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On Minimal Surfaces in Galilean Space

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Abstract: In this paper, we investigated the minimal surfaces in three dimensional Galilean space $\mathbb{G}^3$. We showed that the condition of minimality of a surface area is locally equivalent to the mean curvature vector H vanishes identically. Then, we derived the necessary and sufficient conditions that the minimal surfaces have to satisfy in Galilean space.

Keywords: Minimal surfaces, Area of a surface, Galilean space.

1 Introduction

Minimal surfaces are one of the most interesting subject in mathematics. The study and computation of minimal surfaces has a long history [12]. Lagrange made the first investigation of the minimal surfaces by asking a simple question named as Plateau's problem which concerns with finding a surface of least area that spans a given fixed one-dimensional contour in three-dimensional Euclidean space [15]. Later G. Monge discovered that the condition for minimality of a surface leads to the condition that vanishing mean curvature, and therefore surfaces with $H = 0$ are called "minimal" [6].

The study of minimal surface has arised many interesting applications in other fields in science, such as the interface between crystals and organic matter in the skeletal element of sea urchins can be described as a minimal surface [11]. Moreover, the surfaces with vanishing mean curvature are also studied in other ambient spaces [16]. In Lorentz–Minkowski Space, a spacelike surface has maximal area if its mean curvature vanishes [10]. Moreover, the timelike minimal surfaces are investigated in [7, 8].

Simultaneity gives the characterization of the Galilean space. The simultaneous events (points), which are events that occur at the same time [9]. The Galilean space $\mathbb{G}^3$ has three-dimensions (x, y, z): two dimensions of Euclidean plane (y, z) and one dimension of time (x) [17]. The distance between non-simultaneous events $p = (x, y, z)$ and $q = (x_1, y_1, z_1)$ in $\mathbb{G}^3$ is defined by

$$d_1(p, q) = |x - x_1|\,.$$

On the other hand, the distance between two simultaneous events $p = (x, y, z)$ and $q_1 = (x, y_1, z_1)$ is defined by

$$d_2(p, q_1) = \sqrt{(y - y_1)^2 + (z - z_1)^2}.$$

Moreover, the Galilean space $\mathbb{G}^3$ can be considered as a Cayley-Klein space equipped with the projective metric of signature $(0, 0, +, +)$[5]. There are two types of plane in the Galilean space. Euclidean planes are in the following form $x = k$ $(y = z = 0)$ $k \in \mathbb{R}$ [4]. The other planes are isotropic [2, 3]. The non-isotropic vectors are in the following form $\mathbf{u} = (u_1, u_2, u_3)$, $u_1 \neq 0$. For isotropic vectors $u_1 = 0$ holds. More information about Galilean space can be found in [1, 13, 18]

Let $\mathbf{a} = (x, y, z)$ and $\mathbf{b} = (x_1, y_1, z_1)$ be vectors in the Galilean space. The scalar product $<, >$ is defined by

$$< \mathbf{a}, \mathbf{b} >= x_1 x. \tag{1}$$

In addition, when both of the vectors $\mathbf{p} = (0, y, z)$ and $\mathbf{q} = (0, y_1, z_1)$ are isotropic, the scalar product $<, >_1$ is given by

$$< \mathbf{p}, \mathbf{q} >_1= y y_1 + z z_1. \tag{2}$$

Let M be a surface in $\mathbb{G}^3$ given by parametrization

$$\varphi(v^1, v^2) = (x(v^1, v^2), y(v^1, v^2), z(v^1, v^2)).$$

The isotropic unit normal vector N is defined by

$$N = \frac{\varphi_1 \wedge \varphi_2}{w} \tag{3}$$

where the partial derivatives of the surface M with respect to v^1 and v^2 is denoted by φ_1 and φ_2, respectively and $w = \|\varphi_1 \wedge \varphi_2\|_1$ [14]. The first fundamental form of the surface is given by

$$I = I_1 + \epsilon I_2 \tag{4}$$

where $I_1 = g_{ij}dv^i dv^j$ and $I_2 = h_{ij}dv^i dv^j$. If $I_1 = 0$ then $\epsilon = 1$, in the other cases $\epsilon = 0$. The induced metrics h_{ij} and g_{ij} $(i, j = 1, 2)$ on the surface are given by

$$h_{ij} = \langle \varphi_i, \varphi_j \rangle_1, \quad g_{ij} = \langle \varphi_i, \varphi_j \rangle. \tag{5}$$

The components of the inverse metric are given by

$$g^1 = \frac{x_2}{w}, \quad g^2 = -\frac{x_1}{w} \quad g^{ij} = g^i g^j \tag{6}$$

where the partial derivatives of the first component $x(v^1, v^2)$ of the surface with respect to v^1 and v^2 is denoted by x_1 and x_2, respectively.

