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Detecting Similarities of Bézier Curves for the Groups $LSim(E_2), LSim^+(E_2)$

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Abstract: In this paper, for linear similarity groups, global invariants of plane Bézier curves (plane polynomial curves) in E_2 are introduced. Using complex numbers and the global G-invariants of a plane Bézier curve(a plane polynomial curve), for given two plane Bézier curves (plane polynomial curves) $x(t)$ and $y(t)$, evident forms of all transformations $g \in G$, carrying $x(t)$ to $y(t)$, are obtained.

Keywords: Polynomial curve, Bézier curve, Invariant, Linear similarity group.

1 Introduction

Let E_2 be the 2-dimensional Euclidean space, $G = LSim(E_2)$ be the group of all linear similarities of E_2 and $G = LSim^+(E_2)$ be the group of all orientation-preserving linear similarities of E_2.

In [1], using local differential invariants and Frenet frames of two curves, uniqueness and existence theorems for a curve determined up to a direct similarity of E_2.

For the group $Sim^+(n)$, this theorem shows that a necessary and sufficient conditions for two curves in E_n to be equivalent is that they have same shape curvatures and the other specially conditions.

The complete systems of global G-invariants of a path and a curve in E_2 are obtained. For the groups G, existence and uniqueness theorems for a curve and a path are given in the terms of global G-invariants of a path and a curve in [2].

$LSim(2)$-equivalence of two Bézier curves without using differential invariants of Bézier curves in the terms of control invariants of Bézier curves is proved in [3, 4].

In this work, starting from the ideas in [2–4, 8–11], we address how to compute explicitly an linear similarity transformation which carrying a Bézier curve into another Bézier curve in the terms of control invariants of a Bézier curve for the groups $LSim(E_2)$ and $LSim^+(E_2)$ without using differential invariants of Bézier curves.

2 Preliminaries

The following definitions and propositions are known in [2].

Let R be the field of real numbers and $\mathbb{C}$ be the field of complex numbers. The multiplication in $\mathbb{C}$ has the form $(a_1 + ia_2)(b_1 + ib_2) = (a_1b_1 - a_2b_2) + i(a_1b_2 + a_2b_1)$. We will consider element $a = a_1 + ia_2$ also in the form $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$. For $a = a_1 + ia_2$, denote by P_a the matrix $\begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}$ and consider P_a also as the transformation $P_a : \mathbb{C} \to \mathbb{C}$, where $P_a b = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1b_1 - a_2b_2 \\ a_1b_2 + a_2b_1 \end{pmatrix}$ for all $b = b_1 + ib_2 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{C}$. Then we have the equality

$$ab = P_a b. \tag{1}$$

for all $a, b \in \mathbb{C}$. Let $P(\mathbb{C})$ denote the set of all matrices P_a, where $a \in \mathbb{C}$. We consider on $P(\mathbb{C})$ the following standard matrix operations: the component-wise addition, a scalar multiplication and the multiplication of matrices. Then $P(\mathbb{C})$ is a field, where the unit element is the unit matrix. The following Propositions are known.

Proposition 1. *The mapping $P : \mathbb{C} \to P(\mathbb{C})$, where $P : a \to P_a$ for all $a \in \mathbb{C}$, is an isomorphism of fields.*

For vectors $a = a_1 + ia_2, b = b_1 + ib_2 \in \mathbb{C}$, we put $< a, b >= a_1b_1 + a_2b_2$. Then $< a, b >$ is a bilinear form on E_2 and $< a, a >= a_1^2 + a_2^2$ is a quadratic form on E_2. Put $Q(a) =< a, a >$. We consider the field $\mathbb{C}$ also as the two-dimensional Euclidean space E_2 with the scalar product $< a, b >$. Then $\|a\| = |a| = \sqrt{Q(a)}, \forall a \in \mathbb{C}$.

