On Minimal Surfaces in Galilean Space

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Abstract: In this paper, we investigated the minimal surfaces in three dimensional Galilean space \( G^3 \). We showed that the condition of minimality of a surface area is locally equivalent to the mean curvature vector \( H \) vanishes identically. Then, we derived the necessary and sufficient conditions that the minimal surfaces have to satisfy in Galilean space.

**Keywords:** Minimal surfaces, Area of a surface, Galilean space.

1 Introduction

Minimal surfaces are one of the most interesting subject in mathematics. The study and computation of minimal surfaces has a long history [12]. Lagrange made the first investigation of the minimal surfaces by asking a simple question named as Plateau’s problem which concerns with finding a surface of least area that spans a given fixed one-dimensional contour in three-dimensional Euclidean space [15]. Later G. Monge discovered that the condition for minimality of a surface leads to the condition that vanishing mean curvature, and therefore surfaces with finding a surface of least area that spans a given fixed one-dimensional contour in three-dimensional Euclidean space [15]. Later G. Monge discovered that the condition for minimality of a surface leads to the condition that vanishing mean curvature, and therefore surfaces with vanishing

Let \( a \) and \( b \) be vectors in the Galilean space. The scalar product

\[
G \cdot \frac{a}{G} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}
\]

is defined by

In addition, when both of the vectors \( p = (0, y, z) \) and \( q = (0, y_1, z_1) \) are isotropic, the scalar product \( <,> \) is given by

\[
< p, q > = y y_1 + z z_1.
\]

Let \( M \) be a surface in \( G^3 \) given by parametrization

\[
\varphi(v^1, v^2) = (x(v^1, v^2), y(v^1, v^2), z(v^1, v^2)).
\]

The isotropic unit normal vector \( N \) is defined by

\[
N = \frac{\varphi_1 \wedge \varphi_2}{|w|}
\]
where the partial derivatives of the surface \(M\) with respect to \(v^1\) and \(v^2\) is denoted by \(\varphi_1\) and \(\varphi_2\), respectively and \(w = \|\varphi_1 \wedge \varphi_2\|_1\) [14]. The first fundamental form of the surface is given by

\[
I = I_1 + \epsilon I_2
\]

where \(I_1 = g_{ij}dv^i dv^j\) and \(I_2 = h_{ij} dv^i dv^j\). If \(I_1 = 0\) then \(\epsilon = 1\), in the other cases \(\epsilon = 0\). The induced metrics \(h_{ij}\) and \(g_{ij}\) \((i,j = 1,2)\) on the surface are given by

\[
h_{ij} = \langle \varphi_i, \varphi_j \rangle_1, \quad g_{ij} = \langle \varphi_i, \varphi_j \rangle.
\]

The components of the inverse metric are given by

\[
g^1 = \frac{x_2}{w}, \quad g^2 = -\frac{x_1}{w} g^{ij} = g^j g^j
\]

where the partial derivatives of the first component \(x(v^1, v^2)\) of the surface with respect to \(v^1\) and \(v^2\) is denoted by \(x_1\) and \(x_2\), respectively. In [14], the coefficients \(L_{ij}\) of the second fundamental form, the Gauss curvature \(K\) and mean curvature \(H\) of the surface are given by

\[
L_{ij} = \langle \varphi_i, \varphi_j \rangle x^1 - \langle x, x_i \varphi_j \rangle x^1, N > 1, K = \frac{\det L_{ij}}{w^2}, 2H = g^{ij} L_{ij}.
\]

The partial derivatives of the normal vector is obtained by

\[
N_i = -g^{jm} L_{ij} \varphi_m.
\]

\section{Minimal surfaces in galilean space}

In this section, we will give a mathematical definition of the minimal surface in Galilean space \(G^3\). Since the minimal surfaces locally minimizes area, firstly we need to show that it is also meaningful in Galilean space. Similar to the three dimensional Euclidean space, the norm of the cross product measures the area spanned by two vectors in the three dimensional Galilean space [17]. Therefore we state the following definition.