In [14], the coefficients L_{ij} of the second fundamental form, the Gauss curvature K and mean curvature H of the surface are given by

$$L_{ij} =< \frac{\varphi_{,ij}x_{,1} - x_{,ij}\varphi_{,1}}{x_{,1}}, N >_1, K = \frac{\det L_{ij}}{w^2}, 2H = g^{ij}L_{ij}. \tag{7}$$

The partial derivatives of the normal vector is obtained by

$$N_i = -g^{jm}L_{ij}\varphi_m. \tag{8}$$

2 Minimal surfaces in galilean space

In this section, we will give a mathematical definition of the minimal surface in Galilean space $\mathbb{G}^3$. Since the minimal surfaces locally minimizes area, firstly we need to show that it is also meaningful in Galilean space. Similar to the three dimensional Euclidean space, the norm of the cross product measures the area spanned by two vectors in the three dimensional Galilean space [17]. Therefore we state the following definition.

Let M be the surface parametrized by $\varphi(u, v) = (x(u, v), y(u, v), z(u, v))$. We can see that $\|\varphi_u \wedge \varphi_v\|_1$ is the area of the parallelogram determined by φ_u and φ_v. Therefore, in Galilean space, the area of the surface can be obtained by

$$A(\varphi) = \int\int \|\varphi_u \wedge \varphi_v\|_1 \, dudv. \tag{9}$$

The minimal surface is the problem of minimizing $A(\varphi)$. To do this, let us consider a normal variation of the surface M in Galilean space. Let $t(u, v)$ be any smooth function on such that it vanishes on the boundary of the surface and N be the unit surface normal. For some small λ a normal variation of the surface M^σ can be parametrized by

$$\omega^\sigma(u, v) = \varphi(u, v) + \sigma t(u, v)N(u, v) \tag{10}$$

where $-\lambda < \sigma < \lambda$. This motivates the following theorem:

Theorem 2.1 Let $A(\sigma)$ be the area of the normal variation of the surface M in Galilean space. The critical point of the area of the normal variation M^σ is given by

$$A'(0) = -2\int\int t(u, v)H \|\varphi_u \wedge \varphi_v\|_1 \, dudv \tag{11}$$

where prime denotes differentiation respect to σ.

Proof: The area $A(\sigma)$ of the surface M^σ is given by

$$A(\sigma) = \int\int \|\omega_u^\sigma \wedge \omega_v^\sigma\|_1 \, dudv. \tag{12}$$

On the other hand from (10), the partial derivatives of the normal variation M^σ are obtained as

$$\omega_u^\sigma = \varphi_u + \sigma t_u \mathbf{N} + \sigma t \mathbf{N}_v, \quad \omega_v^\sigma = \varphi_v + \sigma t_v \mathbf{N} + \sigma t \mathbf{N}_u.$$

By using $\mathbf{N}_u \wedge \mathbf{N}_v = 0$ we arrive at

$$\omega_u^\sigma \wedge \omega_v^\sigma = \varphi_u \wedge \varphi_v + t\sigma(\varphi_u \wedge \mathbf{N}_v + \mathbf{N}_u \wedge \varphi_v) + \sigma^2 t(t_u \mathbf{N} \wedge \mathbf{N}_v + t_v \mathbf{N}_u \wedge \mathbf{N}).$$

Taking norms on both sides of above equation, then differentiating with respect to σ, finally putting $\sigma = 0$ into the result gives

$$A'(0) = \int\int \frac{t \langle \varphi_u \wedge \mathbf{N}_v + \mathbf{N}_u \wedge \varphi_v, \varphi_u \wedge \varphi_v \rangle_1}{\|\varphi_{,1} \wedge \varphi_{,2}\|_1} dudv \tag{13}$$

where $\left.\dfrac{dA(\sigma)}{d\sigma}\right|_{\sigma=0} = A'(0)$.

On the other hand, from (8) we have

$$\mathbf{N}_u \wedge \varphi_v = (-g^{11}L_{11} - g^{12}L_{12})\varphi_u \wedge \varphi_v \tag{14}$$

and

$$\varphi_u \wedge \mathbf{N}_v = \varphi_u \wedge \varphi_v(-g^{12}L_{12} - g^{22}L_{22}). \tag{15}$$

Substituting (14) and (15) into (13) gives

$$A'(0) = \int\int t(-2g^{12}L_{12} - g^{22}L_{22} - g^{11}L_{11}) \left\|\varphi_{,1} \wedge \varphi_{,2}\right\|_1 dudv.$$

From (7), we get

$$A'(0) = -2\int\int t(u,v)H \left\|\varphi_{,1} \wedge \varphi_{,2}\right\|_1 dudv$$

which completes the proof.

