Detecting Similarities of Bézier Curves for the Groups $LSim(E_2)$, $LSim^+(E_2)$

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Abstract: In this paper, for linear similarity groups, global invariants of plane Bézier curves (plane polynomial curves) in $E_2$ are introduced. Using complex numbers and the global $G$-invariants of a plane Bézier curve, for given two plane Bézier curves (plane polynomial curves) $x(t)$ and $y(t)$, evident forms of all transformations $g \in G$, carrying $x(t)$ to $y(t)$, are obtained.

Keywords: Polynomial curve, Bézier curve, Invariant, Linear similarity group.

1 Introduction

Let $E_2$ be the 2-dimensional Euclidean space, $G = LSim(E_2)$ be the group of all linear similarities of $E_2$ and $G = LSim^+(E_2)$ be the group of all orientation-preserving linear similarities of $E_2$.

In [1], using local differential invariants and Frenet frames of two curves, uniqueness and existence theorems for a curve determined up to a direct similarity of $E_2$.

For the group $Sim^+(n)$, this theorem shows that a necessary and sufficient conditions for two curves in $E_n$ to be equivalent is that they have same shape curvatures and the other specially conditions.

The complete systems of global $G$-invariants of a path and a curve in $E_2$ are obtained. For the groups $G$, existence and uniqueness theorems for a curve and a path are given in the terms of global $G$-invariants of a path and a curve in [2].

$LSim(2)$-equivalence of two Bézier curves without using differential invariants of Bézier curves in the terms of control invariants of Bézier curves is proved in [3, 4].

In this work, starting from the ideas in [2–4, 8–11], we address how to compute explicitly an linear similarity transformation which carrying a Bézier curve into another Bézier curve in the terms of control invariants of a Bézier curve for the groups $LSim(E_2)$ and $LSim^+(E_2)$ without using differential invariants of Bézier curves.

2 Preliminaries

The following definitions and propositions are known in [2].

Let $B$ be the field of real numbers and $C$ be the field of complex numbers. The multiplication in $C$ has the form $(a_1+ia_2)(b_1+ib_2) = (a_1b_1-a_2b_2) + i(a_1b_2 + a_2b_1)$. We will consider element $a = a_1 + ia_2$ also in the form $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$. For $a = a_1 + ia_2$, denote by $P_a$ the matrix $\begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}$ and consider $P_a$ as the transformation $P_a : C \rightarrow C$, where $P_ab = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1b_1 - a_2b_2 \\ a_1b_2 + a_2b_1 \end{pmatrix}$ for all $b = b_1 + ib_2 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in C$. Then we have the equality

$$ab = P_ab,$$

for all $a, b \in C$. Let $P(C)$ denote the set of all matrices $P_a$, where $a \in C$. We consider on $P(C)$ the following standard matrix operations; the component-wise addition, a scalar multiplication and the multiplication of matrices. Then $P(C)$ is a field, where the unit element is the unit matrix. The following Propositions are known.

Proposition 1. The mapping $P : C \rightarrow P(C)$, where $P : a \rightarrow P_a$ for all $a \in C$, is an isomorphism of fields.

For vectors $a = a_1 + ia_2, b = b_1 + ib_2 \in C$, we put $< a, b > = a_1b_1 + a_2b_2$. Then $< a, b >$ is a bilinear form on $E_2$ and $< a, a > = a_1^2 + a_2^2$ is a quadratic form on $E_2$. Put $Q(a) = < a, a >$. We consider the field $C$ also as the two-dimensional Euclidean space $E_2$ with the scalar product $< a, b >$. Then $\|a\| = |a| = \sqrt{Q(a)}$, $\forall a \in C$. 

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Lemma 1. Let \( a = a_1 + ia_2 \in \mathbb{C} \). Then \( \text{det}(P_a) = Q(a)^2 > 0 \).