Let \(M\) be the surface parametrized by \(\varphi(u,v) = (x(u,v), y(u,v), z(u,v))\). We can see that \(\|\varphi_u \wedge \varphi_v\|_1\) is the area of the parallelogram determined by \(\varphi_u\) and \(\varphi_v\). Therefore, in Galilean space, the area of the surface can be obtained by

\[
A(\varphi) = \int \|\varphi_u \wedge \varphi_v\|_1 \, du dv.
\]

The minimal surface is the problem of minimizing \(A(\varphi)\). To do this, let us consider a normal variation of the surface \(M\) in Galilean space. Let \(t(u,v)\) be any smooth function on such that it vanishes on the boundary of the surface and \(N\) be the unit surface normal. For some small \(\lambda\) a normal variation of the surface \(M^\sigma\) can be parametrized by

\[
\omega^\sigma (u,v) = \varphi (u,v) + \sigma t(u,v) N(u,v)
\]

where \(-\lambda < \sigma < \lambda\). This motivates the following theorem:

\textbf{Theorem 2.1} Let \(A(\sigma)\) be the area of the normal variation of the surface \(M\) in Galilean space. The critical point of the area of the normal variation \(M^\sigma\) is given by

\[
A'(0) = -2 \iint t(u,v) H \|\varphi_u \wedge \varphi_v\|_1 \, du dv
\]

where prime denotes differentiation respect to \(\sigma\).

\textbf{Proof:} The area \(A(\sigma)\) of the surface \(M^\sigma\) is given by

\[
A(\sigma) = \int \|\omega_u^\sigma \wedge \omega_v^\sigma\|_1 \, du dv.
\]

On the other hand from (10), the partial derivatives of the normal variation \(M^\sigma\) are obtained as

\[
\omega_u^\sigma = \varphi_u + \sigma t u N + \sigma t v N, \quad \omega_v^\sigma = \varphi_v + \sigma t u N + \sigma t v N.
\]

By using \(N_u \wedge N_v = 0\) we arrive at

\[
\omega_u^\sigma \wedge \omega_v^\sigma = \varphi_u \wedge \varphi_v + t \sigma (\varphi_u \wedge N_v + N_u \wedge \varphi_v) + \sigma^2 t (t u N \wedge N_v + t v N_u \wedge N).
\]

Taking norms on both sides of above equation, then differentiating with respect to \(\sigma\), finally putting \(\sigma = 0\) into the result gives

\[
A'(0) = \int \frac{t \langle \varphi_u \wedge N_v + N_u \wedge \varphi_v, \varphi_u \wedge \varphi_v \rangle_1}{\|\varphi_u \wedge \varphi_v\|_1} \, du dv
\]

where \(\frac{dA(\sigma)}{d\sigma}_{\sigma=0} = A'(0)\).
On the other hand, from (8) we have
\[ N_u \land \varphi_v = (-g^{11} L_{11} - g^{12} L_{12}) \varphi_u \land \varphi_v \] (14)
and
\[ \varphi_u \land N_v = \varphi_u \land \varphi_v (-g^{12} L_{12} - g^{22} L_{22}). \] (15)

Substituting (14) and (15) into (13) gives
\[ A'(0) = \int \int t(-2g^{12} L_{12} - g^{22} L_{22} - g^{11} L_{11}) \left\| \varphi_1 \land \varphi_2 \right\|_1 dudv. \]

From (7), we get
\[ A'(0) = -2 \int \int t(u, v) H \left\| \varphi_1 \land \varphi_2 \right\|_1 dudv \]
which completes the proof.

As a corollary, one deduces that a surface is a critical point for area under all smooth compactly supported variations if and only if the mean curvature vanishes identically.

**Definition 2.1** In Galilean space, a regular surface \( M \) is called a area minimizing surface (minimal surface) if and only if its mean curvature is zero at each point.

