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A Common Fixed Point Theorem for Generalized-Weakly Contractive Mappings in Multiplicative Metric Spaces

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Abstract

In this paper, we introduce generalized weakly contractive mappings in the setting of multiplicative metric spaces. Further we establish a common fixed point result and proved the existence and uniqueness of a common fixed point. Finally, we provide an example in support of our main finding.

Keywords: Generalized weakly contractive mappings; Multiplicative metric space; Common fixed points.
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1. Introduction

In this paper, $\mathbb{R}^+$ denotes the set of positive real numbers. The Banach contraction principle [4] is one of the most powerful and important tools in pure and applied mathematics to prove the existence and uniqueness of different problems. After publication of [4], R. Kannan[14] and S.K. Chatterjea [7] introduced types of mapping which is independent of Banach. T. Zamfirescu[20] obtained another fixed point result for operators by generalizing [4] , R. Kannan[14] and S.K. Chatterjea [7]. A. Khan *et.al.*[16] introduce an (α, ϑ)-admissibility and (γ, ϱ)-integral-type contraction with applications to new fixed point theorems for the admissible and continuous mapping. M. Shoaib *et.al.*[19] Obtained Fixed point results and its applications to the systems of non-linear integral and differential equations of arbitrary order. M. Grossman and R. Katz [11] gave definitions of a new kind of derivative and integral, moving the roles of subtraction and addition

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to division and multiplication, and thus established a new calculus, called multiplicative calculus. Following this, A. Bashirov *et.al.* [5] studied the concept of multiplicative calculus and proved the fundamental theorem of multiplicative calculus. Furthermore, they gave application of multiplicative calculus, defined multiplicative absolute value, multiplicative distance between two positive real numbers and finally they introduced the notion of multiplicative metric spaces. L. Florack and H. van Assen[10] explored the advantage of multiplicative calculus in biomedical image analysis. A.E. Bashirov *et.al.*[6] discussed the simplicity of solving multiplicative differential equations than ordinary differential equations in different fields. M. Özavsar *et.al.*[17] gave the definition of multiplicative contraction and proved Banach contraction principle in the setting of multiplicative metric spaces and also they studied multiplicative metric topology. The concept of a weakly contractive mapping was introduced by Y.I. Alber and S. Guerre-Delabriere[3]. Following this, many authors obtained generalizations and extensions of the weak contraction principle. For example, B.S. Choudhury *et.al.*[9] introduced generalized weakly contractive mappings in metric space, M. Abbas *et.al.*[2] obtained several fixed and common fixed point results satisfying certain generalized contractive conditions in the framework of multiplicative metric spaces. M. Abbas *et.al.*[1] proved common fixed points for locally contractive mappings and explored the application of multiplicative metric spaces. X. He *et.al.*[12] proved common fixed points for weak commutative mappings on a multiplicative metric space.
In this paper, we introduce generalized weakly contractive mappings, establish a common fixed point theorem for the mappings introduced and prove the existence and uniqueness a common fixed point result in the setting of multiplicative metric spaces.

2. Preliminaries

In 2008, Bashirov et al., defined new kind of spaces, called multiplicative metric spaces in the following way:

Definition 2.1. *A. Bashirov et.al. [5] Let X be a non-empty set. A mapping $d : X \times X \longrightarrow \mathbb{R}^+$ is said to be a multiplicative metric on X if for any $x, y, z \in X$,the following conditions hold:*
i. $d(x, y) \geq 1$ *and* $d(x, y) = 1$ *if and only if* $x = y$.
ii. $d(x, y) = d(y, x)$.
iii. $d(x, y) \leq d(x, z).d(z, y)$.
Then (X, d) is a multiplicative metric space.

Example 2.2. *M. Özavsar et.al. [17] Let $\mathbb{R}^n_+$ be the collections of $n-$ tuples of positive real numbers. Let $d^* : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \longrightarrow R$ be defined as follows:*

$$d^*(x, y) = \left|\frac{x_1}{y_1}\right|^* . \left|\frac{x_2}{y_2}\right|^* ...... \left|\frac{x_n}{y_n}\right|^*,$$

where $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n) \in R^n_+$ and $|.|^* : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is defined by

$$|a|^* = \begin{cases} a & if \;\; a \geq 1 \\ \frac{1}{a} & if \, a < 1. \end{cases}$$

Then it is clear that all conditions of Definition 2.1 are satisfied. Therefore $(\mathbb{R}^n_+, d^*)$ is a multiplicative metric space.

Example 2.3. *M. Sarwar and R. Badshah-e[18] Let $d : \mathbb{R} \times \mathbb{R} \longrightarrow [1, \infty)$ be defined as $d(x, y) = a^{|x-y|}$,*

where $x, y \in R$ and $a > 1$. Then d is a multiplicative metric and (R, d) is a multiplicative metric space. It is taken as usual multiplicative metric spaces for all real numbers.

Example 2.4. *M. Sarwar and R. Badshah-e [18] Let (X, d) be a metric space. Define a mapping d_a on X by*

$$d_a(x,y) = a^{d(x,y)} = \begin{cases} 1 & \text{if } x = y \\ a & \text{if } x \neq y, \end{cases}$$

where $x, y \in X$ and $a > 1$.

Then d_a is a multiplicative metric and (X, d_a) is known as the discrete multiplicative metric space.

