac{n}{2} \rfloor + 2$ *and* $d \in \{1,3,k-7,k+1,2k-5,2k-1,3k-9,3k-1\}$.

They showed that for a fan graph F_n, $n \geq 4$, there exists a super (a,d)-C_3-antimagic labeling for $d = 0,1,2,3,4,5,6,8$ and a super (a,d)-C_4-antimagic labeling for $d = 0,1,2,3,4,5,6,7,11$ and they proposed the following open problem.

Open Problem 1 ([8]). Find a super (a,d)-C_k-antimagic labeling of the fan graph F_n for $d \neq 1,3,k-7,k+1,2k-5,2k-1,3k-9,3k-1$.

In [13] the following proposition was proved.

Proposition 1.2 ([13]). *The fan graph F_n, $n \geq 3$, admits a super (a,d)-C_k-antimagic labeling for $k = 3, 4, \ldots, \lfloor \frac{n}{2} \rfloor + 2$ and $d \in \{1, 2, k-5, k-4, \ldots, k+2, 2k-5, 2k-1\}$.*

In this paper we prove that the fan graphs are super (a,d)-C_k-antimagic for $d \in \{0, 4, 5, 7\}$, $k = 3, 4, \ldots, \lfloor \frac{n}{2} \rfloor + 2$. Further we use the known results for fan graphs to obtain super cycle-antimagic labelings of ladders.

2. Fan graphs

Let n, r be positive integers, $2 \leq r \leq n-1$. Let $n = (t-1)r + s$, where $2 \leq r \leq n-1$ and $0 \leq s < r$. Thus $t = \lceil \frac{n}{r} \rceil$. The set of integers $\{1, 2, \ldots, n\}$, $n \geq 2$, can be arranged as a sequence $\{a_i^{r,n}\}_{i=1}^n$ such that the i^{th} element $a_i^{r,n}$, $i = \alpha r + \beta$, $0 \leq \alpha \leq t-1$, $1 \leq \beta \leq r$, is defined as follows:

$$a_i^{r,n} = \begin{cases} (\beta-1)t + \alpha + 1, & \text{if } 0 \leq \alpha \leq t-1,\ 1 \leq \beta \leq s, \\ (\beta-1)(t-1) + s + \alpha + 1, & \text{if } 0 \leq \alpha \leq t-2,\ s+1 \leq \beta \leq r. \end{cases}$$

That is

$$\{a_i^{r,n}\}_{i=1}^n = \tag{2.1}$$

$$\begin{matrix} \{1, & t+1, & \ldots, (s-1)t+1, & st+1, & (s+1)t, & \ldots, (r-1)t-r+s+2, \\ 2, & t+2, & \ldots, (s-1)t+2, & st+2, & (s+1)t+1, & \ldots, (r-1)t-r+s+3, \\ \vdots & \vdots & \ddots\ \vdots & \vdots & \ddots & \vdots \\ t-1, & 2t-1, & \ldots, st-1, & (s+1)t-1, & (s+2)t-2, & \ldots, n, \\ t, & 2t, & \ldots, st\}. & & & \end{matrix}$$

In [13] it was proved that the sequence $\{a_i^{r,n}\}_{i=1}^n$ has the property that the sum of any r consecutive members of this sequence differs by 1 from the sum of the foregoing r consecutive members of this sequence.

Proposition 2.1 ([13]). *Let n, r be positive integers, $2 \leq r \leq n-1$. The elements of $\{1, 2, \ldots, n\}$ can be arranged as a sequence $\{a_i^{r,n}\}_{i=1}^n$ such that $\sum_{j=i+1}^{i+r} a_{j+r}^{r,n} - \sum_{j=i}^{i+r-1} a_j^{r,n} = 1$ for $1 \leq i \leq n-r+1$.*

We immediately obtain that for $i = 1, 2, \ldots, n-r+1$

$$1 = \sum_{j=i+1}^{i+r} a_{j+r}^{r,n} - \sum_{j=i}^{i+r-1} a_j^{r,n} = a_{i+r}^{r,n} - a_i^{r,n}, \tag{2.2}$$

$$\sum_{j=i}^{i+r-1} a_j^{r,n} = \frac{r(st+n-t+3) - s(n+1)}{2} - 1 + j.$$

Next theorem shows that the fan graph admits super cycle-antimagic labelings for differences $d = 0, 4, 5$ and $d = 7$.

Theorem 2.2. *Let $n \geq 3$ be a positive integer and $3 \leq k \leq \lfloor \frac{n}{2} \rfloor + 2$. Then the fan graph F_n admits a super (a,d)-C_k-antimagic labeling for $d = 0, 4, 5, 7$.*

Proof. Let $n \geq 3$ be a positive integer and $3 \leq k \leq \lfloor \frac{n}{2} \rfloor + 2$. We consider the total labelings f_m, $m = 0, 4, 5, 7$, of F_n defined in the following way. The vertices are labeled with the numbers $1, 2, \ldots, n+1$ such that

$$\begin{aligned} f_m(v_i) &= a_i^{k-1,n}, && \text{for } i = 1, 2, \ldots, n \text{ and } m = 0, 4, 7, \\ f_5(v_i) &= n + 1 - a_i^{k-1,n}, && \text{for } i = 1, 2, \ldots, n, \\ f_m(v_{n+1}) &= n + 1, && \text{for } m = 0, 4, 5, 7, \end{aligned}$$

where $a_i^{k-1,n}$ is an element of the sequence $\{a_i^{k-1,n}\}_{i=1}^n$, see (2.1).