**Theorem 2.2** Suppose a surface \( M \) is the graph of a function of two variables. Then, the surface \( M \) can be parametrized by
\[ \varphi(x, y) = (x, y, f(x, y)). \]

The surface \( M \) in \( G^3 \) is minimal if and only if it can be locally expressed as the graph of a solution of
\[ f_{yy} = 0. \]

**Proof:** The partial derivatives of the surface are obtained as
\[ \varphi_x = (1, 0, f_x), \varphi_y = (0, 1, f_y). \]

Thus, we have the unit normal vector as follows
\[ N = \frac{(0, -f_y, 1)}{\sqrt{1 + f_y^2}}. \]

The components of the second fundamental form are given by
\[ L_{11} = \frac{f_{xx}}{\sqrt{1 + f_y^2}}, L_{12} = \frac{f_{xy}}{\sqrt{1 + f_y^2}}, L_{22} = \frac{f_{yy}}{\sqrt{1 + f_y^2}}. \] (16)

On the other hand, using (6) gives
\[ g^{11} = g^{12} = 0, g^{22} = \frac{1}{1 + f_y^2}. \] (17)

From (7), (16) and (17), we have the Gauss and mean curvatures as follows
\[ K = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_y^2)^2}, \]
\[ 2H = \frac{f_{yy}}{(1 + f_y^2)^2}, \]

Consequently, this surface is minimal if and only if \( f_{yy} \) vanishes. The geometric interpretation of the above expression is that let us consider the \( y \)–parameter curves \( \varphi(x_0, y) = (x_0, y, f(x_0, y)) \) of the surface, hence \( f_{yy} = 0 \) the \( y \)–parameter curves are isotropic lines. Thus we state the following corollary.

**Corollary 2.1** The minimal surfaces in the Galilean space given with a Monge patch are ruled surfaces type C parametrized by
\[ \varphi(x, y) = \alpha(x) + y\beta(x) \]
where \( \alpha(x) = (x, 0, z(x)) \) is the non-isotropic plane curve, \( \beta(x) = (0, 1, w(x)) \) is the isotropic plane line.

**Example 2.1** Let us consider the surface given by
\[ \varphi(x, y) = (x, y, x^3 y + xy). \]

It is easy to see that hence we have \( f_{yy} = 0 \), this surface is a minimal surface shown in Fig.1.
This surface is also a ruled surface of type C parametrized by
\[ \varphi(x, y) = (x, 0, 0) + y(0, 1, x^3 + x). \]

There is not isothermal coordinates in Galilean space. In order to have a similar view of the minimal surface in Galilean space. We give following definition.

**Definition 3.2** In Galilean space, a surface \( \varphi(u, v) = (x(u, v), y(u, v), z(u, v)) \) can be parameterized by using a isothermal-like parameterization as follows
\[ g_{11} = g_{22} = g_{12} = \lambda^2 \tag{18} \]
and Euclid isothermal-like parameterization as follows
\[ h_{11} = h_{22} = \lambda_1^2, \quad h_{12} = 0 \tag{19} \]
where \( g_{11}, g_{22}, g_{12} \) and \( h_{11}, h_{22}, h_{12} \) are the coefficients of the first fundamental forms \( I_1 \) and \( I_2 \), respectively.

**Theorem 2.3** Let \( M \) be a surface described by an isothermal-like and Euclid isothermal-like patch parameterization in Galilean space. Then we have
\[ w^2 = \| \varphi_u \wedge \varphi_v \|^2_1 = 2\lambda^2\lambda_1^2. \tag{20} \]

**Proof:** From (3) we have
\[ \| \varphi_u \wedge \varphi_v \|^2_1 = x_u^2(x_u^2 + y_u^2) + x_v^2(y_u^2 + z_u^2) - 2x_u x_v(z_u y_v + y_u z_v). \tag{21} \]