Remark 2.5. *M. Sarwar and R. Badshah-e [18] We note that multiplicative metrics and metric spaces are independent. Indeed, the mapping d^* defined in Example 2.2 is multiplicative metric but not metric as it does not satisfy triangular inequality. Consider*

$$d^*\left(\frac{1}{3}, \frac{1}{2}\right) + d^*\left(\frac{1}{2}, 3\right) = \frac{3}{2} + 6 = 7.5 < 9 = d^*\left(\frac{1}{3}, 3\right).$$

On the other hand the usual metric on $\mathbb{R}$ is not multiplicative metric as it doesnt satisfy multiplicative triangular inequality, since

$$d(2,3).d(3,6) = 3 < 4 = d(2,6).$$

Definition 2.6. *M. Özavsar et.al. [17] Let (X, d) be a multiplicative metric space, $x \in X$ and $\epsilon > 1$. We now define a set $B_\epsilon(x) = \{y \in X \mid d(x,y) < \epsilon\}$, which is called multiplicative open ball of radius ϵ with center x. Similarly, one can describe multiplicative closed ball as $\bar{B}_\epsilon(x) = \{y \in X \mid d(x,y) \leq \epsilon\}$.*

Definition 2.7. *M. Özavsar et.al. [17] Let (X, d) be a multiplicative metric space. Then a sequence $\{x_n\}$ in X said to be*
(1) multiplicative convergent to x if for every multiplicative open ball $B_\epsilon(x) = \{y \mid d(x,y) < \epsilon\}, \epsilon > 1$, there exists a natural number N such that $n \geq N$, then $x_n \in B_\epsilon(x)$, that is, $d(x_n, x) \longrightarrow 1$ as $n \longrightarrow \infty$.
(2) a multiplicative Cauchy sequence if for all $\epsilon > 1$, there exists $N \in N_0$ such that $d(x_n, x_m) < \epsilon$ for all $m, n > N_0$, that is, $d(x_n, x_m) \longrightarrow 1$ as $n, m \longrightarrow \infty$.

Definition 2.8. *M. Özavsar et.al. [17] We call a multiplicative metric space is complete if every multiplicative Cauchy sequence in it is multiplicative convergent to $x \in X$.*

Theorem 2.9. *M. Özavsar et.al. [17] Let $\{x_n\}$ be a multiplicative Cauchy sequence in a multiplicative metric space (X, d). If the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \longrightarrow *x \in X$ as $n_k \longrightarrow \infty$, then $x_n \longrightarrow *x \in X$ as $n \longrightarrow \infty$.*

Remark 2.10. *S.M. Kang et.al. See [15] The set of positive real numbers $\mathbb{R}_+ = (0, \infty)$ is not complete according to the usual metric. Let $X = \mathbb{R}_+$ and the sequence $\{x_n\} = \{\frac{1}{n}\}$. It is obvious $\{x_n\}$ is a Cauchy sequence in X with respect to usual metric and X is not a complete metric space, since $0 \notin \mathbb{R}_+$. In the case of a multiplicative metric space, we take a sequence $\{x_n\} = \{a^{\frac{1}{n}}\}$, where $a > 1$. Then $\{x_n\}$ is a multiplicative Cauchy sequence since for $n \geq m$,*

$$\begin{aligned} d(x_n, y_m) &= \left|\frac{x_n}{y_m}\right| = \left|\frac{a^{\frac{1}{n}}}{a^{\frac{1}{m}}}\right| = \left|a^{\frac{1}{n}-\frac{1}{m}}\right| \\ &\leq a^{\frac{1}{n}-\frac{1}{m}} < a^{\frac{1}{m}} < \epsilon \; if \; m > \frac{loga}{log\epsilon}, \end{aligned}$$

$$|a| = \begin{cases} a & \text{if } a \geq 1 \\ \frac{1}{a} & \text{if } a < 1 \end{cases}.$$

Also, $\{x_n\} \longrightarrow 1$ as $n \longrightarrow \infty$ and $1 \in \mathbb{R}_+$.

Hence (X, d) is a complete multiplicative metric space.

Definition 2.11. *Let $f, g : X \longrightarrow X$ be self-mappings. A point $x \in X$ is called*
(1) fixed point of f if $fx = x$;
(2) coincidence point of the pair $\{f, g\}$ if $fx = gx$;
(3) common fixed point of the pair $\{f, g\}$ if $x = fx = gx$.

Definition 2.12. *G.Jungck[13] Two self-maps S and T on a nonempty set X are called weakly compatible if they commute at their coincidence point.*

Definition 2.13. *A function $f : X \longrightarrow [0, \infty)$, where X is a metric space, is called lower semi-continuous if, for all $x \in X$ and $x_n \in X$ with $\lim_{n\to\infty} x_n = x$,*
we have

$$f(x) \leq \lim_{n\to\infty} inf f(x_n).$$

Definition 2.14. *M. Abbas et.al.[2] The control functions ψ and ϕ are defined as follows:*
i. $\Psi = \{\psi : [1, \infty) \longrightarrow [1, \infty) \mid \psi$ is a continuous non-decreasing function with $\psi(t) = 1$ if and only if $t = 1\}$.
ii. $\Phi = \{\phi : [1, \infty) \longrightarrow [1, \infty) \mid \phi$ is a lower semi-continuous function with $\phi(t) = 1$ if and only if $t = 1\}$.

S. Cho [8] proved the following fixed point theorem for generalized weakly contractive mappings in metric spaces as follows:

Theorem 2.15. *Let X be complete metric spaces and T satisfies the following conditions:*
$\psi(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)) \leq \psi(m(x, y, d, T, \varphi)) - \phi(l(x, y, d, T, \varphi))$, for all $x, y \in X$, where $\Psi = \{\psi :$ $[0, \infty) \longrightarrow [0, \infty) \mid \psi$ is a continuous and $\psi(t) = 0$ if and only if $t = 0\}$, $\Phi = \{\phi : [0, \infty) \longrightarrow [0, \infty) \mid \phi$ is a lower semi-c
and $\phi(t) = 0$ if and only if $t = 0\}$, $\varphi : X \longrightarrow [0, \infty)$ is lower semi-continuous function,

$$\begin{aligned} m(x, y, d, T, \varphi) \quad = \quad & max\{(d(x, y) + \varphi(x) + \varphi(y), d(x, Tx) + \varphi(x) + \varphi(Tx), \\ & d(y, Ty) + \varphi(y) + \varphi(Ty), \\ & \{d(x, Ty) + \varphi(x) + \varphi(Ty) + d(y, Tx) + \varphi(y) + \varphi(Tx)\}^{\frac{1}{2}}\}. \end{aligned}$$

and

$$l(x, y, d, T, \varphi) = max((d(x, y) + \varphi(x) + \varphi(y), d(y, Ty) + \varphi(y) + \varphi(Ty)).$$

Then there exists $z \in X$ such that $z = Tz$ and $\varphi(z) = 0$.