For f_m, $m = 0, 4, 5, 7$, we assign numbers $n+2, n+3, \dots, 3n$ to the edges of F_n in the following way:

$$\begin{aligned}
f_m(v_i v_{i+1}) &= a_i^{k-2,n-1} + n + 1, && \text{for } i = 1, 2, \dots, n-1 \text{ and } m = 0, 4,\\
f_m(v_i v_{i+1}) &= 2a_i^{k-2,n-1} + n + 1, && \text{for } i = 1, 2, \dots, n-1 \text{ and } m = 5, 7,\\
f_0(v_i v_{n+1}) &= 3n + 1 - i, && \text{for } i = 1, 2, \dots, n,\\
f_4(v_i v_{n+1}) &= 2n + i, && \text{for } i = 1, 2, \dots, n,\\
f_m(v_i v_{n+1}) &= n + 2i, && \text{for } i = 1, 2, \dots, n \text{ and } m = 5, 7,
\end{aligned}$$

where $a_i^{k-2,n-1}$ is an element of the sequence $\{a_i^{k-2,n-1}\}_{i=1}^{n-1}$, see (2.1). It is easy to see that f_m, $m = 0, 4, 5, 7$, is a bijection.

Every cycle C_k on k vertices in a fan graph F_n is of the form $C_k^j = v_j v_{j+1} v_{j+2} \dots v_{j+k-2} v_{n+1} v_j$, where $j = 1, 2, \dots, n-k+2$. As $k = 3, 4, \dots, \lfloor \frac{n}{2} \rfloor + 2$ every edge of F_n belongs at least to one cycle C_k^j.

For the C_k-weight of the cycle C_k^j, $j = 1, 2, \dots, n-k+2$, under a total labeling f_m, $m = 0, 4, 5, 7$, we get.

$$\begin{aligned}
wt_{f_m}(C_k^j) &= \sum_{v \in V(C_k^j)} f_m(v) + \sum_{e \in E(C_k^j)} f_m(e)\\
&= \sum_{s=0}^{k-2} f_m(v_{j+s}) + f_m(v_{n+1}) + \sum_{s=0}^{k-3} f_m(v_{j+s} v_{j+s+1})\\
&\quad + f_m(v_{j+k-2} v_{n+1}) + f_m(v_j v_{n+1}).
\end{aligned}$$

We consider the difference of weights of cycles C_k^{j+1} and C_k^j for $j = 1, 2, \dots, n-k+1$

$$\begin{aligned}
wt_{f_m}(C_k^{j+1}) - wt_{f_m}(C_k^j) &= f_m(v_{j+k-1}) + f_m(v_{j+k-2} v_{j+k-1})\\
&\quad + f_m(v_{j+1} v_{n+1}) + f_m(v_{j+k-1} v_{n+1}) - f_m(v_j)\\
&\quad - f_m(v_j v_{j+1}) - f_m(v_j v_{n+1}) - f_m(v_{j+k-2} v_{n+1}).
\end{aligned} \tag{2.3}$$

Using (2.2) and (2.3) we obtain the following:

$$\begin{aligned}
wt_{f_0}(C_k^{j+1}) - wt_{f_0}(C_k^j) &= a_{j+k-1}^{k-1,n} + (a_{j+k-2}^{k-2,n-1} + n + 1)\\
&\quad + (3n + 1 - (j+1)) + (3n + 1 - (j + k - 1)) - a_j^{k-1,n}\\
&\quad - (a_j^{k-2,n-1} + n + 1) - (3n + 1 - j) - (3n + 1 - (j + k - 2))\\
&= 0.
\end{aligned}$$

Thus under the labeling f_0 all the C_k-weights are the same.

$$\begin{aligned}
wt_{f_4}(C_k^{j+1}) - wt_{f_4}(C_k^j) &= a_{j+k-1}^{k-1,n} + (a_{j+k-2}^{k-2,n-1} + n + 1) + (2n + (j+1))\\
&\quad + (2n + (j + k - 1)) - a_j^{k-1,n} - (a_j^{k-2,n-1} + n + 1)\\
&\quad - (2n + j) - (2n + (j + k - 2)) = 4.
\end{aligned}$$

Thus, under the labeling f_4 the C_k-weights form the arithmetic sequence with the difference 4.

For f_5 we obtain

$$\begin{aligned}
wt_{f_5}(C_k^{j+1}) - wt_{f_5}(C_k^j) &= (n + 1 - a_{j+k-1}^{k-1,n}) + (2a_{j+k-2}^{k-2,n-1} + n + 1)\\
&\quad + (n + 2(j+1)) + (n + 2(j + k - 1)) - (n + 1 - a_j^{k-1,n})\\
&\quad - (2a_j^{k-2,n-1} + n + 1) - (n + 2j) - (n + 2(j + k - 2)) = 5,
\end{aligned}$$

which means that the C_k-weights form the arithmetic sequence with the difference 5.

For the labeling f_7 we get

$$\begin{aligned} wt_{f_7}(C_k^{j+1}) - wt_{f_7}(C_k^{j}) &= a_{j+k-1}^{k-1,n} + (2a_{j+k-2}^{k-2,n-1} + n + 1) + (n + 2(j+1)) \\ &+ (n + 2(j+k-1)) - a_j^{k-1,n} - (2a_j^{k-2,n-1} + n + 1) \\ &- (n + 2j) - (n + 2(j+k-2)) = 7. \end{aligned}$$

The C_k-weights under the total labeling f_7 form the arithmetic sequence with the difference 7. This concludes the proof. □

Figures 1 to 4 give a super (a,d)-C_5-antimagic labeling of the fan F_6 for $d = 0, 4, 5$ and $d = 7$.

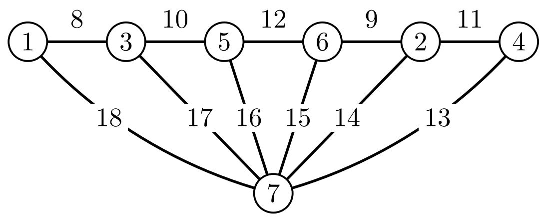

Figure 1. A super $(85,0)$-C_5-antimagic labeling of the fan F_6.

