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Convergence theorems of modified Ishikawa iterations in Banach spaces

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Abstract

In this paper, we introduce the modified iterations of Ishikawa type for nonexpansive mappings (nonexpansive semigroups) to have the strong convergence in a uniformlyconvex Banach space. We study approximation of common fixed point of nonexpansive mappings and nonexpansive semigroups in Banach space by using a new iterative scheme.

Keywords: strong convergence; modified Ishikawa iteration; uniformly convex Banach space; nonexpansive mapping; nonexpansive semigroup.
2010 MSC: 47H09, 47H10, 47J25.

1. Introduction

Let X be a real (or complex) space, if for every $x \in E$, there exists a certain real number, remark as $\|x\|$ with the corresponding to it, and satisfy the following conditions:(i) $\|x\| \geq 0$, $\|x\| = 0$ if and only if $\|x\| = 0$; (ii) $\|\alpha x\| = |\alpha|\|x\|$, where α is a arbitrarily real (or complex) number; (iii) $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in X$. Then, $\|x\|$ is said to be a norm for x and $(X, \|\cdot\|)$ is said to be a normed linear space. A complete normed linear space is a Banach space.

A normed space X is called uniformly convex if for any $\varepsilon \in (0, 2]$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in X$ with $\|x\| = 1$, $\|y\| = 1$ and $\|x - y\| \geq \varepsilon$, then $\|\frac{1}{2}(x + y)\| \leq 1 - \delta$.

Let E be a real Banach space, C a nonempty closed convex subset of E, and $T : C \to C$ a mapping. Recall that T is *nonexpansive mapping*[1] if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Denote by $Fix(T)$ the set of fixed point of T, that is, $F(T) = \{x \in C : Tx = x\}$.

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Iterative methods are often used to solve the fixed point equation $Tx = x$. One classical iteration process was introduced in 1953 by Mann[2] known as Mann iteration process and is defined as follows:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, n \geq 0, \tag{1.1}$$

where the sequence $\{\alpha_n\}$ is chosen in $(0,1)$ and the initial guess $x_0 \in C$ is arbitrarily chosen.

Based on this, other classical iteration process is introduced in 1974 by Ishikawa[3] known as Ishikawa's iteration process and was defined as follows:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)Ty_n, n \geq 0, \end{cases} \tag{1.2}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$ are chosen in $(0,1)$ and the initial guess $x_0 \in C$ is arbitrarily chosen.

There exists a rich literature on the convergence of Ishikawa's iteration for different classes of operators considered on various spaces(see:[[4],[5],[6],[7],[8]]).

In 2013, Chen and Wu[9] introduced the following iteration process of nonexpansive mapping in uniformly convex Banach spaces as follows:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ x_{n+1} = \beta_n u + (1-\beta_n)y_n, n \geq 0, \end{cases}$$

where $u \in C$ is an arbitrary fixed point in C. They obtained some strong convergence theorems.

In fact, after Mann's iteration, Ishikawa's iteration, and Halpern's iteration, kinds of iterative schemes also have been proposed and further developed in recent years. In 2000, M.A.Noor introduced a new iterative scheme and named it as Noor-type iteration(For detail study one may see the paper[10]). In 2011, W.Phuengrattana and S.Suantai introduced a new iterative scheme and named it as SP-type iteration(The detailed content is in [11]).

Motivated and inspired by the research going in these fields, we now suggest and analyze a new modified Ishikawa's iteration for finding the common fixed point of the nonexpansive mappings in Banach space as follows. We suggest and analyze the following iteration:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ z_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ y_n = \beta_n Tx_n + (1-\beta_n)Sz_n, \\ x_{n+1} = \delta_n u + (1-\delta_n)y_n, \ n \geq 0, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$ are three sequences in $[0,1]$, T, S are two nonexpansive mappings.

In 1887, J.Napier studied the function Equation $f(x+y) = f(x)f(y)$, and G.Peano simplified the system of first-order linear differential equations to a simple form as follows:

$$\frac{d}{dt}u(t) = Au(t).$$

From here, mathematical researchers found the development trace of semigroup theory and semigroup theory have been widely developed (more details see [[12],[13],[14],[15]]).