3. Main Results

In this section, we introduce generalized weakly contractive mappings, establish a common fixed point theorem for the mappings introduced and prove the existence and uniqueness a common fixed point result in the setting of multiplicative metric spaces.

Definition 3.1. *Let (X, d) be a multiplicative metric space, let $S, T : X \longrightarrow X$, and let $\varphi : X \to [1, \infty)$ be a lower semi-continuous function. Then S and T are called a generalized weakly contractive mapping if they satisfy the following condition:*

$$\psi(d(Tx, Ty).\varphi(Tx).\varphi(Ty)) \leq \frac{\psi(m(Sx, Sy, d, T, \varphi))}{\phi(l(Sx, Sy, d, T, \varphi))}, \; for \; all x, y \in X, \tag{1}$$

where $\psi \in \Psi, \phi \in \Phi$ and

$$\begin{aligned} m(Sx, Sy, d, T, \varphi) \quad = \quad & max\{d(Sx, Sy).\varphi(Sx).\varphi(Sy), d(Sx, Tx).\varphi(Sx).\varphi(Tx), \\ & d(Sy, Ty).\varphi(Sy).\varphi(Ty), \\ & \{d(Sx, Ty).\varphi(Sx).\varphi(Ty).d(Sy, Tx).\varphi(Sy).\varphi(Tx)\}^{\frac{1}{2}}\} \end{aligned}$$

and

$$l(Sx,Sy,d,T,\varphi) = max(d(Sx,Sy).\varphi(Sx).\varphi(Sy), d(Sy,Ty).\varphi(Sy).\varphi(Ty)). \tag{2}$$

Theorem 3.2. *Let (X,d) be a multiplicative metric space and S and T are generalized weakly contractive mappings. Assume that*
i. $T(X) \subseteq S(X)$;
ii. S and T are weakly compatible mappings. If one of the subspaces $T(X)$ or $S(X)$ is complete, then S and T have a unique common fixed point $z \in X$ such that $z = Tz = Sz$ and $\varphi(z) = 1$.

Proof. Let $x_0 \in X$ be fixed. Since $T(X) \subseteq S(X)$, choose $x_1 \in X$ such that $Sx_1 = Tx_0$. In general, choose $x_{n+1} \in X$ and define a sequence $\{x_n\}$ by $y_{n+1} = Sx_{n+1} = Tx_n$ for all $n = 0,1,2,...$
If $y_n = y_{n+1}$ for some n, we have $Sx_n = Sx_{n+1} = Tx_n$ and x_n is a coincidence point of S and T. Since S and T are weakly compatible, we have

$$TSx_n = STx_n = SSx_n \tag{3}$$

Here, Sx_n is a coincidence point of S and T. Now by setting $x = x_{n+1}$ and $y = Sx_n$ in (2), we have

$$\begin{aligned} m(Sx,Sy,d,T,\varphi) \;=\;& max\{d(Sx_{n+1},SSx_n).\varphi(Sx_{n+1}).\varphi(SSx_n),\\ & d(Sx_{n+1},Tx_{n+1}).\varphi(Sx_{n+1}).\varphi(Tx_{n+1}),\\ & d(SSx_n,TSx_n).\varphi(SSx_n).\varphi(TSx_n),\\ & \{d(Sx_{n+1},TSx_n).\varphi(Sx_{n+1}).\varphi(TSx_n).d(SSx_n,Tx_{n+1}).\varphi(SSx_n).\varphi(Tx_{n+1})\}^{\frac{1}{2}}\} \end{aligned}$$

and

$$l(Sx,Sy,d,T,\varphi) = max(d(Sx_{n+1},SSx_n).\varphi(Sx_{n+1}).\varphi(SSx_n), d(SSx_n,TSx_n).\varphi(SSx_n).\varphi(TSx_n)).$$

Which implies that

$$m(Sx,Sy,d,T,\varphi) \;=\; d(Sx_{n+1},SSx_n).\varphi(Sx_{n+1}).\varphi(SSx_n)$$

and

$$l(Sx,Sy,d,T,\varphi) = d(Sx_{n+1},SSx_n).\varphi(Sx_{n+1}).\varphi(SSx_n).$$

Hence, (1) becomes $\psi(d(Tx_{n+1},TSx_n).\varphi(Tx_{n+1}).\varphi(TSx_n)) \leq \frac{\psi(d(Sx_{n+1},SSx_n).\varphi(Sx_{n+1}).\varphi(SSx_n))}{\phi(d(Sx_{n+1},SSx_n).\varphi(Sx_{n+1}).\varphi(SSx_n))}$. Using (3), we have $\phi(d(Sx_{n+1},SSx_n).\varphi(Sx_{n+1}).\varphi(SSx_n)) = 1$. From this, $d(Sx_{n+1},SSx_n) = 1$ and Sx_n is a fixed point of S. Again using (3), $d(Sx_{n+1},TSx_n) = 1$ and Sx_n is a fixed point of T.
Therefore, Sx_n is a common fixed point of S and T.
Suppose $y_n \neq y_{n+1}$. Plunging $x = x_n$ and $y = x_{n+1}$ in (2) we have,

$$\begin{aligned} m(Sx_n,Sx_{n+1},d,T,\varphi) \;=\;& max\{d(Sx_n,Sx_{n+1}).\varphi(Sx_n).\varphi(Sx_{n+1}), d(Sx_n,Tx_n).\varphi(Sx_n).\varphi(Tx_n),\\ & d(Sx_{n+1},Tx_{n+1}).\varphi(Sx_{n+1}).\varphi(Tx_{n+1}),\\ & \{d(Sx_n,Tx_{n+1}).\varphi(Sx_n).\varphi(Tx_{n+1}).d(Sx_{n+1},Tx_n).\varphi(Sx_{n+1}).\varphi(Tx_n)\}^{\frac{1}{2}}\}\\ \;=\;& max\{(d(y_n,y_{n+1}).\varphi(y_n).\varphi(y_{n+1}), d(y_n,y_{n+1}).\varphi(y_n).\varphi(y_{n+1}),\\ & d(y_{n+1},y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2}),\\ & \{d(y_n,y_{n+2}).\varphi(y_n).\varphi(y_{n+2}).d(y_{n+1},y_{n+1}).\varphi(y_{n+1}).\varphi(y_{n+1})\}^{\frac{1}{2}}\}. \end{aligned}$$