As a corollary, one deduces that a surface is a critical point for area under all smooth compactly supported variations if and only if the mean curvature vanishes identically.

Definition 2.1 In Galilean space, a regular surface M is called a area minimizing surface (minimal surface) if and only if its mean curvature is zero at each point.

Theorem 2.2 Suppose a surface M is the graph of a function of two variables. Then, the surface M can be parametrized by

$$\varphi(x,y) = (x, y, f(x,y)).$$

The surface M in $\mathbb{G}^3$ is minimal if and only if it can be locally expressed as the graph of a solution of

$$f_{yy} = 0.$$

Proof: The partial derivatives of the surface are obtained as

$$\varphi_x = (1, 0, f_x), \varphi_y = (0, 1, f_y).$$

Thus, we have the unit normal vector as follows

$$N = \frac{(0, -f_y, 1)}{\sqrt{1+f_y^2}}.$$

The components of the second fundamental form are given by

$$L_{11} = \frac{f_{xx}}{\sqrt{1+f_y^2}}, L_{12} = \frac{f_{xy}}{\sqrt{1+f_y^2}}, L_{22} = \frac{f_{yy}}{\sqrt{1+f_y^2}}. \tag{16}$$

On the other hand, using (6) gives

$$g^{11} = g^{12} = 0, g^{22} = \frac{1}{1+f_y^2}. \tag{17}$$

From (7), (16) and (17), we have the Gauss and mean curvatures as follows

$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1+f_y^2)^2}$$

and

$$2H = \frac{f_{yy}}{(1+f_y^2)^{\frac{3}{2}}}.$$

Consequently, this surface is minimal if and only if f_{yy} vanishes.

The geometric interpretation of the above expression is that let us consider the $y-$parameter curves $\varphi(x_0, y) = (x_0, y, f(x_0, y))$ of the surface, hence $f_{yy} = 0$ the $y-$parameter curves are isotropic lines. Thus we state the following corollary.

Corollary 2.1 The minimal surfaces in the Galilean space given with a Monge patch are ruled surfaces type C parametrized by

$$\varphi(x,y) = \alpha(x) + y\beta(x)$$

where $\alpha(x) = (x, 0, z(x))$ is the non-isotropic plane curve, $\beta(x) = (0, 1, w(x))$ is the isotropic plane line.

Example 2.1 Let us consider the surface given by

$$\varphi(x,y) = (x, y, x^3y + xy).$$

It is easy to see that hence we have $f_{yy} = 0$, this surface is a minimal surface shown in Fig.1.

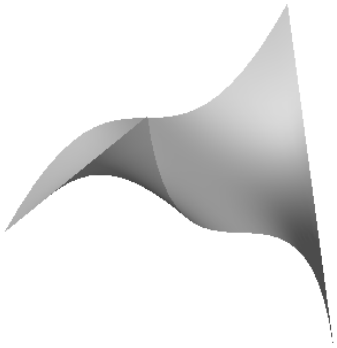

Fig. 1:

This surface is also a ruled surface of type C parametrized by

$$\varphi(x,y) = (x,0,0) + y(0,1,x^3+x).$$

There is not isothermal coordinates in Galilean space. In order to have a similar view of the minimal surface in Galilean space. We give following definition.

Definition 3.2 In Galilean space, a surface $\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$ can be parameterized by using a isothermal-like parameterization as follows

$$g_{11} = g_{22} = g_{12} = \lambda^2 \tag{18}$$

and Euclid isothermal-like parameterization as follows

$$h_{11} = h_{22} = \lambda_1^2, \;\; h_{12} = 0 \tag{19}$$

where g_{11}, g_{22}, g_{12} and h_{11}, h_{22}, h_{12} are the coefficients of the first fundamental forms I_1 and I_2, respectively.

Theorem 2.3 Let M be a surface described by an isothermal-like and Euclid isothermal-like patch parameterization in Galilean space. Then we have

$$w^2 = \|\varphi_u \wedge \varphi_v\|_1^2 = 2\lambda^2\lambda_1^2. \tag{20}$$

Proof: From (3) we have

$$\|\varphi_u \wedge \varphi_v\|_1^2 = x_u^2(z_v^2 + y_v^2) + x_v^2(y_u^2 + z_u^2) - 2x_u x_v(z_u z_v + y_u y_v). \tag{21}$$

It is easy to see that

$$g_{11} = x_u^2, g_{12} = x_u x_v, g_{22} = x_v^2 \tag{22}$$

and

$$h_{11} = y_u^2 + z_u^2, h_{12} = z_u z_v + y_u y_v, h_{22} = z_v^2 + y_v^2. \tag{23}$$

Substituting (22) and (23) into (21) gives

$$w^2 = g_{11}h_{22} + g_{22}h_{11} - 2g_{12}h_{12}.$$

From (18) and (19) we have

$$w^2 = 2\lambda^2\lambda_1^2.$$

Theorem 2.4 Let M be a surface described by an isothermal-like patch in Galilean space. Then the first fundamental form I of the surface is given by

$$I = I_1 + \epsilon I_2 \tag{24}$$

where if $du \neq -dv$ then $\epsilon = 1$, in the other cases $\epsilon = 0$ and I_1, I_2 are obtained as

$$I_1 = \lambda^2(du + dv)^2$$

and

$$I_2 = 2\lambda_1^2 dv^2.$$

Proof: The proof is straightforward.