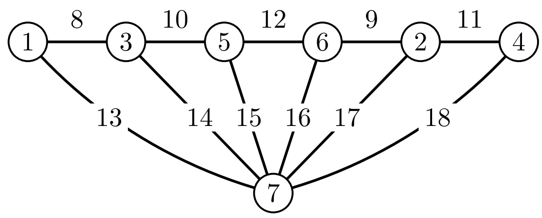

Figure 2. A super $(81,4)$-C_5-antimagic labeling of the fan F_6.

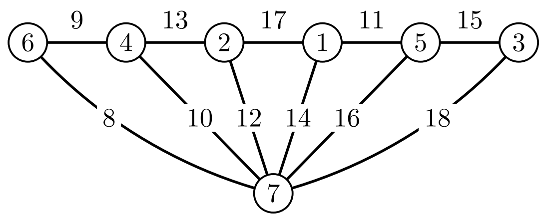

Figure 3. A super $(81,5)$-C_5-antimagic labeling of the fan F_6.

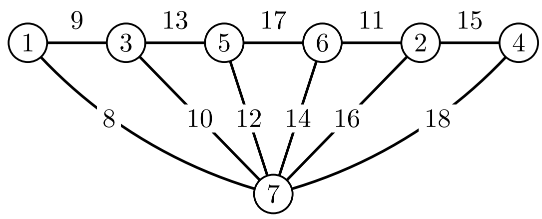

Figure 4. A super $(83,7)$-C_5-antimagic labeling of the fan F_6.

Immediately from Propositions 1.1, 1.2 and Theorem 2.2 we obtain the following result for cycle-antimagicness of fan graphs.

Theorem 2.3. *The fan graph F_n, $n \geq 3$, admits a super (a,d)-C_k-antimagic labeling for $k = 3,4,\ldots,\lfloor \frac{n}{2} \rfloor + 2$ and $d \in \{0,1,2,3,4,5,7,k-7,k-5,k-4,\ldots,k+2,2k-5,2k-1,3k-9,3k-1\}$.*

We note that from the constructions of the given labelings described in the proofs of Propositions 1.1, 1.2 and Theorem 2.2, each of such labelings has the property, that the centre of the fan graph F_n (the vertex v_{n+1}) is labeled with the number $n+1$.

3. Ladders

In this section we deal with the super cycle-antimagicness of Cartesian product $P_n \square P_2$. We construct the required labelings from super cycle-antimagic labelings of fans.

Theorem 3.1. *The ladder $P_n \square P_2$, $n \geq 3$, admits a super (a,d)-$C_{2(k-1)}$-antimagic labeling for $k = 3,4,\ldots,\lfloor \frac{n}{2} \rfloor + 2$ and $d \in \{1,2,3,4,5,6,8,k-8,k-6,k-5,\ldots,k+3,2k-6,2k-4,2k-2,2k,3k-10,3k-8,3k-2,3k\}$.*

Proof. According to Theorem 2.2 and the results proved in [8] and [13] there exists a super (a_d,d)-C_k-antimagic labeling f_d of fan F_n for $k = 3,4,\ldots,\lfloor \frac{n}{2} \rfloor + 2$ and $d \in \{0,2,3,4,7,k-7,k-4,k-3,k-2,k,k+1,k+2,2k-5,2k-1,3k-9,3k-1\}$ having the following properties:

(P1) the vertex v_{n+1} is labeled with the number $n+1$;

(P2) it is possible to denote all subgraphs of F_n isomorphic to C_k by the symbols $\mathcal{C}_k^i$, $i = 1,2,\ldots,n-k+2$, such that

$$wt_{f_d}(\mathcal{C}_k^i) = a_d + (i-1)d;$$

(P3) and the weight of the cycle C_k containing the vertex v_1 of F_n is the smallest one, that is, $v_1 \in V(\mathcal{C}_k^1)$.

We define the total labelings g_1 and g_2 of the $P_n \square P_2$ such that

$$\begin{aligned}
g_m(v_i) &= f_d(v_i), && \text{for } i = 1,2,\ldots,n \text{ and } m = 1,2,\\
g_1(u_i) &= n+i, && \text{for } i = 1,2,\ldots,n-k+2\\
& && \text{and } v_i \in V(\mathcal{C}_k^i),\\
& && \mathcal{C}_k^i = v_i v_{i+1} \ldots v_{i+k-2} v_{n+1} v_i,\\
g_2(u_i) &= 2n+1-i, && \text{for } i = 1,2,\ldots,n-k+2\\
& && \text{and } v_i \in V(\mathcal{C}_k^i),\\
& && \mathcal{C}_k^i = v_i v_{i+1} \ldots v_{i+k-2} v_{n+1} v_i,\\
g_1(u_i) &= n+i, && \text{for } i = n-k+3, n-k+4, \ldots, n,\\
g_2(u_i) &= 2n+1-i, && \text{for } i = n-k+3, n-k+4, \ldots, n,\\
g_m(v_i v_{i+1}) &= f_d(v_i v_{i+1}) + n - 1, && \text{for } i = 1,2,\ldots,n-1 \text{ and } m = 1,2,\\
g_m(v_i u_i) &= f_d(v_i v_{n+1}) + n - 1, && \text{for } i = 1,2,\ldots,n \text{ and } m = 1,2,\\
g_1(u_i u_{i+1}) &= 6n - g_1(u_{i+1}), && \text{for } i = 1,2,\ldots,n-1,\\
g_2(u_i u_{i+1}) &= 6n - 1 - g_2(u_{i+1}), && \text{for } i = 1,2,\ldots,n-1.
\end{aligned}$$

According to Properties (P1) and (P3) the labeling g_m, $m = 1,2$, is a bijective mapping from the set $\{1,2,\ldots,5n-2\}$ and the smallest possible numbers are used to label the vertices of $P_n \square P_2$.