Since

$$\begin{aligned} \{d(y_n,y_{n+2}).\varphi(y_n).\varphi(y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+1})\}^{\frac{1}{2}}\} \;\leq\;& \{d(y_n,y_{n+1}).d(y_{n+1},y_{n+2}).\\ & \varphi(y_n).\varphi(y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+1})\}^{\frac{1}{2}}\}\\ \;\leq\;& max\{d(y_n,y_{n+1}).\varphi(y_n).\varphi(y_{n+1}), d(y_{n+1},y_{n+2}).\\ & \varphi(y_{n+1}).\varphi(y_{n+2})\}, \end{aligned}$$

$$m(Sx, Sy, d, T, \varphi) = max\{d(y_n, y_{n+1}).\varphi(y_n).\varphi(y_{n+1}), d(y_{n+1}, y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2})\} \tag{4}$$

and

$$l(Sx, Sy, d, T, \varphi) = max\{d(y_n, y_{n+1}).\varphi(y_n).\varphi(y_{n+1}), d(y_{n+1}, y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2})\}. \tag{5}$$

Then by using (4) and (5), (1) becomes

$$\psi(d(Tx, Ty).\varphi(Tx).\varphi(Ty)) \leq \frac{\psi(max\{d(y_n, y_{n+1}).\varphi(y_n).\varphi(y_{n+1}), d(y_{n+1}, y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2})\}}{\phi(max\{d(y_n, y_{n+1}).\varphi(y_n).\varphi(y_{n+1}), d(y_{n+1}, y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2})\}} \tag{6}$$

Now suppose $d(y_n, y_{n+1}).\varphi(y_n).\varphi(y_{n+1}) < d(y_{n+1}, y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2})$, for some positive integer n. Then (6) becomes

$$\begin{aligned}\psi(d(y_{n+1}, y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2})) &\leq \frac{\psi(d(y_{n+1}, y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2}))}{\phi(d(y_{n+1}, y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2}))} \\ &< \psi(d(y_{n+1}, y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2})),\end{aligned}$$

which is a contradiction. Thus

$$d(y_{n+1}, y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2}) \leq d(y_n, y_{n+1}).\varphi(y_n).\varphi(y_{n+1}), \tag{7}$$

and (6) becomes

$$\psi(d(y_{n+1}, y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2})) \leq \frac{\psi(d(y_n, y_{n+1}).\varphi(y_n).\varphi(y_{n+1}))}{\phi(d(y_n, y_{n+1}).\varphi(y_n).\varphi(y_{n+1}))}. \tag{8}$$

Hence, the sequence $\{d(y_{n+1}, y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2})\}$ is monotone decreasing. Thus, there exists $r \geq 1$ such that

$$\lim_{n\to\infty}(d(y_{n+1}, y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2})) \to r. \tag{9}$$

Now we show $r = 1$. Assume $r > 1$. Letting $n \to \infty$ in (8), by the continuity of ψ and the lower semi-continuity of ϕ it follows that

$$\psi(r) \leq \frac{\psi(r)}{\lim_{n\to\infty} inf\,\phi(d(y_n, y_{n+1}).\varphi(y_n).\varphi(y_{n+1}))} \leq \frac{\psi(r)}{\phi(r)}.$$

This implies that $\phi(r) \leq \frac{\psi(r)}{\psi(r)} = 1$, which is a contradiction since $r > 1$, from property of ϕ. Hence, $r = 1$ and (9) becomes

$$\lim_{n\to\infty}(d(y_{n+1}, y_{n+2})) \longrightarrow 1, \lim_{n\to\infty}\varphi(y_{n+1}) \to 1, and \lim_{n\to\infty}\varphi(y_{n+2})) \to 1. \tag{10}$$

Now we prove that the sequence $\{y_n\}$ is a multiplicative Cauchy sequence. By using (10), it is sufficient to prove that $\{y_{2n}\}$ is a multiplicative Cauchy sequence. To prove this, suppose $\{y_{2n}\}$ is not a multiplicative Cauchy sequence, that is there exist $\epsilon > 1$ for which we can find two sequences of positive integers $2m(k)$ and $2n(k)$ such that for all positive integer k, $2n(k) > 2m(k) > k$,

$$d(y_{2m(k)}, y_{2n(k)}) \geq \epsilon \; and \; d(y_{2m(k)}, y_{2n(k-2)}) < \epsilon. \tag{11}$$

Now using the triangle inequality,
$\epsilon \leq d(y_{2m(k)}, y_{2n(k)}) \leq d(y_{2m(k)}, y_{2n(k)-2}).d(y_{2n(k)-2}, y_{2n(k)-1}).d(y_{2n(k)-1}, y_{2n(k)})$.
This implies that $\epsilon \leq d(y_{2m(k)}, y_{2n(k)}) < \epsilon.d(y_{2n(k)-2}, y_{2n(k)-1}).d(y_{2n(k)-1}, y_{2n(k)})$.
Letting $k \to \infty$ in the above inequalities and using (9), we have

$$\lim_{k\to\infty}(d(y_{2m(k)}, y_{2n(k)})) \leq \epsilon. \tag{12}$$

Now letting $k \to \infty$ in (11) and using (12), we have

$$\lim_{k\to\infty}(d(y_{2m(k)}, y_{2n(k)})) = \epsilon. \tag{13}$$

Again,

$$d(y_{2m(k)}, y_{2n(k)}) \leq d(y_{2m(k)}, y_{2m(k)+1}).d(y_{2m(k)+1}, y_{2n(k)+1}).d(y_{2n(k)+1}, y_{2n(k)})$$

and

$$d(y_{2m(k)+1}, y_{2n(k)+1}) \leq d(y_{2m(k)+1}, y_{2m(k)}).d(y_{2m(k)}, y_{2n(k)}).d(y_{2n(k)}, y_{2n(k)+1}).$$