A family $\{T(s) : s \geq 0\}$ of mapping of C into itself is called *nonexpansive semigroup* of C, if it satisfies the following conditions:

(1) $T(s_1 + s_2)x = T(s_1)T(s_2)x$ for each $s_1,\ s_2 \geq 0$ and $x \in C$;
(2) $T(0)x = x$ for each $x \in C$;
(3) for each $x \in C, s \to T(s)x$ is continuous;
(4) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for each $s \geq 0$ and $x, y \in C$.

Based on the above, we make use of the notion of nonexpansive semigroup to modify the iterative scheme (1.3) as follows:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ z_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(u)x_n du, \\ y_n = \beta_n \frac{1}{t_n}\int_0^{e_n} T(u)x_n du + (1-\beta_n)\frac{1}{e_n}\int_0^{e_n} S(v)z_n dv, \\ x_{n+1} = \delta_n u + (1-\delta_n)y_n, \ n \geq 0, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$ are three sequences in $[0,1]$, $\{T(u) : u \geq 0\}$ and $\{S(v) : v \geq 0\}$ are two nonexpansive semigroups, $\{t_n\}, \{e_n\}$ are two positive real divergent sequences.

It is the purpose of this paper to develop iterations (1.7) (1.8) in [9] to the processes for two nonexpansive mappings, two nonexpansive semigroups in the frame of uniformly convex Banach space in Section 3.

2. Preliminaries

This section collects some lemmas which will be used in the proofs for the main results in the next section.

Lemma 1. *[16]* Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, n \geq 1.$$

If $\sum\limits_{n=1}^{\infty} \delta_n < \infty$ and $\sum\limits_{n=1}^{\infty} b_n < \infty$, then
(1) $\lim\limits_{n\to\infty} a_n$ exists;
(2) $\lim\limits_{n\to\infty} a_n = 0$ whenever $\liminf\limits_{n\to\infty} a_n = 0$.

Lemma 2. *[17]* Suppose that E is a uniformly convex Banach space and $0 < t_n < 1$ for all $n \in N$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\limsup\limits_{n\to\infty} \|x_n\| \leq r$, $\limsup\limits_{n\to\infty} \|y_n\| \leq r$ and $\lim\limits_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$, then $\lim\limits_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 3. *[18]* A mapping T:C$\to C$ with nonempty fixed point set F in C will be said to satisfy Condition (I): If there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x,F))$ *for all* $x \in C$, *where* $d(x,F) = \inf\{\|x-p\| : p \in F\}$.

Lemma 4. *[19] Let C be a nonempty closed convex subset of a uniformly convex Banach space E, D a bounded closed convex subset of C and $\Im = \{T(t) : t \geq 0\}$ a nonexpansive semigroup on C, such that $Fix(\Im) \neq \emptyset$. For each $h \geq 0$,then*

$$\lim_{t\to\infty} \sup_{x\in D} \|\frac{1}{t}\int_0^t T(u)xdu - T(h)\frac{1}{t}\int_0^t T(u)xdu\|.$$

3. Main results

In this part, we prove our main theorems for finding a common fixed point of nonexpansive mappings and nonexpansive semigroups in Banach spaces.

Theorem 1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let T, S be two nonexpansive commuting mappings of C satisfy Condition (I) and $F(T) \cap F(S) \neq \emptyset$. Given $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequences in (0,1) such that $\sum\limits_{n=1}^{\infty} \beta_n < \infty$, $\sum\limits_{n=1}^{\infty} \delta_n < \infty$ for all $n \geq 1$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by the algorithm (1.3), then $\{x_n\}_{n=0}^{\infty}$ strongly converges to a common fixed point of T, S.*