An endomorphism \( \psi \) of a vector space \( \mathbb{C} \) is called an involution of the field \( \mathbb{C} \) if \( \psi(\psi(a)) = a \) and \( \psi(ab) = \psi(a)\psi(b) \) for all \( a, b \in \mathbb{C} \). For an element \( a = a_1 + ia_2 \in \mathbb{C} \), we set \( \pi = a_1 - ia_2 \).

Proposition 3. The mapping \( a \to \pi \) is an involution of the field \( \mathbb{C} \). In addition, for an arbitrary element \( a = a_1 + ia_2 \in \mathbb{C} \), equalities \( a + \pi = 2a_1, <a, a> = a_1^2 + a_2^2 \in R \) hold.

Proposition 4. Let \( x \in \mathbb{C} \). Then the element \( x^{-1} \) exists if and only if \( Q(x) \neq 0 \). In the case \( Q(x) \neq 0 \), equalities \( x^{-1} = \frac{\pi}{Q(x)} \) and \( Q(x^{-1}) = \frac{1}{Q(x)} \) hold.

Put \( W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). We will use \( W \) also for the writing of the element \( \pi \) in the form \( \pi = W z \).

Proposition 5. \( Q(Wx) = Q(x) \) for all \( x \in \mathbb{C} \) and \( <Wx, Wy> = <x, y> \) for all \( x, y \in \mathbb{C} \).

Put \( \mathbb{C}^* = \{ z \in \mathbb{C} \mid Q(z) \neq 0 \} \). \( \mathbb{C}^* \) is a group with respect to the multiplication operation in the field \( \mathbb{C} \). Let \( a = a_1 + ia_2 \in \mathbb{C}^* \) that is \( |a| \neq 0 \). Put

\[
P_a^+ = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}.
\]

Proposition 6. Let \( a = a_1 + ia_2 \in \mathbb{C}^* \). Then the equality \( P_a = |a| P_a^+ \) holds, where \( P_a^+ \in SO(2) \).

Put \( S(\mathbb{C}^*) = \{ z \in \mathbb{C} \mid Q(z) = 1 \} \). \( P(\mathbb{C}^*) = \{ P_z \mid z \in \mathbb{C}^* \} \) and \( P(S(\mathbb{C}^*)) = \{ P_z \mid z \in S(\mathbb{C}^*) \} \). \( S(\mathbb{C}^*) \) is a subgroup of the group \( \mathbb{C}^* \) and \( S(\mathbb{C}^*) = \{ e^{i\varphi} \mid \varphi \in R \} \). Denote the set of all matrices \( \{ gW \mid g \in P(\mathbb{C}^*) \} \) by \( P(\mathbb{C}^*)W \), where \( gW \) is the multiplication of matrices \( g \) and \( W \).

Theorem 1. (see [7, p.172]) The following equalities are hold:

(i) \( LSim^+ (E_2) = \{ P_a \mid E_2 \to E_2 | a \in \mathbb{C}^* \} = P(\mathbb{C}^*) \).

(ii) \( LSim^- (E_2) = \{ P_aW : E_2 \to E_2 | a \in \mathbb{C}^* \} = P(\mathbb{C}^*)W \).

(iii) \( LSim(E_2) = LSim^+ (E_2) \cup LSim^- (E_2) \).

Proposition 7. (i) Let \( u, v \in \mathbb{C} \). Assume that \( Q(u) \neq 0 \). Then the element \( vu^{-1} \) exists, the following equalities hold:

\[
vu^{-1} = \frac{<u, v>}{Q(u)} + \frac{|uv|}{Q(u)}
\]

and

\[
P_{vu^{-1}} = \begin{pmatrix} <u, v> & |uv| \\ \frac{Q(u)}{Q(v)} & \frac{Q(u)}{Q(v)} \end{pmatrix}.
\]

(ii) Assume that \( Q(u) \neq 0 \). Then \( \text{det}(P_{vu^{-1}}) = \left( \frac{<u, v>}{Q(u)} \right)^2 + \left( \frac{|uv|}{Q(u)} \right)^2 \neq 0 \) if and only if \( Q(v) \neq 0 \).