Letting $k \to \infty$, using (10) and (13), we have

$$\lim_{k\to\infty}(d(y_{2m(k)+1}, y_{2n(k)+1})) = \epsilon. \tag{14}$$

Again,

$$\begin{aligned} d(y_{2n(k)+2}, y_{2m(k)+1}) &\leq d(y_{2n(k)+2}, y_{2n(k)+1}).d(y_{2n(k)+1}, y_{2m(k)+1}),\\ d(y_{2n(k)+1}, y_{2m(k)+1}) &\leq d(y_{2n(k)+1}, y_{2n(k)+2}).d(y_{2n(k)+2}, y_{2m(k)+1}) \end{aligned}$$

and similarly,

$$\begin{aligned} d(y_{2m(k)}, y_{2n(k)+1}) &\leq d(y_{2m(k)}, y_{2n(k)}).d(y_{2n(k)}, y_{2n(k)+1}),\\ d(y_{2m(k)}, y_{2n(k)}) &\leq d(y_{2m(k)}, y_{2n(k)+1}).d(y_{2n(k)+1}, y_{2n(k)}). \end{aligned}$$

Letting $k \to \infty$ in the above inequalities, using (10), (13) and (14), we have

$$\lim_{k\to\infty}(d(y_{2n(k)+2}, y_{2m(k)+1})) = \epsilon, \tag{15}$$

$$\lim_{k\to\infty}(d(y_{2m(k)}, y_{2n(k)+2})) = \epsilon, \tag{16}$$

$$\lim_{k\to\infty}(d(y_{2m(k)}, y_{2n(k)+1})) = \epsilon. \tag{17}$$

By setting $x = x_{2m(k)}$ and $y = x_{2n(k)+1}$ in (2), we have

$$\begin{aligned} m(Sx_{2m(k)}, Sx_{2n(k)+1}, d, T, \varphi) &= max\{d(Sx_{2m(k)}, Sx_{2n(k)+1}).\varphi(Sx_{2m(k)}).\varphi(Sx_{2n(k)+1}),\\ &\quad d(Sx_{2m(k)}, Tx_{2m(k)}).\varphi(Sx_{2m(k)}).\varphi(Tx_{2m(k)}),\\ &\quad d(Sx_{2n(k)+1}, Tx_{2n(k)+1}).\varphi(Sx_{2n(k)+1}).\varphi(Tx_{2n(k)+1}),\\ &\quad \{d(Sx_{2m(k)}, Tx_{2n(k)+1}).\varphi(y_{2m(k)}).\varphi(Ty_{2n(k)+1}).\\ &\quad d(Sx_{2n(k)+1}, Tx_{2m(k)}).\varphi(Sx_{2n(k)+1}).\varphi(Tx_{2m(k)})\}^{\frac{1}{2}}\}\\ &= max\{(d(y_{2m(k)}, y_{2n(k)+1}).\varphi(y_{2m(k)}).\varphi(y_{2n(k)+1}),\\ &\quad d(y_{2m(k)}, y_{2m(k)+1}).\varphi(y_{2m(k)}).\varphi(y_{2m(k)+1}),\\ &\quad d(y_{2n(k)+1}, y_{2m(k)+2}).\varphi(y_{2n(k)+1}).\varphi(y_{2m(k)+2}),\\ &\quad \{d(y_{2m(k)}, y_{2m(k)+2}).\varphi(y_{2m(k)}).\varphi(y_{2m(k)+2}).\\ &\quad d(y_{2n(k)+1}, y_{2m(k)+1}).\varphi(y_{2n(k)+1}).\varphi(y_{2m(k)+1})\}^{\frac{1}{2}}\} \end{aligned}$$

and

$$\begin{aligned} l(Sx_{2m(k)}, Sx_{2n(k)+1}, d, T, \varphi) &= max(d(Sx_{2m(k)}, Sx_{2n(k)+1}).\varphi(Sx_{2m(k)}.\varphi(Sx_{2n(k)+1}),\\ &\quad d(Sx_{2n(k)+1}, Tx_{2n(k)+1}).\varphi(Sx_{2n(k)+1}).\varphi(Tx_{2n(k)+1})\\ &= max(d(y_{2m(k)}, y_{2n(k)+1}).\varphi(y_{2m(k)}.\varphi(y_{2n(k)+1}),\\ &\quad d(y_{2n(k)+1}, y_{2n(k)+2}).\varphi(y_{2n(k)+1}).\varphi(y_{2n(k)+2}). \end{aligned}$$

Letting $k \to \infty$ in the above inequalities, using (9), and (13) - (17), we have

$$\lim_{k\to\infty}(m(Sx_{2m(k)}, Sx_{2n(k)+1}, d, T, \varphi)) = \epsilon \ and \ \lim_{k\to\infty}(l(Sx_{2m(k)}, Sx_{2n(k)+1}, d, T, \varphi)) = \epsilon. \tag{18}$$

Thus from (1), we have

$$\psi(d(Tx_{2m(k)}, Tx_{2n(k)+1}).\varphi(Tx_{2m(k)}).\varphi(Tx_{2n(k)+1})) \leq \frac{\psi(m(Sx_{2m(k)}, Sx_{2n(k)+1}, d, T, \varphi))}{\phi(l(Sx_{2m(k)}, Sx_{2n(k)+1}, d, T, \varphi))},$$

or

$$\psi(d(y_{2m(k)+1}, y_{2n(k)+2}).\varphi(y_{2m(k)+1}).\varphi(y_{2n(k)+2})) \leq \frac{\psi(m(Sx_{2m(k)}, Sx_{2n(k)+1}, d, T, \varphi))}{\phi(l(Sx_{2m(k)}, Sx_{2n(k)+1}, d, T, \varphi))}.$$