Theorem 2.5 Let M be a surface described by an isothermal-like patch given by parameterization $\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$ in Galilean space. The surface M is minimal if and only if the partial derivatives of the surface satisfy the following equation

$$\varphi_{uu} - 2\varphi_{uv} + \varphi_{vv} = 0.$$

Proof: From (18) and (22), it follows that

$$x_u(u,v) = \lambda, \;\; x_v(u,v) = \lambda \tag{25}$$

where the partial derivatives of the first component $x(u,v)$ of the surface with respect to u and v is denoted by x_u and x_v, respectively.

Combining (6), (25) and (20) we arrive at

$$g^{11}=\frac{\lambda^2}{w^2},g^{22}=\frac{\lambda^2}{w^2},g^{12}=-\frac{\lambda^2}{w^2}. \tag{26}$$

Substituting (26) into (7) we get

$$\frac{2Hw^2}{\lambda^2}=L_{11}-2L_{12}+L_{22}.$$

Which implies that

$$\frac{2Hw^2}{\lambda^2}=\langle\varphi_{uu}-2\varphi_{uv}+\varphi_{vv},N\rangle_1\,.$$

It follows that

$$\frac{2Hw^2N}{\lambda^2}=\varphi_{uu}-2\varphi_{uv}+\varphi_{vv}.$$

Thus, the surface is minimal if and only if

$$\varphi_{uu}-2\varphi_{uv}+\varphi_{vv}=0$$

holds.

Example 2.2 Let us consider the surface given by

$$\varphi(u,v)=(u+v,u^2-v^2,u-v).$$

From (5) we get

$$g_{11}=g_{12}=g_{22}=1.$$

In addition, it is easy to see that

$$\varphi_{uu}=(0,2,0),\varphi_{uv}=(0,0,0),\varphi_{vv}=(0,-2,0).$$

Hence we have $\varphi_{uu}-2\varphi_{uv}+\varphi_{vv}=0$, the surface is a minimal surface shown in Fig.2.

Fig. 2:

Special cases:

- If $x_u=\lambda$ and $x_v=0$ then, from (6) we have

$$g^{11}=g^{12}=0,g^{22}=\frac{1}{w^2}. \tag{27}$$

It is easy to see that

$$L_{22}=\frac{1}{w}\langle(0,z_vx_u,-y_vx_u),(0,y_{vv},z_{vv})\rangle_1\,. \tag{28}$$

Substituting (27) and (28) into (7) we get

$$2Hw^3=\langle\varphi_{vv},N\rangle_1\,.$$

Thus, the surface is minimal if and only if φ_{vv} vanishes. One of the interesting example of this case is that the ruled surface of type *C*.

Example 2.3 Assume that the ruled surface of type *C* is parametrized by

$$\Phi(u,v)=(u,u^2+v\cos u,v\sin u)$$

where $r(u)=(u,u^2,0)$ is the directrix and $a(u)=(0,\cos u,\sin u)$ is the generator.

Observe that $x_u=1$, $x_v=0$ and $\Phi_{vv}=0$. Thus this surface is a minimal surface shown in Fig.3.

- If $x_u=0$ and $x_v=\lambda$ then, similar to the previous case, the surface is minimal if and only if $\varphi_{uu}=0$ holds. One of the exciting example of this case is the helicoid parametrized by

$$\varphi(u,v)=(v,u\cos v,u\sin v).$$

- If $x_u=x_v=0$ then, the surface is a part of a plane with vanishing mean curvature.

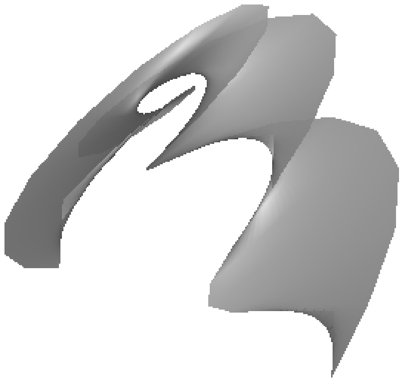

Fig. 3:

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