3 Control invariants of planar Bézier curve

A planar Bézier curve is a parametric curve(or a I-path, where \( I = [0, 1] \)) whose points \( x(t) \) are defined by \( x(t) = \sum_{i=0}^{m} p_i B_{i,m}(t) \), where the \( p_i \in E_2 \) are control points and \( B_{i,m}(t) \) are Bernstein basis polynomials.(for more details, see [6].)

A planar polynomial curve is a parametric curve whose points \( x(t) \) are defined by \( x(t) = \sum_{i=0}^{m} a_i t^i \), where the \( a_i \in \mathbb{E}_2 \) are monomial control points.(for more details, see [6, p.181].)

All polynomial curves can be represented in Bézier form. The following lemma is given in [6, p.181].

Lemma 1. The following equalities

\[
a_i = \sum_{j=0}^{i} (-1)^{i-j} \frac{m!}{(m-i)! j!(i-j)!} b_j
\]

hold for all \( i = 1, 2, \ldots, m \) and \( i \geq j \).

Let \( G = LSim(E_2), LSim^+ (E_2) \).
Definition 2. (see [5]) A function \( f(z_0, z_1, \ldots, z_m) \) of control points \( z_0, z_1, \ldots, z_m \) in \( E_2 \) will be called G-invariant if \( f(Fz_0, Fz_1, \ldots, Fz_m) = f(z_0, z_1, \ldots, z_m) \) for all \( F \in G \).

A G-invariant function \( f(b_0, b_1, \ldots, b_m) \) of control points \( b_0, b_1, \ldots, b_m \) of a Bézier curve \( x(t) = \sum_{j=0}^{m} b_j B_j,m(t) \) will be called a control G-invariant of \( x(t) \), where \( B_j,m(t) \) are Bernstein basis polynomials. A G-invariant function \( f(a_0, a_1, \ldots, a_m) \) of monomial control points \( a_0, a_1, \ldots, a_m \) of a polynomial curve \( x(t) = \sum_{j=0}^{m} a_j t^j \) will be called a monomial G-invariant of \( x(t) \).

Definition 2. (see [5]) Bézier curves \( x(t) \) and \( y(t) \) in \( E_2 \) will be called G-similar and written \( x \sim^G y \) if there exists \( F \in G \) such that \( y(t) = Fx(t) \) for all \( t \in [0, 1] \).

Since Bézier curves can be introduced by control points, we will define the problem of G-similarity of points in \( E_2 \).

Definition 3. (see [5]) \( m \)-uples \( \{z_1, z_2, \ldots, z_m\} \) and \( \{w_1, w_2, \ldots, w_m\} \) of points in \( E_2 \) will be called G-similar and written by \( \{z_1, z_2, \ldots, z_m\} \sim^G \{w_1, w_2, \ldots, w_m\} \) if there exists \( F \in G \) such that \( wz_j = Fz_j \) for all \( j = 1, 2, \ldots, m \).

Let \( u, v \) be points in \( E_2 \). We denote the the matrix of column-vectors \( u, v \) by \( [u \, v] \) and its determinant by \( |u \, v| \).

Example 1. Since \( \langle u, v \rangle = \langle u, v \rangle \) for all \( g \in LSim(E_2) \), we obtain that the function \( \langle u, v \rangle \) of points \( u, v \in E_2 \) is \( LSim(E_2) \)-invariant. Similarly, the function \( \langle u, v \rangle \) is \( LSim^+(E_2) \)-invariant.

Example 2. Let \( x(t) \) and \( y(t) \) be Bézier curves of degrees of \( m \) and \( k \), respectively. Assume that \( x \sim^L E_2 \) \( y \). Then \( m = k \) that is the degree of a Bézier curve \( x(t) \) is \( LSim(E_2) \)-invariant.