Proposition 2. *(i) Equalities $Q(a) = det(P_a)$, $Q(ab) = Q(a)Q(b)$, $|ab| = |a|\,|b|$, $Q(a) = det(P_a) =$ hold for all $a, b \in \mathbb{C}$.*
(ii) Let $a = a_1 + ia_2 \in \mathbb{C}^*$. Then $det(P_a) = Q(a) > 0$.

An endomorphism ψ of a vector space $\mathbb{C}$ is called an involution of the field $\mathbb{C}$ if $\psi(\psi(a)) = a$ and $\psi(ab) = \psi(a)\psi(b)$ for all $a, b \in \mathbb{C}$. For an element $a = a_1 + ia_2 \in \mathbb{C}$, we set $\overline{a} = a_1 - ia_2$.

Proposition 3. *The mapping $a \to \overline{a}$ is an involution of the field $\mathbb{C}$. In addition, for an arbitrary element $a = a_1 + ia_2 \in \mathbb{C}$, equalities $a + \overline{a} = 2a_1, < a, a >= a\overline{a} = a_1^2 + a_2^2 \in R$ hold.*

Proposition 4. *Let $x \in \mathbb{C}$. Then the element x^{-1} exists if and only if $Q(x) \neq 0$. In the case $Q(x) \neq 0$, equalities $x^{-1} = \frac{\overline{x}}{Q(x)}$ and $Q(x^{-1}) = \frac{1}{Q(x)}$ hold.*

Let $W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We will use W also for the writing of the element $\overline{z}$ in the form $\overline{z} = Wz$.

Proposition 5. *$Q(Wx) = Q(x)$ for all $x \in \mathbb{C}$ and $< Wx, Wy >=< x, y >$ for all $x, y \in \mathbb{C}$.*

Put $\mathbb{C}^* = \{z \in \mathbb{C} \mid Q(z) \neq 0\}$. $\mathbb{C}^*$ is a group with respect to the multiplication operation in the field $\mathbb{C}$. Let $a = a_1 + ia_2 \in \mathbb{C}^*$ that is $|a| \neq 0$. Put

$$P_a^+ = \begin{pmatrix} \frac{a_1}{|a|} & \frac{-a_2}{|a|} \\ \frac{a_2}{|a|} & \frac{a_1}{|a|} \end{pmatrix}.$$

Proposition 6. *Let $a = a_1 + ia_2 \in \mathbb{C}^*$. Then the equality $P_a = |a|\, P_a^+$ holds, where $P_a^+ \in SO(2)$.*

Put $S(\mathbb{C}^*) = \{z \in \mathbb{C} \mid Q(z) = 1\}$, $P(\mathbb{C}^*) = \{P_z \mid z \in \mathbb{C}^*\}$ and $P(S(\mathbb{C}^*)) = \{P_z \mid z \in S(\mathbb{C}^*)\}$. $S(\mathbb{C}^*)$ is a subgroup of the group $\mathbb{C}^*$ and $S(\mathbb{C}^*) = \left\{e^{i\varphi} \mid \varphi \in R\right\}$. Denote the set of all matrices $\{gW \mid g \in P(\mathbb{C}^*)\}$ by $P(\mathbb{C}^*)W$, where gW is the multiplication of matrices g and W.

Theorem 1. *(see [7, p.172]) The following equalities are hold:*

(i) $LSim^+(E_2) = \{P_a : E_2 \to E_2 | a \in \mathbb{C}^*\} = P(\mathbb{C}^*)$.
(ii) $LSim^-(E_2) = \{P_a W : E_2 \to E_2 | a \in \mathbb{C}^*\} = P(\mathbb{C}^*)W$.
(iii) $LSim(E_2) = LSim^+(E_2) \cup LSim^-(E_2)$.