Proof. The first step, we need to show that $\{x_n\}$ is bounded, if we take an arbitrary fixed point q of $F(T)\cap F(S)$, we have

$$\begin{aligned}\|z_n-q\| &= \|\alpha_n x_n+(1-\alpha_n)Tx_n-q\|\\ &\le \alpha_n\|x_n-q\|+(1-\alpha_n)\|Tx_n-q\|\\ &\le \|x_n-q\|\end{aligned}$$

and

$$\begin{aligned}\|y_n-q\| &= \|\beta_n Tx_n+(1-\beta_n)Sz_n-q\|\\ &\le \beta_n\|Tx_n-q\|+(1-\beta_n)\|Sz_n-q\|\\ &\le \beta_n\|x_n-q\|+(1-\beta_n)\|z_n-q\|\\ &\le \|x_n-q\|.\end{aligned}$$

Then, we obtain that

$$\begin{aligned}\|x_{n+1}-q\| &= \|\delta_n u+(1-\delta_n)y_n-q\|\\ &\le \delta_n\|u-y_n\|+\|y_n-q\|\\ &\le \delta_n(\|u-q\|+\|y_n-q\|)+\|y_n-q\|\\ &\le (1+\delta_n)\|x_n-q\|+\delta_n\|u-q\|.\end{aligned} \tag{3.1}$$

By Lemma 1 and $\lim_{n\to\infty}\delta_n<\infty$, thus $\lim_{n\to\infty}\|x_n-q\|$ exists. Denote

$$\lim_{n\to\infty}\|x_n-q\|=l.$$

Hence, $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$.

The second step is to show that

$$\lim_{n\to\infty}=\|Tx_n-x_n\|=0,\ \lim_{n\to\infty}=\|Sx_n-x_n\|=0.$$

Now, in fact, we can show that

$$\begin{aligned}\|x_{n+1}-q\| &= \|\delta_n u+(1-\delta_n)y_n-q\|\\ &= \|\delta_n(u-y_n)+\|y_n-q\|\\ &\le \delta_n\|u-q\|+\|y_n-q\|.\end{aligned} \tag{3.2}$$

Owing to $\sum_{n=1}^{\infty}\delta_n<\infty$, we have

$$\lim_{n\to\infty}\|x_n-q\|\le\lim_{n\to\infty}\inf\|y_n-q\|. \tag{3.3}$$

In addition, by $\sum_{n=1}^{\infty}\beta_n<\infty$, from the following

$$\|z_n-q\|\ge\frac{1}{1-\beta_n}[\|y_n-q\|-\beta_n\|x_n-q\|],$$

this implies

$$\lim_{n\to\infty}\|z_n-q\|\ge\lim_{n\to\infty}\sup\{\frac{1}{1-\beta_n}[\|y_n-q\|-\beta_n\|x_n-q\|]\}. \tag{3.4}$$

Because $\|z_n-q\|\le\|x_n-q\|$, $\|y_n-q\|\le\|x_n-q\|$, then we have

$$\lim_{n\to\infty}\sup\|z_n-q\|\le\lim_{n\to\infty}\|x_n-q\| \tag{3.5}$$

and

$$\lim_{n\to\infty}\sup\|y_n-q\|\le\lim_{n\to\infty}\|x_n-q\|. \tag{3.6}$$

Thus, by (3.5)(3.6) this shows that

$$\lim_{n\to\infty} \|y_n - q\| = \lim_{n\to\infty} \|x_n - q\| = l, \tag{3.7}$$

and by (3.4)(3.5)(3.7) we obtain

$$\lim_{n\to\infty} \|z_n - q\| = \lim_{n\to\infty} \|x_n - q\| = l. \tag{3.8}$$

By $\|Tx_n - q\| \le \|x_n - q\|$, it implies that

$$\lim_{n\to\infty} \sup \|Tx_n - q\| \le l,$$

and through $\|Sx_n - q\| \le \|x_n - q\|$ we have

$$\lim_{n\to\infty} \sup \|Sx_n - q\| \le l.$$

Therefore, by (3.8) we have

$$\begin{aligned} l &= \lim_{n\to\infty} \|z_n - q\| \\ &= \lim_{n\to\infty} \|\alpha_n x_n + (1-\alpha_n)Tx_n - q\| \\ &= \lim_{n\to\infty} \|\alpha_n(x_n - q) + (1-\alpha_n)(Tx_n - q)\|. \end{aligned}$$

