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A Study on Generalized Tubes

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Abstract: In this paper, we consider generalized tubes, which we refer to in the paper as hereafter GTs, according to q-frame in Euclidean space E^3. First, we give a parametric representation of directional generalized tubes (DGTs). Since GT class is divided by two important subclasses, we investigate geometric properties of these two classes with respect to the q-frame.

Keywords: Adapted frame, Frenet frame, Generalized tubes.

1 Introduction

A canal surface is introduced by the French mathematician Gaspard Monge in 1850. Canal surface can be defined as envelope of a nonparameter set of spheres, centered at a spine curve $\alpha(s)$ with radius $r(s)$. As a special case of such a surface is called a pipe surface or tubular surface at a constant function radius $r(s)$with spine curve $\alpha(s)$ [12]. Moreover, tubular surfaces are used in many applications such as CAGD, shape reconstruction, transition surfaces between pipes,robotic path planning etc [16]. A generalized tube (or GT) is the surface constructed by sweeping some planar closed curve along an arbitrary 3D space curve. One good reason to understand GTs are used in many man-made objects. It has a spine that is generally not straight, and includes some bending and twisting and maybe even some knots. Many cables, garden hoses,wires, poles, objects fit into this category of objects [8]. Horaud and Brady present a method for recovering the underlying GC such that a cross-section curve is found that extremizes compactness subject to an orthogonality constraint between the cross-section plane and the GC axis [10]. Recovery of surfaces of revolution, an important subclass of SHGCs, is considered by Richetin et al. in [13] and LaVest [11]. In Ulupinar and Nevatia [14], general definitions are given for both parallel and mirror symmetry. In Ulupinar [15], SHGC contours are combined with various heuristic constraints to recover 3D shape from contour.

Let $\alpha : (a, b) \to E^3$ be a curve that is parametrized by arc lenght s. Denote by $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ the moving Frenet frame along the unit speed curve $\alpha(s)$. A parameterization of the GT is written as in [8]

$$\psi(s, \theta) = \alpha(s) + r(\theta)\,(\mathbf{n}(s)\cos\theta + \mathbf{b}(s)\sin\theta) \tag{1}$$

where $\theta \in (0, 2\pi)$. For the GT to be a regular, well-defined surface, we impose the additional restrictions that r is twice differentiable, $r(\theta) > 0$ and $r(0) = r(2\pi)$.

Theorem 1.1. The directions of the parameter curves at a non-umbilical point on a patch are in the direction of the principal directions if and only if $F = M = 0$ at the point, where F and M are the respective first and second fundamental coefficients. At an umbilical point, every direction is a principal direction.

Theorem 1.2. All straight lines on a surface are geodesics. A curve not a straight line is a geodesic if and only if the osculating plane of the curve is perpendicular to the tangent plane to the surface at each point [6].

The tube surface can be parameterized using Frenet frame. However, this frame is undefined wherever the curvature vanishes, such as at points of inflection or along straight sections of the curve [3]. Therefore, new frames have been investigated as an alternative to Frenet frame such as Bishop (parallel transport), q-frames etc [1, 5]. The directional q-frame offers two key advantages over the Frenet frame The first, it is well defined even if the curve has vanishing second derivative and the second, it avoids the unnecessary twist around the tangent. Also, the directional q-frame is easier than the rotation minimizing frames, one of them is Bishop frame to calculate [5]. The q-frame of a regular curve $\alpha(t)$ as follows

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{n}_q = \frac{\mathbf{t} \wedge \mathbf{k}}{\|\mathbf{t} \wedge \mathbf{k}\|}, \mathbf{b}_q = \mathbf{t} \wedge \mathbf{n}_q \tag{2}$$

where $\mathbf{t}$ is the unit tangent vector, $\mathbf{n}_q$ is quasi-normal and $\mathbf{b}_q$ is quasi-binormal vector. Also, k is the projection vector and is usually chosen as $k = (0, 0, 1)$ [5]. The q-frame and Frenet frame along a space curve are shown in Figure 1.

The variation equations of the directional q-frame are given by

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \|\alpha'\| \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} \tag{3}$$

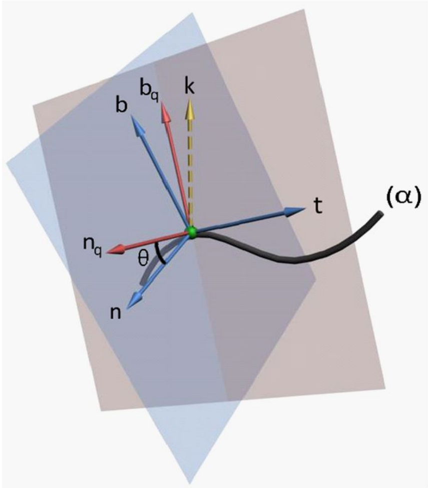

Fig. 1: The q-frame and Frenet frame.

where the q-curvatures are expressed as follows

$$k_1 = \frac{\langle \mathbf{t}', \mathbf{n}_q \rangle}{\|\alpha'\|}, k_2 = \frac{\langle \mathbf{t}', \mathbf{b}_q \rangle}{\|\alpha'\|}, k_3 = -\frac{\langle \mathbf{n}_q, \mathbf{b}_q' \rangle}{\|\alpha'\|}. \tag{4}$$

Dede et al. have been defined the q-frame for tubular surface modeling, called D-tubular surface and gave a parametric representation of these surface in Euclidean and Minkowski spaces[4, 7] The D-tubular surface, at a distance r from the spine curve $\alpha(s)$, may be represented as

$$\psi^r(s,v) = \alpha(s) + r(s)(\cos v\mathbf{n}_q + \sin v\mathbf{b}_q). \tag{5}$$

2 A study on generalized tubes

In this study, we consider generalized tubes, which we refer to in the paper as hereafter GTs, according to q-frame in E^3. we give a parametric representation of directional generalized tubes (DGTs).