### 4 Similarity of planar Bézier curves

Theorem 2. Let \( x(t) = \sum_{j=0}^{m} a_j t^j = \sum_{j=0}^{m} a_j B_j,m(t) \) and \( y(t) = \sum_{j=0}^{m} c_j t^j = \sum_{j=0}^{m} c_j B_j,m(t) \) be Bézier curves in \( E_2 \) of degree \( m \), where \( m \geq 1 \). Then following conditions are equivalent:

(i) \( x(t) \sim^L E_2 \) \( y(t) \)

(ii) \( \{p_0, p_1, \ldots, p_m\} \sim^L \{q_0, q_1, \ldots, q_m\} \)

(iv) \( \{a_0, a_1, \ldots, a_m\} \sim \{c_0, c_1, \ldots, c_m\} \)

Theorem 3. Let \( x(t) = \sum_{j=0}^{m} a_j t^j = \sum_{j=0}^{m} a_j B_j,m(t) \) and \( y(t) = \sum_{j=0}^{m} c_j t^j = \sum_{j=0}^{m} c_j B_j,m(t) \) be Bézier curves in \( E_2 \) of degree \( m \), where \( m \geq 1 \). Then following conditions are equivalent:

(i) \( x(t) \sim^L E_2 \) \( y(t) \)

(ii) \( \{p_0, p_1, \ldots, p_m\} \sim^L \{q_0, q_1, \ldots, q_m\} \)

(iv) \( \{a_0, a_1, \ldots, a_m\} \sim^L \{c_0, c_1, \ldots, c_m\} \)

Remark 1. In Theorems 2 and 3, we have considered the problem of G-similarity of polynomial curves in the case \( m \geq 1 \). For the case \( m = 0 \), the problem of G-similarity of polynomial curves \( x(t) = a_0 y(t) \) reduces to the problem of G-similarity of points \( a_0 \) and \( c_0 \) in \( E_2 \). For the groups \( G = LSim(E_2), LSim^+(E_2) \), it is obvious that \( a_0 \sim c_0 \) for all \( a_0 \) and \( c_0 \) in \( E_2 \). In what follows, \( m \geq 1 \). The case \( m = 0 \) is easily considered.

Theorem 4. Let \( A = \{a_0, \ldots, a_m\} \) and \( C = \{c_0, \ldots, c_m\} \) be two systems in \( E_2 \) such that \( a_k \neq 0, c_k \neq 0 \), where \( k \in \{0, 1, \ldots, m\} \). Then, \( A \) and \( C \) are \( LSim^+(E_2) \)-similar if and only if

\[
\begin{pmatrix}
\langle a_1, a_k \rangle \\
\langle a_k, a_k \rangle \\
\langle a_k, c_k \rangle
\end{pmatrix} \geq
\begin{pmatrix}
\langle c_1, a_k \rangle \\
\langle c_k, a_k \rangle \\
\langle c_k, c_k \rangle
\end{pmatrix}
\] (4)

for all \( i = 0, 1, \ldots, k-1, k+1, k+2, \ldots, m \). Moreover, there exists the unique element \( F \in LSim^+(E_2) \) such that \( c_j = Fa_j \) for all \( j = 0, 1, \ldots, m \), where the matrix \( F \) can be written as

\[
F = \begin{pmatrix}
\frac{\langle a_1, a_k \rangle}{\langle a_k, a_k \rangle} & -\frac{\langle a_1, c_k \rangle}{\langle a_k, a_k \rangle} \\
\frac{\langle c_1, a_k \rangle}{\langle a_k, a_k \rangle} & -\frac{\langle c_1, c_k \rangle}{\langle a_k, a_k \rangle}
\end{pmatrix}.
\] (5)