By Lemma 2, this shows that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{3.9}$$

By $\|Sz_n - q\| \le \|z_n - q\| \le \|x_n - q\| \le l$, then we have $\|Sz_n - q\| \le l$, thus by Lemma 2 this shows that

$$\begin{aligned} l &= \lim_{n\to\infty} \|y_n - q\| \\ &= \lim_{n\to\infty} \|\beta_n Tx_n + (1-\beta_n)Sz_n - q\| \\ &= \lim_{n\to\infty} \|\beta_n(Tx_n - q) + (1-\beta_n)(Sz_n - q)\|, \end{aligned}$$

that is

$$\lim_{n\to\infty} \|Tx_n - Sz_n\| = 0. \tag{3.10}$$

In fact, we can obtain that

$$\begin{aligned} \lim_{n\to\infty} \|z_n - x_n\| &= \lim_{n\to\infty} \|\alpha_n x_n + (1-\alpha_n)Tx_n - x_n\| \\ &= \lim_{n\to\infty} \|(1-\alpha_n)Tx_n - (1-\alpha_n)x_n\| \\ &\le \lim_{n\to\infty} (1-\alpha_n)\|Tx_n - x_n\| \to 0. \end{aligned} \tag{3.11}$$

Now, by (3.9)(3.10)(3.11) we have

$$\begin{aligned} \lim_{n\to\infty} \|Sx_n - x_n\| &\le \lim_{n\to\infty} \{\|Sx_n - Sz_n\| + \|Sz_n - Tx_n\| + \|Tx_n - x_n\|\} \\ &\le \lim_{n\to\infty} \{\|x_n - z_n\| + \|Sz_n - Tx_n\| + \|Tx_n - x_n\|\} \to 0. \end{aligned} \tag{3.12}$$

Thus, by (3.9) and (3.12), we obtain that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0, \ \lim_{n\to\infty} \|Sx_n - x_n\| = 0. \tag{3.13}$$

Moreover, given by (3.1) and $\sum_{n=1}^{\infty} \delta_n < \infty$ shows

$$\begin{aligned}\|x_{n+m} - q\| &= \|\delta_{n+m-1}u + (1-\delta_{n+m-1})y_n - q\| \\ &\le (1+\delta_{n+m-1})\|x_{n+m-1} - q\| + \delta_{n+m-1}\|u-q\| \\ &\le e^{\delta_{n+m-1}}\|x_{n+m-1} - q\| + C_{n+m-1} \\ &\le e^{\delta_{n+m-1}}e^{\delta_{n+m-2}}\|x_{n+m-2} - q\| + e^{\delta_{n+m-1}}C_{n+m-2} + C_{n+m-1} \\ &\le e^{\delta_{n+m-1}}e^{\delta_{n+m-2}}\|x_{n+m-2} - q\| + e^{\delta_{n+m-1}}(C_{n+m-2} + C_{n+m-1}) \\ &\le \cdots\cdots \\ &\le e^{\sum_{i=n}^{n+m-1}\delta_i}\|x_n - q\| + e^{\sum_{i=n}^{n+m-1}\delta_i}(\sum_{i=n}^{n+m-1} C_i) \\ &\le M(\|x_n - q\| + \sum_{i=1}^{\infty} C_i),\end{aligned}$$

that is

$$\|x_{n+m} - q\| \le M(\|x_n - q\| + \sum_{i=1}^{\infty} C_i), \tag{3.14}$$

where $M = \sum_{i=n}^{n+m-1} \delta_i,\ C_i = \delta_i\|u_n - q\|$.

The third step is to show that the sequence $\{x_n\}$ is cauchy.