Proposition 7. *(i) Let $u, v \in \mathbb{C}$. Assume that $Q(u) \neq 0$. Then the element vu^{-1} exists, the following equalities hold:*

$$vu^{-1} = \frac{< u, v >}{Q(u)} + i\frac{[uv]}{Q(u)}$$

and

$$P_{vu^{-1}} = \begin{pmatrix} \frac{<u,v>}{Q(u)} & -\frac{[uv]}{Q(u)} \\ \frac{[uv]}{Q(u)} & \frac{<u,v>}{Q(u)} \end{pmatrix}. \tag{2}$$

(ii) Assume that $Q(u) \neq 0$. Then $det(P_{vu^{-1}}) = (\frac{<u,v>}{Q(u)})^2 + (\frac{[uv]}{Q(u)})^2 \neq 0$ if and only if $Q(v) \neq 0$.

3 Control invariants of planar Bézier curve

A planar Bézier curve is a parametric curve(or a I-path, where $I = [0, 1]$) whose points $x(t)$ are defined by $x(t) = \sum_{i=0}^m p_i B_{i,m}(t)$, where the $p_i \in E_2$ are control points and $B_{i,m}(t)$ are Bernstein basis polynomials.(for more details, see [6].)

A planar polynomial curve is a parametric curve whose points $x(t)$ are defined by $x(t) = \sum_{i=0}^m a_i t^i$, where the $a_i \in E_2$ are monomial control points.(for more details, see [6, p.181].)
All polynomial curves can be represented in Bézier form. The following lemma is given in [6, p.181].

Lemma 1. *The following equalities*

$$a_i = \sum_{j=0}^i (-1)^{i-j} \frac{m!}{i!(m-i)!} \frac{i!}{j!(i-j)!} b_j \tag{3}$$

hold for all $i = 1, 2, \ldots, m$ and $i \geq j$.

Let $G = LSim(E_2), LSim^+(E_2)$.

Definition 1. *(see [5]) A function $f(z_0, z_1, \dots, z_m)$ of points $z_0, z_1, \dots, z_m$ in E_2 will be called G-invariant if $f(Fz_0, Fz_1, \dots, Fz_m) = f(z_0, z_1, \dots, z_m)$ for all $F \in G$.*
A G-invariant function $f(b_0, b_1, \dots, b_m)$ of control points $b_0, b_1, \dots, b_m$ of a Bézier curve $x(t) = \sum_{j=0}^{m} b_j B_{j,m}(t)$ will be called a control G-invariant of $x(t)$, where $B_{j,m}(t)$ are Bernstein basis polynomials. A G-invariant function $f(a_0, a_1, \dots, a_m)$ of monomial control points $a_0, a_1, \dots, a_m$ of a polynomial curve $x(t) = \sum_{j=0}^{m} a_j t^j$ will be called a monomial G-invariant of $x(t)$.

Definition 2. *(see [5]) Bézier curves $x(t)$ and $y(t)$ in E_2 will be called G -similar and written $x \overset{G}{\sim} y$ if there exists $F \in G$ such that $y(t) = Fx(t)$ for all $t \in [0, 1]$.*

Since Bézier curves can be introduced by control points, we will define the problem of G-similarity of points in E_2.

Definition 3. *(see [5]) m-uples $\{z_1, z_2, \dots, z_m\}$ and $\{w_1, w_2, \dots, w_m\}$ of points in E_2 will be called G-similar and written by $\{z_1, z_2, \dots, z_m\} \overset{G}{\sim} \{w_1, w_2, \dots, w_m\}$ if there exists $F \in G$ such that $w_j = Fz_j$ for all $j = 1, 2, \dots, m$.*

Let u, v be points in E_2. We denote the the matrix of column-vectors u, v by $\|u\ v\|$ and its determinant by $[u\ v]$.