We denote the $C_{2(k-1)}$ cycle $v_j v_{j+1} \ldots v_{j+k-2} u_{j+k-2} u_{j+k-3} \ldots u_j v_j$, $j = 1,2,\ \ldots, n-k+2$, in $P_n \square P_2$ by the symbol $C_{2(k-1)}^j$ if $\mathcal{C}_k^j = v_j v_{j+1} v_{j+2} \ldots v_{j+k-2} v_{n+1} v_j$ is the cycle in F_n.

For the $C_{2(k-1)}$-weight of the cycle $C^j_{2(k-1)}$, $j = 1, 2, \ldots, n-k+2$, in $P_n \square P_2$ under a total labeling g_m, $m = 1, 2$, we get

$$
\begin{aligned}
wt_{g_m}(C^j_{2(k-1)}) =& \sum_{v \in V(C^j_{2(k-1)})} g_m(v) + \sum_{e \in E(C^j_{2(k-1)})} g_m(e) \\
=& \sum_{s=0}^{k-2} g_m(v_{j+s}) + \sum_{s=0}^{k-2} g_m(u_{j+s}) + \sum_{s=0}^{k-3} g_m(v_{j+s}v_{j+s+1}) \\
&+ \sum_{s=0}^{k-3} g_m(u_{j+s}u_{j+s+1}) + g_m(v_j u_j) + g_m(v_{j+k-2}u_{j+k-2}).
\end{aligned}
$$

Using Property (P2) we obtain

$$
\begin{aligned}
wt_{g_1}(C^j_{2(k-1)}) =& \sum_{s=0}^{k-2} g_1(v_{j+s}) + \sum_{s=0}^{k-2} g_1(u_{j+s}) + \sum_{s=0}^{k-3} g_1(v_{j+s}v_{j+s+1})) \\
&+ \sum_{s=0}^{k-3} g_1(u_{j+s}u_{j+s+1}) + g_1(v_j u_j) + g_1(v_{j+k-2}u_{j+k-2}) \\
=& \sum_{s=0}^{k-2} f_d(v_{j+s}) + \sum_{s=0}^{k-2} g_1(u_{j+s}) \\
&+ \sum_{s=0}^{k-3} (f_d(v_{j+s}v_{j+s+1}) + n - 1) + \sum_{s=0}^{k-3} (6n - g_1(u_{j+s+1})) \\
&+ (f_d(v_j v_{n+1}) + n - 1) + (f_d(v_{j+k-2}v_{n+1}) + n - 1) \\
=& \sum_{s=0}^{k-2} f_d(v_{j+s}) + g_1(u_j) + \sum_{s=0}^{k-3} g_1(u_{j+s+1}) \\
&+ \sum_{s=0}^{k-3} f_d(v_{j+s}v_{j+s+1}) + (k-2)(n-1) + (k-2)6n \\
&- \sum_{s=0}^{k-3} g_1(u_{j+s+1}) + f_d(v_j v_{n+1}) + f_d(v_{j+k-2}v_{n+1}) \\
&+ 2n - 2 \\
=& \Bigg(\sum_{s=0}^{k-2} f_d(v_{j+s}) + \sum_{s=0}^{k-3} f_d(v_{j+s}v_{j+s+1}) + f_d(v_j v_{n+1}) \\
&+ f_d(v_{j+k-2}v_{n+1}) \Bigg) + g_1(u_j) + n(7k - 12) - k \\
=& \Big(wt_{f_d}(\mathcal{C}^j_k) - (n+1) \Big) + g_1(u_j) + n(7k - 12) - k \\
=& (a_d + (j-1)d) + (n + j) + n(7k - 13) - k - 1 \\
=& a_d + n(7k - 12) - k - 1 - d + j(d + 1).
\end{aligned}
$$

For the $C_{2(k-1)}$-weights under the labeling g_2 we get

$$\begin{aligned}
wt_{g_2}(C^j_{2(k-1)}) =& \sum_{s=0}^{k-2} g_2(v_{j+s}) + \sum_{s=0}^{k-2} g_2(u_{j+s}) + \sum_{s=0}^{k-3} g_2(v_{j+s}v_{j+s+1}) \\
&+ \sum_{s=0}^{k-3} g_2(u_{j+s}u_{j+s+1}) + g_2(v_j u_j) + g_2(v_{j+k-2}u_{j+k-2}) \\
=& \sum_{s=0}^{k-2} f_d(v_{j+s}) + \sum_{s=0}^{k-2} g_2(u_{j+s}) \\
&+ \sum_{s=0}^{k-3} (f_d(v_{j+s}v_{j+s+1}) + n - 1) \\
&+ \sum_{s=0}^{k-3} (6n - 1 - g_2(u_{j+s+1})) \\
&+ (f_d(v_j v_{n+1}) + n - 1) + (f_d(v_{j+k-2}v_{n+1}) + n - 1) \\
=& \sum_{s=0}^{k-2} f_d(v_{j+s}) + g_2(u_j) + \sum_{s=0}^{k-3} g_2(u_{j+s+1}) \\
&+ \sum_{s=0}^{k-3} f_d(v_{j+s}v_{j+s+1}) + (k-2)(n-1) + (k-2)(6n-1) \\
&- \sum_{s=0}^{k-3} g_2(u_{j+s+1}) + f_d(v_j v_{n+1}) + f_d(v_{j+k-2}v_{n+1}) \\
&+ 2n - 2 \\
=& \Bigg(\sum_{s=0}^{k-2} f_d(v_{j+s}) + \sum_{s=0}^{k-3} f_d(v_{j+s}v_{j+s+1}) + f_d(v_j v_{n+1}) \\
&+ f_d(v_{j+k-2}v_{n+1}) \Bigg) + g_2(u_j) + n(7k-12) - 2k + 2 \\
=& \left(wt_{f_d}(\mathcal{C}^j_k) - (n+1) \right) + g_2(u_j) + n(7k-12) - 2k + 2 \\
=& (a_d + (j-1)d) + (2n + 1 - j) + n(7k-13) - 2k + 1 \\
=& a_d + n(7k-11) - 2k + 2 - d + j(d-1).
\end{aligned}$$