Theorem 5. Let \( A = \{a_0, \ldots, a_m\} \) and \( C = \{c_0, \ldots, c_m\} \) be two systems in \( E_2 \) such that \( a_k \neq 0, c_k \neq 0 \) for \( k \in \{0, 1, \ldots, m\} \) and \( \text{rank} A = \text{rank} C = 1 \). Then, \( A \) and \( C \) are \( LSim(E_2) \)-similar if and only if

\[
\begin{pmatrix}
\langle a_1, a_k \rangle \\
\langle a_k, a_k \rangle \\
\langle a_k, c_k \rangle
\end{pmatrix} \geq
\begin{pmatrix}
\langle c_1, a_k \rangle \\
\langle c_k, a_k \rangle \\
\langle c_k, c_k \rangle
\end{pmatrix}
\] (6)

for all \( i = 0, 1, \ldots, k-1, k+1, k+2, \ldots, m \). Moreover, there exists the unique element \( H \in LSim(E_2) \) such that \( c_j = Ha_j \) for all \( j = 0, 1, \ldots, m \), where the matrix \( H \) has the form (5).
Remark 2. Let $A = \{a_0, \ldots, a_m\}$. In the case $\text{rank} A = 2$, denote by $\text{index} A$ smallest of $s$, $0 \leq s \leq m$, such that $a_s \neq \lambda a_k$ for all $\lambda \in E_2$ and $a_k \neq 0$. The number $\text{index} A$ is $LSim(E_2)$-invariant.

Theorem 6. Let $A = \{a_0, \ldots, a_m\}$ and $C = \{c_0, \ldots, c_m\}$ be two systems in $E_2$ such that $a_k \neq 0$, $c_k \neq 0$, $\text{rank} A = \text{rank} C = 2$ and $\text{index} A = \text{index} C = l$ for $k, l \in \{0, 1, \ldots, m\}$, $l \neq k$. Then, $A$ and $C$ are $LSim(E_2)$-similar if and only if

$$
\begin{align*}
<& a_i, a_k > &<& c_i, c_k > \\
<& a_k, a_k > &<& c_k, c_k > \\
(&< a_i, a_k > &<& c_i, c_k > \\
<& a_k, a_k > &<& c_k, c_k > &>&
\end{align*}
$$

\begin{equation}
(7)
\end{equation}

for all $i = 0, 1, \ldots, m$, $i \neq k$ and $i \neq l$. Moreover, there exists the unique element $M \in LSim(E_2)$ such that $c_j = Ma_j$ for all $j = 1, \ldots, m$. Then there exist following cases:

(i) In the case $\frac{[a_i, a_k]}{<a_i, a_k>} = \frac{[c_i, c_k]}{<c_i, c_k>}$, the matrix $M \in LSim^+(E_2)$ and it has the form (5).

(ii) In the case $\frac{[a_i, a_k]}{<a_i, a_k>} = -\frac{[c_i, c_k]}{<c_i, c_k>}$, the matrix $MW \in LSim(E_2)$ and it can be represented by

$$
M = \left( \begin{array}{ccc}
& <W_{a_k, c_k}> & <W_{a_k, c_k}> \\
& Q(a_k) & Q(a_k) \\
& W_{a_k, c_k} & Q(a_k) \\
& Q(a_k) & Q(a_k)
\end{array} \right).
$$

\begin{equation}
(8)
\end{equation}

Theorem 7. (i) Let $x(t) = \sum_{j=0}^{m} a_j t^j$ and $y(t) = \sum_{j=0}^{m} c_j t^j$ be two polynomial curves in $E_2$ of degree $m$, where $m \geq 1$ such that $x(t) \sim_{LSim^+(E_2)} y(t)$. Then, the equalities (4) in Theorem 4 hold.

(ii) Conversely, if $x(t) = \sum_{j=0}^{m} a_j t^j$ and $y(t) = \sum_{j=0}^{m} c_j t^j$ are two polynomial curves in $E_2$ of degree $m$, where $m \geq 1$ such that the equalities (4) in Theorem 4 hold, then $x(t) \sim_{LSim^+(E_2)} y(t)$. Moreover, there exists the unique $F \in LSim^+(E_2)$ such that $y(t) = Fx(t)$ for all $t \in [0, 1]$ and $F$ has the form (5).