Example 1. *Since $\frac{<g(u),g(v)>}{<g(u),g(u)>} = \frac{<u,v>}{<u,u>}$ for all $g \in LSim(E_2)$, we obtain that the function $\frac{<u,v)>}{<u,u>}$ of points $u, v \in E_2$ is $LSim(E_2)$-invariant. Similarly, the function $\frac{[u\ v]}{<u,u>}$ is $LSim^+(E_2)$-invariant.*

Example 2. *Let $x(t)$ and $y(t)$ be Bézier curves of degrees of m and k, respectively. Assume that $x \overset{LSim(E_2)}{\sim} y$. Then $m = k$ that is the degree of a Bézier curve $x(t)$ is $LSim(E_2)$-invariant.*

4 Similarity of planar Bézier curves

Theorem 2. *Let $x(t) = \sum_{j=0}^{m} a_j t^j = \sum_{j=0}^{m} p_j B_{j,m}(t)$ and $y(t) = \sum_{j=0}^{m} c_j t^j = \sum_{j=0}^{m} q_j B_{j,m}(t)$ be Bézier curves in E_2 of degree m, where $m \geq 1$. Then following conditions are equivalent:*

(i) $x(t) \overset{LSim(E_2)}{\sim} y(t)$
(ii) $\{p_0, p_1, \dots, p_m\} \overset{LSim(E_2)}{\sim} \{q_0, q_1, \dots, q_m\}$
(iv) $\{a_0, a_1, \dots, a_m\} \overset{LSim(E_2)}{\sim} \{c_0, c_1, \dots, c_m\}$

Theorem 3. *Let $x(t) = \sum_{j=0}^{m} a_j t^j = \sum_{j=0}^{m} p_j B_{j,m}(t)$ and $y(t) = \sum_{j=0}^{m} c_j t^j = \sum_{j=0}^{m} q_j B_{j,m}(t)$ be Bézier curves in E_2 of degree m, where $m \geq 1$. Then following conditions are equivalent:*

(i) $x(t) \overset{LSim^+(E_2)}{\sim} y(t)$
(ii) $\{p_0, p_1, \dots, p_m\} \overset{LSim^+(E_2)}{\sim} \{q_0, q_1, \dots, q_m\}$
(iv) $\{a_0, a_1, \dots, a_m\} \overset{LSim^+(E_2)}{\sim} \{c_0, c_1, \dots, c_m\}$

Remark 1. *In Theorems 2 and 3, we have considered the problem of G-similarity of polynomial curves in the case $m \geq 1$. For the case $m = 0$, the problem of G-similarity of polynomial curves $x(t) = a_0$ and $y(t) = c_0$ reduces to the problem of G-similarity of points a_0 and c_0 in E_2. For the groups $G = LSim(E_2), LSim^+(E_2)$, it is obvious that $a_0 \overset{G}{\sim} c_0$ for all a_0 and c_0 in E_2. In what follows, $m \geq 1$. The case $m = 0$ is easily considered.*

Theorem 4. *Let $A = \{a_0, \dots, a_m\}$ and $C = \{c_0, \dots, c_m\}$ be two systems in E_2 such that $a_k \neq 0$, $c_k \neq 0$, where $k \in \{0, 1, \dots, m\}$. Then, A and C are $LSim^+(E_2)$-similar if and only if*

$$\begin{cases} \dfrac{< a_i, a_k >}{< a_k, a_k >} = \dfrac{< c_i, c_k >}{< c_k, c_k >}, \\ \dfrac{[a_i\ a_k]}{< a_k, a_k >} = \dfrac{[c_i\ c_k]}{< c_k, c_k >} \end{cases} \tag{4}$$

for all $i = 0, 1, \dots k-1, k+1, k+2, \dots, m$. Moreover, there exists the unique element $F \in LSim^+(E_2)$ such that $c_j = Fa_j$ for all $j = 0, 1, \dots, m$, where the matrix F can be written as

$$F = \begin{pmatrix} \frac{<a_k,c_k>}{Q(a_k)} & -\frac{[a_k c_k]}{Q(a_k)} \\ \frac{[a_k c_k]}{Q(a_k)} & \frac{<a_k,c_k>}{Q(a_k)} \end{pmatrix}. \tag{5}$$