The directional generalized tube (DGT) centered at a spine curve $\alpha(s)$ with radius $r(v) = r$ may be represented as

$$\psi^r(s,v) = \alpha(s) + r(v)\left(\cos(v)\,\mathbf{n_q} + \sin(v)\,\mathbf{b_q}\right) \tag{6}$$

where $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q, k\}$ is the q-frame of the spine curve $\alpha(s)$. Since the DGT should be a regular, well-defined surface, we take the additional restrictions that r is twice differentiable, $r(v) > 0$ and $r(0) = r(2\pi)$. The partial derivatives of $\psi^r(s,v)$, with respect to v and s, respectively, are determined by

$$\psi^r_v = (r'\cos v - r\sin v)\mathbf{n}_q + (r'\sin v - r\cos v)\mathbf{b}_q \tag{7}$$

and

$$\psi^r_s(s,v) = \begin{array}{c} (1 - r(k_1\cos(v) + k_2\sin(v)))\mathbf{t} \\ -rk_3\left(\sin(v)\,\mathbf{n_q} - \cos(v)\,\mathbf{b_q}\right) \end{array}. \tag{8}$$

It follows that the unit normal vector of the DGT is

$$U = \frac{1}{A}\left(\begin{array}{c} rr'k_3t + (1 - r(k_1\cos(v) + k_2\sin(v)))\cdot \\ \left[(r'\sin v + r\cos v)\mathbf{n}_q - (r'\cos v - r\sin v)\mathbf{b}_q\right] \end{array}\right) \tag{9}$$

where $A = \sqrt{(1 - r(k_1\cos(v) + k_2\sin(v)))^2(r^2 + r'^2) + r^2r'^2k_3^2}$.

From (7) and (8), the coefficients $E = \langle \psi^r_v, \psi^r_v \rangle$, $F = \langle \psi^r_s, \psi^r_v \rangle$ and $G = \langle \psi^r_s, \psi^r_s \rangle$ of the first fundamental form are calculated by

$$E = r'^2 + r^2,\ F = r^2k_3, \tag{10}$$

and

$$G = (1 - r(k_1\cos(v) + k_2\sin(v)))^2 + r^2k_3^2. \tag{11}$$

By using $L = \langle \psi^r_{vv}, U \rangle$, $M = \langle \psi^r_{vs}, U \rangle$ and $N = \langle \psi^r_{ss}, U \rangle$, the coefficients of second fundamental form are obtained as

$$L = \frac{1}{A}\left[(1 - r(k_1\cos(v) + k_2\sin(v)))\left(rr'' - r^2 - 2r'^2\right)\right] \tag{12}$$

$$M = -\frac{1}{A}k_3\left(\begin{array}{l} r^2(1 - r(k_1\cos(v) + k_2\sin(v))) \\ +r'\left(r^2(1 - r(k_1\sin(v) - k_2\cos(v))) + r'\right) \end{array}\right), \tag{13}$$

$$N=\frac{1}{A}\left[\begin{array}{c} k_3r^2r'(k_1k_3\sin v-k_1\cos v)- \\ (1-r(k_1\cos(v)+k_2\sin(v)))(r^2k_3^2+rr'k_3') \\ +k_2(1-r(k_1\cos(v)+k_2\sin(v)))^2\left(r'\sin v+r\cos v\right) \end{array}\right] \tag{14}$$

In this study, we consider according to the lines of curvature in terms of natural parametrization of DGTs. Thus, we check whether s and v parameter curves of a DGT are its lines of curvature.

Theorem 2.1. The parameter curves of a DGT are lines of curvature if and only if k_3 vanishes.

Proof: If the k_3 vanishes, then by substituting $k_3=0$ into the above expressions for M and F , we obtain $M=F=0$. To show the "only if" part of the theorem, assume the DGT parameter curves are lines of curvature. Then from the Theorem 1.1., we have $M=F=0$. But since $F=r^2k_3=0$ and by definition $u>0$, it follows that $k_3=0$.

Parameter curve is also a geodesic. Thus, we are interested in when DGT parameter curves are either lines of curvature or geodesics. We only determine for the DGT v- parameter curves.

Theorem 2.2. The non-linear cross-sections of a DGT are geodesics if and only if the DGT is either a ZDGT(zero gaussian directional generalized tube) or a CDGT(constant cross section directional generalized tubes).

Proof: The osculating plane of a DGT cross-section at some point $\alpha(s)$ on the DGT axis is always perpendicular to $t(s)$, the tangent at $\alpha(s)$, by construction. But then, for a point $\psi^r(s,v)$ on the DGT surface, the osculating plane of the cross-section curve is perpendicular to the tangent plane exactly when $t(s)$ is perpendicular to $U(s,v)$. Since $U_t=\langle U,t\rangle=rr'k_3=0$, $r>0$, it follows from the geodesic theorem that $U_t=0$ exactly when either $r'=0$ or $k_3=0$.

Theorem 2.3. The non-linear cross-sections of a DGT are lines of curvature if the DGT is either a ZDGT or a CDGT.

Proof: This follows directly from the fact that a geodesic is planar if and only if it is a line of curvature.

Corollary 2.1. There seems to be a relationship between the parametric curves and the set of intrinsic directions as defined for both ZDGTs and CDGTs. This relationship implies that, in some sense, they are "natural subclasses" of DGTs, and should be studied further with respect to their surface.

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