Letting $k \to \infty$, using (10), (15) and (18), applying continuity of ψ and lower semi-continuity of ϕ, we have

$$\psi(\varepsilon) \leq \frac{\psi(\varepsilon)}{\lim_{k\to\infty} inf \phi(l(Sx_{2m(k)}, Sx_{2n(k)+1}, d, T, \varphi))} \leq \frac{\psi(\varepsilon)}{\phi(\varepsilon)}.$$

This implies that,

$$\psi(\varepsilon) \leq \frac{\psi(\varepsilon)}{\phi(\varepsilon)} < \psi(\varepsilon),$$

which is a contradiction from property of ϕ.
Therefore $\{y_{2n}\}$ is a multiplicative Cauchy sequence. Hence by (10), $\{y_n\}$ is a multiplicative Cauchy sequence. Now since $S(X)$ is a complete subspace of X, it has multiplicative convergent subsequence of $\{y_n\}$. That is, there exists $p \in X$ such that

$$Sp = z. \tag{19}$$

As $\{y_n\}$ is a multiplicative Cauchy sequence containing a convergent multiplicative subsequence, therefore the sequence $\{y_n\}$ also converges to $z \in X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z. \tag{20}$$

Since φ is lower semi- continuous, $\varphi(z) \leq \lim_{n\to\infty} inf \varphi(y_n) = 1$. But $\varphi(z) \geq 1$, which implies that $\varphi(z) = 1$. Now we show $Tp = z$.
By setting $x = x_n$ and $y = p$ in (2), we have

$$\begin{aligned} m(Sx_n, Sp, d, T, \varphi) \quad = \quad & \max\{(d(Sx_n, Sp).\varphi(Sx_n).\varphi(Sp), d(Sx_n, Tx_n).\varphi(Sx_n).\varphi(Tx_n), \\ & d(Sp, Tp).\varphi(Sp).\varphi(Tp), \\ & \{d(Sx_n, Tp).\varphi(Sx_n).\varphi(Tp).d(Sp, Tx_n).\varphi(Sp).\varphi(Tx_n)\}^{\frac{1}{2}}\} \end{aligned}$$

and

$$l(Sx_n, Sp, d, T, \varphi) = \max\{d(Sx_n, Sp).\varphi(Sx_n).\varphi(Sp), d(Sp, Tp).\varphi(Sp).\varphi(Tp)\}.$$

Letting $n \to \infty$, using (20) and applying lower semi-continuty of φ, we have

$$\begin{aligned} \lim_{n\to\infty} m(Sx_n, Sp, d, T, \varphi) \quad = \quad & \max\{(d(z, Sp).\varphi(z).\varphi(Sp), d(z, z).\varphi(z).\varphi(z), \\ & d(Sp, Tp).\varphi(Sp).\varphi(Tp), \\ & \{d(z, Tp).\varphi(z).\varphi(Tp).d(Sp, z).\varphi(Sp).\varphi(z)\}^{\frac{1}{2}}\} \\ = \quad & d(z, Tp).\varphi(Tp) \end{aligned}$$

and

$$l(Sx_n, Sp, d, T, \varphi) = d(z, Tp).\varphi(Tp). \tag{21}$$

Then using (1), we have

$$\psi(d(Tx_n, Tp).\varphi(Tx_n).\varphi(Tp)) \leq \frac{\psi(m(Sx_n, Sp, d, T, \varphi))}{\phi(l(Sx_n, Sp, d, T, \varphi))}.$$

Letting $n \to \infty$, using (21) and by applying the continuity of ψ, the lower semi-continuity of ϕ, we have $\psi(d(z,Tp).\varphi(Tp)) \leq \frac{\psi(d(z,Tp).\varphi(Tp)}{\phi(d(z,Tp).\varphi(Tp)}$, which implies that $\phi(d(z,Tp).\varphi(Tp)) = 1$. Then from property of ϕ, we have

$$d(z,Tp).\varphi(Tp) = 1.$$

$$Hence, d(z,Tp) = 1 \ implies \ that \ z = Tp \ and \ \varphi(Tp) = 1. \tag{22}$$

Therefore, from (19) and (22), we have $Sp = Tp = z$.
Since S and T are weakly compatible, we have

$$STp = TSp = Sz = Tz. \tag{23}$$

Now we show $Tz = z$.
Again by setting $x = z$ and $y = x_n$ in (2), we have

$$\begin{aligned} m(Sz, Sx_n, d, T, \varphi) &= \max\{d(Sz, Sx_n).\varphi(Sz).\varphi(Sx_n), d(Sz, Tz).\varphi(Sz).\varphi(Tz), \\ &\quad d(Sx_n, Tx_n).\varphi(Sx_n).\varphi(Tx_n), \\ &\quad \{d(Sz, Tx_n).\varphi(Sz).\varphi(Tx_n).d(Sx_n, Tz).\varphi(Sx_n).\varphi(Tz)\}^{\frac{1}{2}}\}. \end{aligned}$$

and

$$l(Sz, Sx_n, d, T, \varphi) = max\{d(Sz, Sx_n).\varphi(Sz).\varphi(Sx_n), d(Sx_n, Tx_n).\varphi(Sx_n).\varphi(Tx_n)\}.$$

Letting $n \to \infty$, using (23) and applying lower semi-continuity of φ, we have

$$\begin{aligned} m(Tz, z, d, T, \varphi) &= \max\{(d(Tz, z).\varphi(Tz).\varphi(z), d(Tz, Tz).\varphi(Tz).\varphi(Tz), \\ &\quad d(z, z).\varphi(z).\varphi(z), \\ &\quad \{d(Tz, z).\varphi(Tz).\varphi(z).d(z, Tz).\varphi(z).\varphi(Tz)\}^{\frac{1}{2}}\} \\ &= d(Tz, z).\varphi(Tz). \end{aligned}$$

and

$$l(Tz, z, d, T, \varphi) = d(Tz, z).\varphi(Tz). \tag{24}$$

Then using (1), we have $\psi(d(Tz, Tx_n).\varphi(Tz).\varphi(Tx_n)) \leq \frac{\psi(m(Sz, Sx_n, d, T, \varphi))}{\phi(l(Sz, Sx_n, d, T, \varphi))}$. Letting $n \to \infty$, using (23), (24) and applying lower semi-continuity of φ, we have

$$\psi(d(Tz, z).\varphi(Tz)) \leq \frac{\psi(d(Tz, z).\varphi(Tz))}{\phi(d(Tz, z).\varphi(Tz))}.$$

Which implies that $\phi(d(Tz, z).\varphi(Tz)) = 1$. Then from property of ϕ, we have
$d(Tz, z) = 1$ and hence, $Tz = z$. Therefore z is a fixed point of T. Using (23), $Tz = z = Sz$.
Hence z is a common fixed point of T and S.