Theorem 5. *Let $A = \{a_0, \dots, a_m\}$ and $C = \{c_0, \dots, c_m\}$ be two systems in E_2 such that $a_k \neq 0$, $c_k \neq 0$ for $k \in \{0, 1, \dots, m\}$ and $rankA = rankC = 1$. Then, A and C are $LSim(E_2)$-similar if and only if*

$$\frac{< a_i, a_k >}{< a_k, a_k >} = \frac{< c_i, c_k >}{< c_k, c_k >} \tag{6}$$

for all $i = 0, 1, \dots k-1, k+1, k+2, \dots, m$. Moreover, there exists the unique element $H \in LSim(E_2)$ such that $c_j = Ha_j$ for all $j = 0, 1, \dots, m$, where the matrix H has the form (5).

Remark 2. *Let $A = \{a_0, \ldots, a_m\}$. In the case $rankA = 2$, denote by $indexA$ smallest of s, $0 \leq s \leq m$, such that $a_s \neq \lambda a_k$ for all $\lambda \in E_2$ and $a_k \neq 0$. The number $indexA$ is $LSim(E_2)$-invariant.*

Theorem 6. *Let $A = \{a_0, \ldots, a_m\}$ and $C = \{c_0, \ldots, c_m\}$ be two systems in E_2 such that $a_k \neq 0$, $c_k \neq 0$, $rankA = rankC = 2$ and $indexA = indexC = l$ for $k, l \in \{0, 1, \ldots, m\}$, $l \neq k$. Then, A and C are $LSim(E_2)$-similar if and only if*

$$\begin{cases} \dfrac{< a_i, a_k >}{< a_k, a_k >} = \dfrac{< c_i, c_k >}{< c_k, c_k >} \\ \left(\dfrac{[a_l \ a_k]}{< a_k, a_k >} \right)^2 = \left(\dfrac{[c_l \ c_k]}{< c_k, c_k >} \right)^2 \\ \dfrac{[a_i \ a_k]}{[a_l \ a_k]} = \dfrac{[c_i \ c_k]}{[c_l \ c_k]} \end{cases} \tag{7}$$

for all $i = 0, 1, \ldots, \ldots, m$, $i \neq k$ and $i \neq l$. Moreover, there exists the unique element $M \in LSim(E_2)$ such that $c_j = Ma_j$ for all $j = 1, \ldots, m$. Then there exist following cases:

(i) *In the case $\frac{[a_l \ a_k]}{<a_k, a_k>} = \frac{[c_l \ c_k]}{<c_k, c_k>}$, the matrix $M \in LSim^+(E_2)$ and it has the form (5).*
(ii) *In the case $\frac{[a_l \ a_k]}{<a_k, a_k>} = -\frac{[c_l \ c_k]}{<c_k, c_k>}$, the matrix $MW \in LSim(E_2)$ and it can be represented by*

$$M = \begin{pmatrix} \frac{<Wa_k, c_k>}{Q(a_k)} & -\frac{[Wa_k c_k]}{Q(a_k)} \\ \frac{[Wa_k c_k]}{Q(a_k)} & \frac{<Wa_k, c_k>}{Q(a_k)} \end{pmatrix}. \tag{8}$$

Theorem 7. (i) *Let $x(t) = \sum_{j=0}^{m} a_j t^j$ and $y(t) = \sum_{j=0}^{m} c_j t^j$ be two polynomial curves in E_2 of degree m,where $m \geq 1$ such that $x(t) \overset{LSim^+(E_2)}{\sim} y(t)$. Then, the equalities (4) in Theorem 4 hold.*
(ii) *Conversely, if $x(t) = \sum_{j=0}^{m} a_j t^j$ and $y(t) = \sum_{j=0}^{m} c_j t^j$ are two polynomial curves in E_2 of degree m, where $m \geq 1$ such that the equalities (4) in Theorem 4 hold, then $x(t) \overset{LSim^+(E_2)}{\sim} y(t)$. Moreover, there exists the unique $F \in LSim^+(E_2)$ such that $y(t) = Fx(t)$ for all $t \in [0, 1]$ and F has the form (5).*