As $d \in \{0, 2, 3, 4, 7, k-7, k-5, k-4, k-3, k-2, k, k+1, k+2, 2k-5, 2k-1, 3k-9, 3k-1\}$, we immediately obtain that the ladder $P_n \square P_2$, $n \geq 3$, admits a super (a, d^*)-$C_{2(k-1)}$-antimagic labeling for $k = 3, 4, \ldots, \lfloor \frac{n}{2} \rfloor + 2$ and $d^* \in \{1, 2, 3, 4, 5, 6, 8, k-8, k-6, k-5, \ldots, k+3, 2k-6, 2k-4, 2k-2, 2k, 3k-10, 3k-8, 3k-2, 3k\}$. □

Figure 5 illustrates a super $(218, 1)$-C_8-antimagic labeling of the ladder $P_6 \square P_2$ obtained from $(85, 0)$-C_5-antimagic labeling of the fan F_6, (see Figure 1), by using the construction described in the previous theorem.

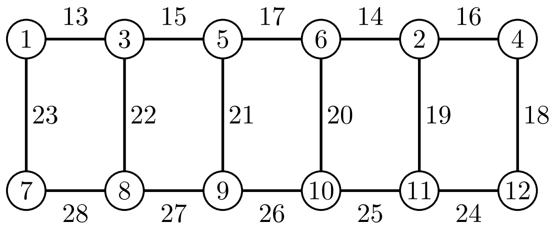

Figure 5. A super $(218, 1)$-C_8-antimagic labeling of the ladder $P_6\Box P_2$.

The following theorems show that the ladder also admits super cycle-antimagic labelings with differences $d = 0, 10, 12, 14, 16$.

Theorem 3.2. *Let $n \geq 3$ be a positive integer and $2 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$. Then the ladder $P_n\Box P_2$ admits a super C_{2k}-magic labeling.*

Proof. Let $n \geq 3$ be a positive integer and $2 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$. We consider the total labeling f of ladder $P_n\Box P_2$ defined in the following way.

$$\begin{aligned} f(v_i) &= i, && \text{for } i = 1, 2, \dots, n,\\ f(u_i) &= n + i, && \text{for } i = 1, 2, \dots, n,\\ f(v_iv_{i+1}) &= 3n - i, && \text{for } i = 1, 2, \dots, n-1,\\ f(u_iu_{i+1}) &= 4n - 1 - i, && \text{for } i = 1, 2, \dots, n-1,\\ f(v_iu_i) &= 5n - 1 - i, && \text{for } i = 1, 2, \dots, n. \end{aligned}$$

It is easy to see that the labeling f is a bijection and the vertices are labeled with numbers $1, 2, \dots, 2n$.

Every cycle C_{2k} on $2k$ vertices in $P_n\Box P_2$ is of the form $C_{2k}^j = v_jv_{j+1}v_{j+2}\dots v_{j+k-1}$ $u_{j+k-1}u_{j+k-2}\dots u_jv_j$, where $j = 1, 2, \dots, n-k+1$. As $k = 2, 3, \dots, \lfloor \frac{n}{2} \rfloor + 1$ every edge of $P_n\Box P_2$ belongs at least to one cycle C_{2k}^j.

For the C_{2k}-weight of the cycle C_{2k}^j, $j = 1, 2, \dots, n-k+1$, under the total labeling f we get.

$$\begin{aligned} wt_f(C_{2k}^j) &= \sum_{v\in V(C_{2k}^j)} f(v) + \sum_{e\in E(C_{2k}^j)} f(e)\\ &= \sum_{s=0}^{k-1} f(v_{j+s}) + \sum_{s=0}^{k-1} f(u_{j+s}) + \sum_{s=0}^{k-2} f(v_{j+s}v_{j+s+1})\\ &\quad + \sum_{s=0}^{k-2} f(u_{j+s}u_{j+s+1}) + f(v_{j+k-1}u_{j+k-1}) + f(v_ju_j). \end{aligned}$$

We consider the difference of weights of cycles C_{2k}^{j+1} and C_{2k}^j for $j = 1, 2, \dots, n-k$

$$\begin{aligned} wt_f({C_{2k}}^{j+1}) - wt_f(C_{2k}^j) &= f(v_{j+k}) + f(u_{j+k}) + f(v_{j+k-1}v_{j+k})\\ &+ f(u_{j+k-1}u_{j+k}) + f(v_{j+1}u_{j+1}) + f(v_{j+k}u_{j+k}) - f(v_j)\\ &- f(u_j) - f(v_jv_{j+1}) - f(u_ju_{j+1}) - f(v_ju_j) - f(v_{j+k-1}u_{j+k-1})\\ =&(j+k) + (n+j+k) + \big(3n - (j+k-1)\big)\\ &+ \big(4n - 1 - (j+k-1)\big) + \big(5n - 1 - (j+1)\big)\\ &+ \big(5n - 1 - (j+k)\big) - j - (n+j) - (3n-j) - (4n-1-j)\\ &- (5n-1-j) - \big(5n - 1 - (j+k-1)\big) = 0. \end{aligned}$$

Thus the C_{2k}-weights are the same. This means that the labeling f is super C_{2k}-magic. This concludes the proof. □

Figure 6 depicts a super C_8-magic labeling of the ladder $P_6 \square P_2$.