Uniqueness.
Suppose there is another common fixed point of T and S say u with $Tu = u$ and $Su = u$. Setting $x = z$ and $y = u$ in (2) and applying semi-continuity of φ, we have

$$\begin{aligned} m(Sz, Su, d, T, \varphi) &= \max\{d(Sz, Su).\varphi(Sz).\varphi(Su), d(Sz, Tz).\varphi(Sz).\varphi(Tz), \\ &\quad d(Su, Tu).\varphi(Su).\varphi(Tu), \\ &\quad \{d(Sz, Tu).\varphi(Sz).\varphi(Tu).d(Su, Tz).\varphi(Su).\varphi(Tz)\}^{\frac{1}{2}}\} \\ &= \max\{(d(z, u).\varphi(z).\varphi(u), d(z, z).\varphi(z).\varphi(z), \\ &\quad d(u, u).\varphi(u).\varphi(u), \\ &\quad \{d(z, u).\varphi(z).\varphi(u).d(u, z).\varphi(u).\varphi(z)\}^{\frac{1}{2}}\} \\ &= d(z, u). \end{aligned}$$

and

$$\begin{aligned} l(Sz,Su,d,T,\varphi) &= \max\{d(Sz,Su).\varphi(Sz).\varphi(Su), d(Su,Tu).\varphi(Su).\varphi(Tu)\} \\ &= \max\{d(z,u).\varphi(z).\varphi(u), d(u,u).\varphi(u).\varphi(u)\} \\ &= d(z,u). \end{aligned}$$

Using (1), we have

$$\begin{aligned} \psi(d(Tz,Tu).\varphi(Tz).\varphi(Tu)) \le \psi(d(z,u).\varphi(z).\varphi(u)) &\le \frac{\psi(m(Sz,Su,d,T,\varphi))}{\phi(l(Sz,Su,d,T,\varphi))} \\ &\le \frac{\psi(d(z,u))}{\phi(d(z,u))}. \end{aligned}$$

By applying lower semi-continuity of φ to the left side, we have

$$\psi(d(z,u)) \le \frac{\psi(d(z,u))}{\phi(d(z,u))}.$$

Which implies that $\phi(d(z,u)) = 1$. Then from property of ϕ, we have $d(z,u) = 1$ and hence, $z = u$. Therefore, T and S have a unique common fixed point z. □

The following is an example in support of our main result.

Example 3.3. *Let $X = [1,\infty)$ with the usual multiplicative metric d. Define S and $T : X \longrightarrow X$ by*

$$S(x) = \begin{cases} x & if \ 1 \le x \le 5; \\ 12 & if \ x > 5; \end{cases}$$

and

$$T(x) = \begin{cases} \sqrt{x} & if \ 1 \le x \le 5 \\ 5 & if \ x > 5 \end{cases}$$

for all $x \in X$. Let $\phi, \psi : [1,\infty) \longrightarrow [1,\infty)$ defined by $\psi(t) = t^2$ for $t \in [1,\infty)$,

$$\varphi(t) = \begin{cases} 2t & if \ t > 5 \\ t & if \ t \le 5 \end{cases}$$

and

$$\phi(t) = \begin{cases} t^{\frac{1}{2}} & if \ t > 5 \\ 1 & if \ t \le 5. \end{cases}$$

Now we show condition (1) as follows.
Case 1: *Let $x, y \in (5,\infty)$. Then*

$$\begin{aligned} (i).\ \psi(d(Tx,Ty).\varphi(Tx).\varphi(Ty)) &= \psi(d(5,5).\varphi(5).\varphi(5)) \\ &= \psi\left(\left(\left|\frac{5}{5}\right|^*\right).10.10\right) \\ &= \psi(100) = 10000. \\ (ii).\ \ d(Sx,Sy).\varphi(Sx).\varphi(Sy) &= d(12,12).\varphi(12).\varphi(12) \\ &= \left|\frac{12}{12}\right|^* .24.24 = 576. \\ (iii).\ \ d(Sx,Tx).\varphi(Sx).\varphi(Tx) &= d(12,5).\varphi(12).\varphi(5) \\ &= \left|\frac{12}{5}\right|^* .24.10 = 576. \end{aligned}$$

Similarly,

$$(iv).\quad d(Sy,Ty).\varphi(Sy).\varphi(Ty) = 576.$$
$$(v).\quad (d(Sx,Ty).\varphi(Sx).\varphi(Ty).d(Sy,Tx).\varphi(Sy).\varphi(Tx))^{\frac{1}{2}} = 576.$$

Thus, using $(ii),(iii),(iv)$ and (v), (2) becomes

$$m(Sx,Sy,d,T,\varphi) = \max\{576,576,576,576\} = 576$$

and

$$l(Sx,Sy,d,T,\varphi) = 576. \tag{25}$$

Hence, using (i) and (25), (1) becomes $10,000 \leq \frac{\psi(576)}{\phi(576)} \leq 13,824.$