Since $q \in F(T) \cap F(S)$ arbitrarily and $\lim_{n\to\infty} \|x_n - q\|$ exists, let $F = F(T) \cap F(S)$, then by Lemma 3 knowing that $d(x_n, F)$ exists. From Lemma 3 and (3.13), we get

$$\lim_{n\to\infty} f(d(x_n, F)) \le \lim_{n\to\infty} \|Tx_n - x_n\| = 0$$

and

$$\lim_{n\to\infty} f(d(x_n, F)) \le \lim_{n\to\infty} \|Sx_n - x_n\| = 0.$$

Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfy $f(0) = 0,\ f(r) > 0$ for all $r \in (0,\infty)$, therefore we have

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Let $\varepsilon > 0$ be given arbitrarily and since $\lim_{n\to\infty} d(x_n, F) = 0$, $\sum_{i=0}^{\infty} C_i < \infty$, therefore exists a constant n_0 such that for all $n \ge n_0$, we have

$$d(x_n, F) \le \frac{\varepsilon}{3M} \ and \ \sum_{j=n_0}^{\infty} C_j \le \frac{\varepsilon}{6M},$$

especially,

$$d(x_{n_0}, F) \le \frac{\varepsilon}{3M}.$$

So there must exists $p_1 \in F$ such that

$$d(x_{n_0}, p_1) \le \frac{\varepsilon}{3M}.$$

From (3.14), it can be obtained that when $n \ge n_{n_0}$,

$$\begin{aligned}\|x_{n+m} - x_n\| &\le \|x_{n+m} - p_1\| + \|x_n - p_1\| \\ &\le 2M(\|x_{n_0} - p_1\| + \sum_{j=n_0}^{n_0+m-1} C_j) \\ &\le 2M(\frac{\varepsilon}{3M} + \frac{\varepsilon}{6M}) = \varepsilon.\end{aligned}$$

This implies $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in a closed subset C of a Banach space E.

Thus, the fourth step is to show that the sequence $\{x_n\}_{i=0}^{\infty}$ converges to a point in C. Suppose that $\lim_{n\to\infty} x_n = p$.

For all $\varsigma > 0$, since $\lim_{n\to\infty} x_n = p$, thus there exists a number n_1 such that when $n_2 \geq n_1$, then

$$\|x_{n_2} - p\| \leq \frac{\varsigma}{4}. \tag{3.15}$$

In fact, $\lim_{n\to\infty} d(x_n, F) = 0$ implies that making use of the number n_2 above, when $n \geq n_2$, we have $d(x_n, F) \leq \frac{\varsigma}{8}$. Thus, there must exist $p_2 \in F$ such that

$$\|x_{n_2} - p_2\| = d(x_{n_2}, p_2) = \frac{\varsigma}{8}. \tag{3.16}$$

From (3.15) and (3.16), we get

$$\begin{aligned}
\|Tp - p\| &= \|Tp - p_2 + Tx_{n_2} - p_2 + p_2 - x_{n_2} + x_{n_2} - p + p_2 - Tx_{n_2}\| \\
&\leq \|Tp - p_2\| + \|x_{n_2} - p_2\| + \|x_{n_2} - p\| + 2\|Tx_{n_2} - p_2\| \\
&\leq \|p - p_2\| + 3\|x_{n_2} - p_2\| + +\|x_{n_2} - p\| \\
&\leq \|x_{n_2} - p\| + \|x_{n_2} - p_2\| + 3\|x_{n_2} - p_2\| + \|x_{n_2} - p\| \\
&= 4\|x_{n_2} - p_2\| + 2\|x_{n_2} - p\| \\
&\leq \frac{4\varsigma}{8} + \frac{2\varsigma}{4} = \varsigma.
\end{aligned} \tag{3.17}$$

On the other hand, we can use the similar way to show that

$$\|Sp - p\| \leq \varsigma.$$

Since the positive number ς is given arbitrarily, then $Tp = p$ and $Sp = p$. So, the iterative sequence $\{x_n\}$ converges strongly (in norm) to a common fixed point of T and S. This completes the proof. □