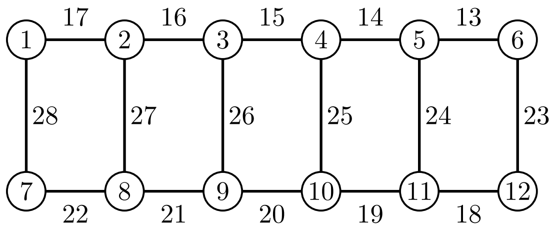

Figure 6. A super C_8-magic labeling of the ladder $P_6 \square P_2$.

Theorem 3.3. *Let $n \geq 3$ be a positive integer and $2 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$. Then the ladder $P_n \square P_2$ admits a super $(a, 10)$-C_{2k}-antimagic labeling.*

Proof. Let $n \geq 3$ be a positive integer and $2 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$. We consider the total labeling f of ladder $P_n \square P_2$ defined in the following way. The vertices are labeled with numbers $1, 2, \ldots, 2n$ such that

$$f(v_i) = 2a_i^{k,n}, \qquad \text{for } i = 1, 2, \ldots, n,$$
$$f(u_i) = 2a_i^{k,n} - 1, \qquad \text{for } i = 1, 2, \ldots, n,$$

where $a_i^{k,n}$ is an element of the sequence $\{a_i^{k,n}\}_{i=1}^{n}$, see (2.1).

The labeling f assigns numbers $2n+1, 2n+2, \ldots, 5n-2$ to the edges of $P_n \square P_2$ in the following way:

$$f(v_i v_{i+1}) = 2a_i^{k-1,n-1} + 2n, \qquad \text{for } i = 1, 2, \ldots, n-1,$$
$$f(u_i u_{i+1}) = 2a_i^{k-1,n-1} + 2n - 1, \qquad \text{for } i = 1, 2, \ldots, n-1,$$
$$f(v_i u_i) = 4n - 2 + i, \qquad \text{for } i = 1, 2, \ldots, n,$$

where $a_i^{k-1,n-1}$ is an element of the sequence $\{a_i^{k-1,n-1}\}_{i=1}^{n-1}$, see (2.1). It is easy to see that the labeling f is a bijection.

As in the proof of the previous theorem we consider the difference of weights of cycles C_{2k}^{j+1} and C_{2k}^{j} for $j = 1, 2, \ldots, n-k$ and we use (2.2)

$$\begin{aligned} wt_f({C_{2k}}^{j+1}) - wt_f(C_{2k}^{j}) = {} & 2a_{j+k}^{k,n} + \left(2a_{j+k}^{k,n} - 1\right) + \left(2a_{j+k-1}^{k-1,n-1} + 2n\right) \\ & + \left(2a_{j+k-1}^{k-1,n-1} + 2n - 1\right) + \left(4n - 2 + (j+1)\right) \\ & + \left(4n - 2 + (j+k)\right) - 2a_j^{k,n} - \left(2a_j^{k,n} - 1\right) - \left(2a_j^{k-1,n-1} + 2n\right) \\ & - \left(2a_j^{k-1,n-1} + 2n - 1\right) - \left(4n - 2 + j\right) - \left(4n - 2 + (j + k - 1)\right) \\ = {} & 10. \end{aligned}$$

The C_{2k}-weights under the total labeling f form the arithmetic sequence with the difference 10. This concludes the proof. □

Figure 7 illustrates a super $(210, 10)$-C_8-antimagic labeling of the ladder $P_6 \square P_2$.

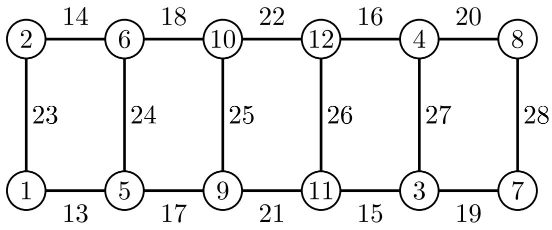

Figure 7. A super $(210, 10)$-C_8-antimagic labeling of the ladder $P_6 \square P_2$.

Theorem 3.4. *Let $n \geq 3$ be a positive integer and $2 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$. Then the ladder $P_n \square P_2$ admits a super (a, d)-C_{2k}-antimagic labeling for $d = 12, 14, 16$.*

Proof. Let $n \geq 3$ be a positive integer and $2 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$. We consider the total labeling f_m, $m = 12, 14, 16$, of ladder $P_n \square P_2$ defined in the following way. The vertices are labeled with numbers $1, 2, \dots, 2n$ such that

$$\begin{aligned} f_m(v_i) &= a_i^{k,n}, && \text{for } i = 1, 2, \dots, n \text{ and } m = 12, 14, \\ f_{16}(v_i) &= 2a_i^{k,n}, && \text{for } i = 1, 2, \dots, n, \\ f_{12}(u_i) &= 2n + 1 - a_i^{k,n}, && \text{for } i = 1, 2, \dots, n, \\ f_{14}(u_i) &= a_i^{k,n} + n, && \text{for } i = 1, 2, \dots, n, \\ f_{16}(u_i) &= 2a_i^{k,n} - 1, && \text{for } i = 1, 2, \dots, n, \end{aligned}$$

where $a_i^{k,n}$ is an element of the sequence $\{a_i^{k,n}\}_{i=1}^{n}$, see (2.1).