Case 2. *Let $x,y \in [1,5]$ with $x \geq y$. Then*

$$\begin{aligned}
(i).\ \psi(d(Tx,Ty).\varphi(Tx).\varphi(Ty)) &= \psi\left(d\left(x^{\frac{1}{2}},y^{\frac{1}{2}}\right).\varphi\left(x^{\frac{1}{2}}\right).\varphi\left(y^{\frac{1}{2}}\right)\right)\\
&= \psi\left(\left|\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}\right|^{*}.x^{\frac{1}{2}}.y^{\frac{1}{2}}\right) = x^2.\\
(ii).\quad d(Sx,Sy).\varphi(Sx).\varphi(Sy) &= d(x,y).\varphi(x).\varphi(y)\\
&= \left|\frac{x}{y}\right|^{*}.x.y = x^2.\\
(iii).\quad d(Sx,Tx).\varphi(Sx).\varphi(Tx) &= d(x,\sqrt{x}).\varphi(x).\varphi(\sqrt{x})\\
&= \left|\frac{x}{\sqrt{x}}\right|^{*}.x.\sqrt{x} = x^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(iv).\quad d(Sy,Ty).\varphi(Sy).\varphi(Ty) &= y^2\\
(v).\quad (d(Sx,Ty).\varphi(Sx).\varphi(Ty).d(Sy,Tx).\varphi(Sy).\varphi(Tx))^{\frac{1}{2}} &= xy.
\end{aligned}$$

Hence, (1) becomes

$$1 \leq x^2, for\ x \geq y,\ where\ equality\ holds\ for\ x = 1$$

and

$$1 \leq y^2, for\ x < y.$$

Case 3. Let $x \in (5,\infty)$ and $y \in [1,5]$.

$$\begin{aligned}
(i).\ \psi(d(Tx,Ty).\varphi(Tx).\varphi(Ty)) &= \psi(d(5,y^{\frac{1}{2}}).\varphi(5).\varphi(y^{\frac{1}{2}}))\\
&= \psi\left(\left|\frac{5}{y^{\frac{1}{2}}}\right|^{*}.5.y^{\frac{1}{2}}\right) = 625.\\
(ii).\quad d(Sx,Sy).\varphi(Sx).\varphi(Sy) &= d(12,y).\varphi(12).\varphi(y)\\
&= \left|\frac{12}{y}\right|^{*}.24.y = 288.\\
(iii).\quad d(Sx,Tx).\varphi(Sx).\varphi(Tx) &= d(12,5).\varphi(12).\varphi(5)\\
&= \left|\frac{12}{5}\right|^{*}.24.5 = 288.\\
(iv).\quad d(Sy,Ty).\varphi(Sy).\varphi(Ty) &= d(y,\sqrt{y}).\varphi(y).\varphi(\sqrt{y})\\
&= \left|\frac{y}{\sqrt{y}}\right|^{*}.y.\sqrt{y} = y^2.
\end{aligned}$$

Similarly,

$$(v).\quad (d(Sx,Ty).\varphi(Sx).\varphi(Ty).d(Sy,Tx).\varphi(Sy).\varphi(Tx)))^{\frac{1}{2}} = 60\sqrt{2}$$

Here,

$$m(Sx,Sy,d,T,\varphi) = max\{288, y^2, 288, 60\sqrt{2}\} = 288$$

and

$$l(Sx,Sy,d,T,\varphi) = \max\{288, 288\} = 288. \tag{26}$$

Hence, using (i) and (26), (1) becomes $625 \leq 4887.68$.

Case 4. Let $y \in (5,\infty)$ *and* $x \in [1,5]$.

$$\begin{aligned}
(i).\ \psi(d(Tx,Ty).\varphi(Tx).\varphi(Ty)) &= \psi(d(\sqrt{x},5).\varphi(\sqrt{x}).\varphi(5)) \\
&= \psi(\left|\frac{\sqrt{x}}{5}\right|^{*}.\sqrt{x}.5) = 625. \\
(ii).\quad d(Sx,Sy).\varphi(Sx).\varphi(Sy) &= d(x,12).\varphi(x).\varphi(12) \\
&= \left|\frac{x}{12}\right|^{*}.x.24 = 288. \\
(iii).\quad d(Sx,Tx).\varphi(Sx).\varphi(Tx) &= d(x,\sqrt{x}).\varphi(x).\varphi(\sqrt{x}) \\
&= \left|\frac{x}{\sqrt{x}}\right|^{*}.x.\sqrt{x} = x^2. \\
(iv).\quad d(Sy,Ty).\varphi(Sy).\varphi(Ty) &= d(12,5).\varphi(12).\varphi(5) \\
&= \left|\frac{12}{5}\right|^{*}.24.5 = 288.
\end{aligned}$$

Similarly,

$$(v).\quad (d(Sx,Ty).\varphi(Sx).\varphi(Ty).d(Sy,Tx).\varphi(Sy).\varphi(Tx)))^{\frac{1}{2}} = 60\sqrt{2}$$

Here,

$$m(Sx,Sy,d,T,\varphi) = \max\{288, x^2, 288, 60\sqrt{2}\} = 288$$

and

$$l(Sx,Sy,d,T,\varphi) = \max\{288, 288\} = 288. \tag{27}$$

Hence, using (i) and (27), (1) becomes

$$625 \leq 4887.68.$$

Therefore, condition (1) is satisfied.
Next $Sx = Tx$ *at* $x = 1$ *and* $STx = TSx = 1$. *This shows that* S *and* T *are weakly compatible. Again* $T(X) \subseteq S(X)$.
Thus all conditions of the Theorem 3.2. are satisfied and $x = 1$ *is a unique common fixed point of* S *and* T.

4. Conclusion

In this paper, we have discussed the historical back ground of multiplicative calculus with its applications in different fields and simplicity of its operation. Next, we have explored the properties of multiplicative metric spaces with its some of topological spaces, development of contraction and weak contraction in multiplicative metric spaces and also the independence of metric spaces and multiplicative metric spaces has been discussed. We introduced generalized weakly contractive mappings, establish a common fixed point theorem for the mappings introduced and prove the existence and uniqueness a common fixed point result in the setting of multiplicative metric spaces. We have supported the result of this work by example.

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