Theorem 2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $\Im_1 = \{T(t) : t \geq 0\}, \Im_2 = \{S(e) : e \geq 0\}$ be two nonexpansive semigroups on C satisfy Condition (I) and $\bigcap_{i=1}^{2} F(\Im_i) \neq \emptyset$. Given a point $u \in C$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequences in (0,1) such that $\sum\limits_{n=1}^{\infty} \beta_n < \infty$, $\sum\limits_{n=1}^{\infty} \delta_n < \infty$ for all $n \geq 1$, $\{t_n\}, \{e_n\}$ are two positive real divergent sequences. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by the algorithm (1.4), then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a common fixed point of $\bigcap_{i=1}^{2} F(\Im_i)$.*

Proof. The first step is to show that the sequence $\{x_n\}$ is bounded, if we take an arbitrary fixed point q of $\bigcap_{i=1}^{2} F(\Im_i)$, we have

$$\begin{aligned}
\|z_n - q\| &= \|\alpha_n x_n + (1 - \alpha_n)\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - q\| \\
&\leq \alpha_n\|x_n - q\| + (1 - \alpha_n)\|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - q\| \\
&\leq \alpha_n\|x_n - q\| + (1 - \alpha_n)\frac{1}{t_n}\int_0^{t_n} \|T(u)x_n - q\| du \\
&\leq \alpha_n\|x_n - q\| + (1 - \alpha_n)\frac{1}{t_n}\int_0^{t_n} \|x_n - q\| du \\
&= \alpha_n\|x_n - q\| + (1 - \alpha_n)\|x_n - q\| \\
&= \|x_n - q\|
\end{aligned}$$

and

$$\begin{aligned}
\|y_n - q\| &= \|\beta_n \frac{1}{t_n}\int_0^{e_n} T(u)x_n du + (1-\beta_n)\frac{1}{e_n}\int_0^{e_n} S(v)z_n dv - q\| \\
&\le \beta_n \|\frac{1}{t_n}\int_0^{e_n} T(u)x_n du - q\| + (1-\beta_n)\|\frac{1}{e_n}\int_0^{e_n} S(v)z_n dv - q\| \\
&\le \beta_n \frac{1}{t_n}\int_0^{e_n} \|T(u)x_n - q\| du + (1-\beta_n)\frac{1}{e_n}\int_0^{e_n} \|S(v)z_n - q\| dv \\
&\le \beta_n \frac{1}{t_n}\int_0^{e_n} \|x_n - q\| du + (1-\beta_n)\frac{1}{e_n}\int_0^{e_n} \|z_n - q\| dv \\
&= \beta_n\|x_n - q\| + (1-\beta_n)\|z_n - q\| \\
&\le \|x_n - q\|.
\end{aligned}$$

Then, we obtain that

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\delta_n u + (1-\delta_n)y_n - q\| \\
&\le \delta_n\|u - y_n\| + \|y_n - q\| \\
&\le \delta_n(\|u - q\| + \|y_n - q\|) + \|y_n - q\| \\
&\le (1+\delta_n)\|x_n - q\| + \delta_n\|u - q\|.
\end{aligned} \tag{3.18}$$

Now, an induction yields

$$\|x_n - q\| \le \max\{\|x_0 - q\|, \|u - q\|\}, \ n \ge 0.$$

Hence, $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$. Let D be the subset os C,

$$D = \{x \in C : \|x - q\| \le \max\{\|x_0 - q\|, \|u - q\|\}\}.$$

By Lemma 1 and $\sum_{n=1}^{\infty} \delta_n < \infty$, by (3.18) thus $\lim_{n\to\infty} \|x_n - q\|$ exists. Let

$$\lim_{n\to\infty} \|x_n - q\| = b.$$

The second step is to show that

$$\lim_{n\to\infty} = \|T(h)x_n - x_n\| = 0, \ \lim_{n\to\infty} = \|S(k)x_n - x_n\| = 0.$$

Now, in fact, we can show that

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\delta_n u + (1-\delta_n)y_n - q\| \\
&= \|\delta_n(u - y_n) + \|y_n - q\| \\
&\le \delta_n\|u - q\| + \|y_n - q\|.
\end{aligned} \tag{3.19}$$