Theorem 8. (i) Let $x(t) = \sum_{j=0}^{m} a_j t^j$ and $y(t) = \sum_{j=0}^{m} c_j t^j$ be two polynomial curves in $E_2$ of degree $m$, where $m \geq 1$ such that $x(t) \sim_{LSim(E_2)} y(t)$. Then, the equalities (7) in Theorem 6 hold.

(ii) Conversely, if $x(t) = \sum_{j=0}^{m} a_j t^j$ and $y(t) = \sum_{j=0}^{m} c_j t^j$ are two polynomial curves in $E_2$ of degree $m$, where $m \geq 1$ such that the equalities (7) in Theorem 6 hold, then $x(t) \sim_{LSim(E_2)} y(t)$. Moreover, there exists the unique $F \in LSim(E_2)$ such that $y(t) = Fx(t)$ for all $t \in [0, 1]$. Then,

(a) In the case $\frac{[a_i, a_k]}{<a_i, a_k>} = \frac{[c_i, c_k]}{<c_i, c_k>}$, $F$ has the form (5).

(b) In the case $\frac{[a_i, a_k]}{<a_i, a_k>} = -\frac{[c_i, c_k]}{<c_i, c_k>}$, $F$ has the form (8).

Theorem 9. (i) Let $x(t) = \sum_{j=0}^{m} p_j B_{j,m}(t)$ and $y(t) = \sum_{j=0}^{m} q_j B_{j,m}(t)$ be two Bézier curves in $E_2$ of degree $m$, where $m \geq 1$ such that $x(t) \sim_{LSim^+(E_2)} y(t)$. Then by Lemma 1, the equalities (4) in Theorem 4 hold.

(ii) Conversely, if $x(t) = \sum_{j=0}^{m} p_j B_{j,m}(t)$ and $y(t) = \sum_{j=0}^{m} q_j B_{j,m}(t)$ be two Bézier curves in $E_2$ of degree $m$, where $m \geq 1$ such that the equalities (4) in Theorem 4 and Lemma 1 hold, then $x(t) \sim_{LSim^+(E_2)} y(t)$. Moreover, there exists the unique $F \in LSim^+(E_2)$ such that $y(t) = Fx(t)$ for all $t \in [0, 1]$ and $F$ in the terms of the equalities given in Lemma 1 has the form (5).

Theorem 10. (i) Let $x(t) = \sum_{j=0}^{m} p_j B_{j,m}(t)$ and $y(t) = \sum_{j=0}^{m} q_j B_{j,m}(t)$ be two Bézier curves in $E_2$ of degree $m$, where $m \geq 1$ such that $x(t) \sim_{LSim(E_2)} y(t)$. Then by Lemma 1, the equalities (7) in Theorem 6 hold.

(ii) Conversely, if $x(t) = \sum_{j=0}^{m} p_j B_{j,m}(t)$ and $y(t) = \sum_{j=0}^{m} q_j B_{j,m}(t)$ be two Bézier curves in $E_2$ of degree $m$, where $m \geq 1$ such that the equalities (7) in Theorem 6 and Lemma 1 hold, then $x(t) \sim_{LSim(E_2)} y(t)$. Moreover, there exists the unique $F \in LSim(E_2)$ such that $y(t) = Fx(t)$ for all $t \in [0, 1]$. Then,

(a) In the case $\frac{[p_i, p_k]}{<p_i, p_k>} = \frac{[q_i, q_k]}{<q_i, q_k>}$, $F$ in the terms of the equalities given in Lemma 1 has the form (5).

(b) In the case $\frac{[p_i, p_k]}{<p_i, p_k>} = -\frac{[q_i, q_k]}{<q_i, q_k>}$, $F$ in the terms of the equalities given in Lemma 1 has the form (8).

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5 References