Theorem 8. (i) *Let $x(t) = \sum_{j=0}^{m} a_j t^j$ and $y(t) = \sum_{j=0}^{m} c_j t^j$ be two polynomial curves in E_2 of degree m,where $m \geq 1$ such that $x(t) \overset{LSim(E_2)}{\sim} y(t)$. Then, the equalities (7) in Theorem 6 hold.*
(ii) *Conversely, if $x(t) = \sum_{j=0}^{m} a_j t^j$ and $y(t) = \sum_{j=0}^{m} c_j t^j$ are two polynomial curves in E_2 of degree m, where $m \geq 1$ such that the equalities (7) in Theorem 6 hold, then $x(t) \overset{LSim(E_2)}{\sim} y(t)$. Moreover, there exists the unique $F \in LSim(E_2)$ such that $y(t) = Fx(t)$ for all $t \in [0, 1]$. Then,*
(a) *In the case $\frac{[a_l \ a_k]}{<a_k, a_k>} = \frac{[c_l \ c_k]}{<c_k, c_k>}$, F has the form (5).*
(b) *In the case $\frac{[a_l \ a_k]}{<a_k, a_k>} = -\frac{[c_l \ c_k]}{<c_k, c_k>}$, F has the form (8).*

Theorem 9. (i) *Let $x(t) = \sum_{j=0}^{m} p_j B_{j,m}(t)$ and $y(t) = \sum_{j=0}^{m} q_j B_{j,m}(t)$ be two Bézier curves in E_2 of degree m, where $m \geq 1$ such that $x(t) \overset{LSim^+(E_2)}{\sim} y(t)$. Then by Lemma 1, the equalities (4) in Theorem 4 hold.*
(ii) *Conversely, if Let $x(t) = \sum_{j=0}^{m} p_j B_{j,m}(t)$ and $y(t) = \sum_{j=0}^{m} q_j B_{j,m}(t)$ be two Bézier curves in E_2 of degree m, where $m \geq 1$ such that the equalities (4) in Theorem 4 and Lemma 1 hold, then $x(t) \overset{LSim^+(E_2)}{\sim} y(t)$. Moreover, there exists the unique $F \in LSim^+(E_2)$ such that $y(t) = Fx(t)$ for all $t \in [0, 1]$ and F in the terms of the equalities given in Lemma 1 has the form (5) .*

Theorem 10. (i) *Let $x(t) = \sum_{j=0}^{m} p_j B_{j,m}(t)$ and $y(t) = \sum_{j=0}^{m} q_j B_{j,m}(t)$ be two Bézier curves in E_2 of degree m, where $m \geq 1$ such that $x(t) \overset{LSim(E_2)}{\sim} y(t)$. Then by Lemma 1, the equalities (7) in Theorem 6 hold.*
(ii) *Conversely, if $x(t) = \sum_{j=0}^{m} p_j B_{j,m}(t)$ and $y(t) = \sum_{j=0}^{m} q_j B_{j,m}(t)$ be two Bézier curves in E_2 of degree m, where $m \geq 1$ such that the equalities (7) in Theorem 6 and Lemma1 hold, then $x(t) \overset{LSim(E_2)}{\sim} y(t)$. Moreover, there exists the unique $F \in LSim(E_2)$ such that $y(t) = Fx(t)$ for all $t \in [0, 1]$. Then,*
(a) *In the case $\frac{[p_l \ p_k]}{<p_k, p_k>} = \frac{[q_l \ q_k]}{<q_k, q_k>}$, F in the terms of the equalities given in Lemma 1 has the form (5).*
(b) *In the case $\frac{[p_l \ p_k]}{<p_k, p_k>} = -\frac{[q_l \ q_k]}{<q_k, q_k>}$, F in the terms of the equalities given in Lemma 1 has the form (8).*

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