By $\sum_{n=1}^{\infty} \delta_n < \infty$, by (3.19) we have

$$\lim_{n\to\infty} \|x_n - q\| \le \lim_{n\to\infty} \inf \|y_n - q\|. \tag{3.20}$$

Since $\|y_n - q\| \le \|x_n - q\|$, this implies that

$$\lim_{n\to\infty} \sup \|y_n - q\| \le \lim_{n\to\infty} \inf \|x_n - q\| = b. \tag{3.21}$$

At the same time, by virtue of $\|z_n - q\| \le \|x_n - q\|$, this implies that

$$\lim_{n\to\infty} \sup \|z_n - q\| \le \lim_{n\to\infty} \inf \|x_n - q\| = b. \tag{3.22}$$

Then, the above inequalities (3.20)(3.21) shows

$$\lim_{n\to\infty} \sup \|y_n - q\| = \lim_{n\to\infty} \|x_n - q\| = b. \tag{3.23}$$

In addition, by $\sum_{n=1}^{\infty} \beta_n < \infty$, from the following

$$\|z_n - q\| \geq \frac{1}{1-\beta_n}[\|y_n - q\| - \beta_n\|x_n - q\|],$$

this implies

$$\lim_{n\to\infty} \|z_n - q\| \geq \lim_{n\to\infty} \sup\{\frac{1}{1-\beta_n}[\|y_n - q\| - \beta_n\|x_n - q\|]\}.$$

So, by (3.23) we get

$$\lim_{n\to\infty} \|z_n - q\| \geq \lim_{n\to\infty} \|y_n - q\| = b. \tag{3.24}$$

Then, by (3.22) and (3.24)

$$\lim_{n\to\infty} \|z_n - q\| = \lim_{n\to\infty} \|x_n - q\| = b. \tag{3.25}$$

Moreover, $\|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - q\| \leq \|x_n - q\|$ implies that

$$\lim_{n\to\infty} \sup \|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - q\| \leq b.$$

Therefore, we have

$$\begin{aligned} l &= \lim_{n\to\infty} \|z_n - q\| \\ &= \lim_{n\to\infty} \|\alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - q\| \\ &= \lim_{n\to\infty} \|\alpha_n(x_n - q) + (1-\alpha_n)(\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - q)\|. \end{aligned}$$

So, given by Lemma 2 shows that

$$\lim_{n\to\infty} \|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - x_n\| = 0. \tag{3.26}$$

Now,

$$\begin{aligned} \|x_n - T(h)x_n\| &\leq \|x_n - \frac{1}{t_n}\int_0^{t_n} T(u)x_n du\| + \|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - T(h)\frac{1}{t_n}\int_0^{t_n} T(u)x_n du\| \\ &\quad + \|T(h)\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - T(h)x_n\| \\ &\leq 2\|x_n - \frac{1}{t_n}\int_0^{t_n} T(u)x_n du\| + \|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - T(h)\frac{1}{t_n}\int_0^{t_n} T(u)x_n du\|. \end{aligned} \tag{3.27}$$

By virtue of Lemma 4, we get

$$\lim_{n\to\infty} \sup_{x_n\in D} \|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - T(h)\frac{1}{t_n}\int_0^{t_n} T(u)x_n du\| = 0,$$

for every $h \in [0, +\infty)$. From (3.27), we obtain

$$\lim_{n\to\infty} \sup_{x_n\in D} \|x_n - T(h)x_n\| = 0, \tag{3.28}$$

for every $h \in [0, +\infty)$.