The labeling f_m assigns numbers $2n + 1, 2n + 2, \dots, 5n - 2$ to the edges of $P_n \square P_2$ in the following way

$$\begin{aligned} f_m(v_i v_{i+1}) &= 3a_i^{k-1,n-1} + 2n, && \text{for } i = 1, 2, \dots, n-1 \\ & && \text{and } m = 12, 14, 16, \\ f_m(u_i u_{i+1}) &= 3a_i^{k-1,n-1} + 2n - 1, && \text{for } i = 1, 2, \dots, n-1 \\ & && \text{and } m = 12, 14, 16, \\ f_m(v_i u_i) &= 2n - 2 + 3i, && \text{for } i = 1, 2, \dots, n \text{ and } m = 12, 14, 16, \end{aligned}$$

where $a_i^{k-1,n-1}$ is an element of the sequence $\{a_i^{k-1,n-1}\}_{i=1}^{n-1}$, see (2.1). It is easy to see that the labeling f_m is a bijection.

Again, we consider the difference of weights of cycles C_{2k}^{j+1} and C_{2k}^{j} for $j = 1, 2, \dots, n-k$, under the labelings f_m, $m = 12, 14, 16$.

For $m = 12$ we get

$$\begin{aligned} wt_{f_{12}}(C_{2k}^{j+1}) - wt_{f_{12}}(C_{2k}^{j}) &= a_{j+k}^{k,n} + (2n + 1 - a_{j+k}^{k,n}) + (3a_{j+k-1}^{k-1,n-1} + 2n) \\ &+ (3a_{j+k-1}^{k-1,n-1} + 2n - 1) + (2n - 2 + 3(j + 1)) \\ &+ (2n - 2 + 3(j + k)) - a_j^{k,n} - (2n + 1 - a_j^{k,n}) \\ &- (3a_j^{k-1,n-1} + 2n) - (3a_j^{k-1,n-1} + 2n - 1) \\ &- (2n - 2 + 3j) - (2n - 2 + 3(j + k - 1)) = 12. \end{aligned}$$

The C_{2k}-weights under the total labeling f_{12} form the arithmetic sequence with the difference 12.

If $m = 14$ then

$$\begin{aligned} wt_{f_{14}}(C_{2k}^{j+1}) - wt_{f_{14}}(C_{2k}^{j}) &= a_{j+k}^{k,n} + (a_{j+k}^{k,n} + n) + (3a_{j+k-1}^{k-1,n-1} + 2n) \\ &+ (3a_{j+k-1}^{k-1,n-1} + 2n - 1) + (2n - 2 + 3(j+1)) \\ &+ (2n - 2 + 3(j+k)) - a_{j}^{k,n} - (a_{j}^{k,n} + n) \\ &- (3a_{j}^{k-1,n-1} + 2n) - (3a_{j}^{k-1,n-1} + 2n - 1) \\ &- (2n - 2 + 3j) - (2n - 2 + 3(j+k-1)) = 14. \end{aligned}$$

The C_{2k}-weights under the total labeling f_{14} form the arithmetic sequence with the difference 14.

Finally, for the labeling f_{16} we obtain

$$\begin{aligned} wt_{f_{16}}(C_{2k}^{j+1}) - wt_{f_{16}}(C_{2k}^{j}) &= 2a_{j+k}^{k,n} + (2a_{j+k}^{k,n} - 1) + (3a_{j+k-1}^{k-1,n-1} + 2n) \\ &+ (3a_{j+k-1}^{k-1,n-1} + 2n - 1) + (2n - 2 + 3(j+1)) \\ &+ (2n - 2 + 3(j+k)) - 2a_{j}^{k,n} - (2a_{j}^{k,n} - 1) \\ &- (3a_{j}^{k-1,n-1} + 2n) - (3a_{j}^{k-1,n-1} + 2n - 1) \\ &- (2n - 2 + 3j) - (2n - 2 + 3(j+k-1)) = 16. \end{aligned}$$

Thus the C_{2k}-weights under the total labeling f_{16} form the arithmetic sequence with the difference 16. This concludes the proof. □

Figures 8, 9 and 10 give a super (a,d)-C_8-antimagic labeling of the ladder $P_6 \square P_2$ for $d = 12, 14$ and $d = 16$.

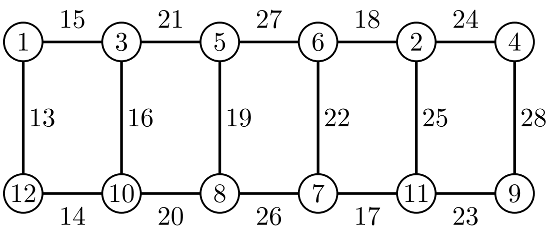

Figure 8. A super $(210,12)$-C_8-antimagic labeling of the ladder $P_6 \square P_2$.

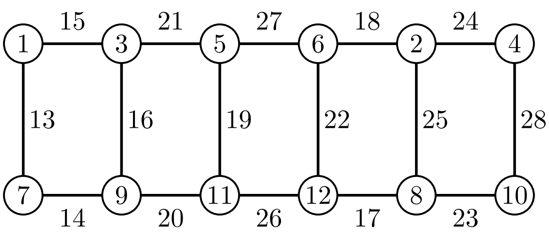

Figure 9. A super $(212,14)$-C_8-antimagic labeling of the ladder $P_6 \square P_2$.

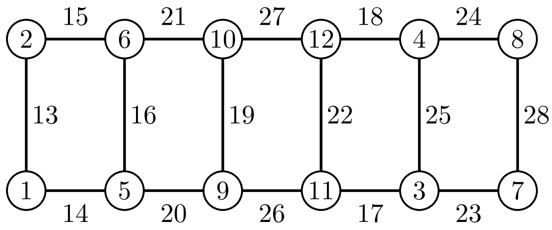

Figure 10. A super $(214, 16)$-C_8-antimagic labeling of the ladder $P_6 \square P_2$.

4. Conclusion

In this paper we prove the existence of super (a, d)-cycle-antimagic labelings of fan graphs and ladders. Combining the results obtained in this paper and in [8,13] we get that the fan graph F_n, $n \geq 3$, admits a super (a, d)-C_k-antimagic labeling for $k = 3, 4, \ldots, \lfloor \frac{n}{2} \rfloor + 2$ and $d \in \{0, 1, 2, 3, 4, 5, 7, k-7, k-5, k-4, \ldots, k+2, 2k-5, 2k-1, 3k-9, 3k-1\}$. Further, we prove that the ladder $P_n \square P_2$, $n \geq 3$, admits a super (a, d)-$C_{2(k-1)}$-antimagic labeling for $k = 3, 4, \ldots, \lfloor \frac{n}{2} \rfloor + 2$ and $d \in \{0, 1, 2, 3, 4, 5, 6, 8, 10, 12, 14, 16, k-8, k-6, k-5, \ldots, k+3, 2k-6, 2k-4, 2k-2, 2k, 3k-10, 3k-8, 3k-2, 3k\}$.

Though there are results on super (a, d)-cycle-antimagic labelings of fan graphs and ladders for various differences d, still some values of d for which the problem of existence of the corresponding labelings is not solved is a scope for further research.

Acknowledgment. This work was supported by the Slovak Research and Development Agency under the contract No. APVV-15-0116 and by VEGA 1/0233/18.

References

[1] A. Ahmad, M. Bača, M. Lascsáková and A. Semaničová–Feňovčíková, *Super magic and antimagic labelings of disjoint union of plane graphs*, Sci. Int. **24** (1), 21–25, 2012.

[2] S. Arumugam, M. Miller, O. Phanalasy and J. Ryan, *Antimagic labeling of generalized pyramid graphs*, Acta Math. Sinica - English Series, **30**, 283–290, 2014.

[3] M. Bača, L. Brankovic and A. Semaničová-Feňovčíková, *Labelings of plane graphs containing Hamilton path*, Acta Math. Sinica - English Series, **27** (4), 701–714, 2011.

[4] M. Bača, Z. Kimáková, A. Semaničová-Feňovčíková and M.A. Umar, *Tree-antimagicness of disconnected graphs*, Mathematical Problems in Engineering, **2015**, Article ID 504251, 1–4, 2015.

[5] M. Bača and M. Miller, *Super edge-antimagic graphs: A wealth of problems and some solutions*, Brown Walker Press, Boca Raton, Florida, 2008.

[6] M. Bača, M. Miller, O. Phanalasy and A. Semaničová-Feňovčíková, *Super d-antimagic labelings of disconnected plane graphs*, Acta Math. Sinica - English Series, **26** (12), 2283–2294, 2010.

[7] M. Bača, M. Miller, J. Ryan and A. Semaničová-Feňovčíková, *On H-antimagicness of disconnected graphs*, Bull. Aust. Math. Soc. **94** (2), 201–207, 2016.

[8] M. Bača, A. Ovais, A. Semaničová-Feňovčíková and M.A. Umar, *Fans are cycle-antimagic*, Australas. J. Combin. **68** (1), 94–105, 2017.

[9] H. Enomoto, A.S. Lladó, T. Nakamigawa and G. Ringel, *Super edge-magic graphs*, SUT J. Math. **34**, 105–109, 1998.

[10] A. Gutiérrez and A.S. Lladó, *Magic coverings*, J. Combin. Math. Combin. Comput. **55**, 43–56, 2005.

[11] N. Inayah, A.N.M. Salman and R. Simanjuntak, *On (a, d)-H-antimagic coverings of graphs*, J. Combin. Math. Combin. Comput. **71**, 273–281, 2009.

[12] N. Inayah, R. Simanjuntak, A.N.M. Salman and K.I.A. Syuhada, *On (a,d)-H-antimagic total labelings for shackles of a connected graph H*, Australasian J. Combin. **57**, 127–138, 2013.

[13] P. Jeyanthi, N.T. Muthuraja, A. Semaničová-Feňovčíková and S.J. Dharshikha, *More classes of super cycle-antimagic graphs*, Australas. J. Combin. **67** (1), 46–64, 2017.

[14] A. Kotzig and A. Rosa, *Magic valuations of finite graphs*, Canad. Math. Bull. **13**, 451–461, 1970.

[15] A. Lladó and J. Moragas, *Cycle-magic graphs*, Discrete Math. **307**, 2925–2933, 2007.

[16] K.W. Lih, *On magic and consecutive labelings of plane graphs*, Utilitas Math. **24**, 165–197, 1983.

[17] A.M. Marr and W.D. Wallis, *Magic Graphs*, Birkhäuser, New York, 2013.

[18] T.K. Maryati, A.N.M. Salman and E.T. Baskoro, *Supermagic coverings of the disjoint union of graphs and amalgamations*, Discrete Math. **313**, 397–405, 2013.

[19] T.K. Maryati, A.N.M. Salman, E.T. Baskoro, J. Ryan and M. Miller, *On H-supermagic labelings for certain shackles and amalgamations of a connected graph*, Utilitas Math. **83**, 333–342, 2010.

[20] A.A.G. Ngurah, A.N.M. Salman and L. Susilowati, *H-supermagic labelings of graphs*, Discrete Math. **310**, 1293–1300, 2010.

[21] A.N.M. Salman, A.A.G. Ngurah and N. Izzati, *On (super)-edge-magic total labelings of subdivision of stars S_n*, Utilitas Math. **81**, 275–284, 2010.

[22] A. Semaničová–Feňovčíková, M. Bača, M. Lascsáková, M. Miller and J. Ryan, *Wheels are cycle-antimagic*, Electron. Notes Discrete Math. **48**, 11–18, 2015.

[23] R. Simanjuntak, M. Miller and F. Bertault, *Two new (a,d)-antimagic graph labelings*, Proc. Eleventh Australas. Workshop Combin. Alg. (AWOCA), 179–189, 2000.