By $\|\frac{1}{e_n}\int_0^{e_n} S(v)z_n dv - q\| \le \|z_n - q\| \le \|x_n - q\| \le b$, then we have $\|\frac{1}{e_n}\int_0^{e_n} S(v)z_n dv - q\| \le l$, thus by Lemma 2 this shows that

$$\begin{aligned}
l &= \lim_{n\to\infty} \|y_n - q\| \\
&= \lim_{n\to\infty} \|\beta_n \frac{1}{t_n}\int_0^{t_n} T(u)x_n du - q + (1-\beta_n)\frac{1}{e_n}\int_0^{e_n} S(v)z_n dv - q\| \\
&= \lim_{n\to\infty} \|\beta_n(\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - q) + (1-\beta_n)(\frac{1}{e_n}\int_0^{e_n} S(v)z_n dv - q)\|,
\end{aligned}$$

that is

$$\lim_{n\to\infty} \|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - \frac{1}{e_n}\int_0^{e_n} S(v)z_n dv\| = 0. \tag{3.29}$$

In fact, by (3.26) and (3.29) we can obtain that

$$\begin{aligned}
\|\frac{1}{e_n}\int_0^{e_n} S(v)z_n dv - x_n\| &\le \|\frac{1}{e_n}\int_0^{e_n} S(v)z_n dv - \frac{1}{t_n}\int_0^{t_n} T(u)x_n du\| \\
&\quad + \|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - x_n\| \to 0 (n \to \infty)
\end{aligned}$$

and

$$\begin{aligned}
\|\frac{1}{e_n}\int_0^{e_n} S(v)x_n dv - x_n\| &\le \|\frac{1}{e_n}\int_0^{e_n} S(v)x_n dv - \frac{1}{e_n}\int_0^{e_n} S(v)z_n dv\| \\
&\quad + \|\frac{1}{e_n}\int_0^{e_n} S(v)z_n dv - \frac{1}{t_n}\int_0^{t_n} T(u)x_n du\| + \|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - x_n\| \\
&\le \|x_n - z_n\| + \|\frac{1}{e_n}\int_0^{e_n} S(v)z_n dv - \frac{1}{t_n}\int_0^{t_n} T(u)x_n du \\
&\quad + \|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - x_n\| \to 0.
\end{aligned}$$

Further we have

$$\begin{aligned}
\|S(k)\frac{1}{e_n}\int_0^{e_n} S(v)z_n dv - \frac{1}{t_n}\int_0^{t_n} T(u)x_n du\| &\le \|S(k)\frac{1}{e_n}\int_0^{e_n} S(v)z_n dv - \frac{1}{e_n}\int_0^{e_n} S(v)z_n dv\| \\
&\quad + \|\frac{1}{e_n}\int_0^{e_n} S(v)z_n dv - \frac{1}{t_n}\int_0^{t_n} T(u)x_n du\| \to 0.
\end{aligned}$$

From the above, we get

$$\begin{aligned}
\|S(k)x_n - x_n\| &\le \|S(k)x_n - S(k)\frac{1}{e_n}\int_0^{e_n} S(v)z_n dv\| + \|S(k)\frac{1}{e_n}\int_0^{e_n} S(v)z_n dv - \frac{1}{t_n}\int_0^{t_n} T(u)x_n du\| \\
&\quad + \|\frac{1}{t_n}\int_0^{t_n} T(u)x_n du - x_n\| \to 0.
\end{aligned}$$

Thus, we obtain that

$$\lim_{n\to\infty} \|T(h)x_n - x_n\| = 0, \ \lim_{n\to\infty} \|S(k)x_n - x_n\| = 0, for\ every\ h, k \in [0, \infty).$$

Since $\{T(t) : t \ge 0\}, \{S(e) : e \ge 0\}$ are two nonexpansive semigroups and $\{t_n\}, \{e_n\}$ are two positive real divergent sequences, then for all $h, k \ge 0$ and the bounded closed subset D of C containing $\{x_n\}$ such that

$$\lim_{n\to\infty} \|x_n - T(h)x_n\| \le \lim_{n\to\infty} \sup_{x_n \in D} \|x_n - T(h)x_n\| = 0$$

and

$$\lim_{n\to\infty} \|x_n - S(k)x_n\| \le \lim_{n\to\infty} \sup_{x_n\in D} \|x_n - S(k)x_n\| = 0.$$

As in the proof of Theorem 1, we have $\{x_n\}$ converges strongly to p ($p \in \bigcap_{i=1}^{2} F(\Im_i)$. This completes the proof. □

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