It is easy to see that
\[ g_{11} = x_u^2, g_{12} = x_u x_v, g_{22} = x_v^2 \tag{22} \]
and
\[ h_{11} = y_u^2 + z_u^2, h_{12} = z_u z_v + y_u y_v, h_{22} = z_v^2 + y_v^2. \tag{23} \]
Substituting (22) and (23) into (21) gives
\[ w^2 = g_{11} h_{22} + g_{22} h_{11} - 2g_{12} h_{12}. \]
From (18) and (19) we have
\[ w^2 = 2\lambda^2\lambda_1^2. \]

**Theorem 2.4** Let \( M \) be a surface described by an isothermal-like patch in Galilean space. Then the first fundamental form \( I \) of the surface is given by
\[ I = I_1 + \epsilon I_2 \tag{24} \]
where if \( du \neq -dv \) then \( \epsilon = 1 \), in the other cases \( \epsilon = 0 \) and \( I_1, I_2 \) are obtained as
\[ I_1 = \lambda^2(du + dv)^2 \]
and
\[ I_2 = 2\lambda_1^2 dv^2. \]

**Proof:** The proof is straightforward.

**Theorem 2.5** Let \( M \) be a surface described by an isothermal-like patch given by parameterization \( \varphi(u, v) = (x(u, v), y(u, v), z(u, v)) \) in Galilean space. The surface \( M \) is minimal if and only if the partial derivatives of the surface satisfy the following equation
\[ \varphi_{uu} - 2\varphi_{uv} + \varphi_{vv} = 0. \]

**Proof:** From (18) and (22), it follows that
\[ x_u(u, v) = \lambda, \quad x_v(u, v) = \lambda \tag{25} \]
where the partial derivatives of the first component \( x(u, v) \) of the surface with respect to \( u \) and \( v \) is denoted by \( x_u \) and \( x_v \), respectively.
Combining (6), (25) and (20) we arrive at
\[
g^{11} = \frac{\lambda^2}{w^2}, \quad g^{22} = \frac{\lambda^2}{w^2}, \quad g^{12} = -\frac{\lambda^2}{w^2}. \tag{26}
\]
Substituting (26) into (7) we get
\[
\frac{2Hw^2}{\lambda^2} = L_{11} - 2L_{12} + L_{22}.
\]
Which implies that
\[
\frac{2Hw^2}{\lambda^2} = \langle \varphi_{uu} - 2\varphi_{uv} + \varphi_{vv}, N \rangle_1.
\]
It follows that
\[
\frac{2Hw^2N}{\lambda^2} = \varphi_{uu} - 2\varphi_{uv} + \varphi_{vv}.
\]
Thus, the surface is minimal if and only if
\[
\varphi_{uu} - 2\varphi_{uv} + \varphi_{vv} = 0
\]
holds.

**Example 2.2** Let us consider the surface given by
\[
\varphi(u, v) = (u + v, u^2 - v^2, u - v).
\]
From (5) we get
\[
g_{11} = g_{12} = g_{22} = 1.
\]
In addition, it is easy to see that \(\varphi_{uu} = (0, 2, 0), \varphi_{uv} = (0, 0, 0), \varphi_{vv} = (0, -2, 0)\).
Hence we have \(\varphi_{uu} - 2\varphi_{uv} + \varphi_{vv} = 0\), the surface is a minimal surface shown in Fig.2.

![Fig. 2:](image)

**Special cases:**

- If \(x_u = \lambda\) and \(x_v = 0\) then, from (6) we have
  \[
g^{11} = g^{12} = 0, \quad g^{22} = \frac{1}{w^2}. \tag{27}
\]
  It is easy to see that
  \[
  L_{22} = \frac{1}{w} \langle (0, z_u x_u - y_u x_u), (0, y_u x_u, z_u x_u) \rangle_1. \tag{28}
  \]
  Substituting (27) and (28) into (7) we get
  \[
  2Hw^3 = \langle \varphi_{uu}, N \rangle_1.
  \]
  Thus, the surface is minimal if and only if \(\varphi_{uu}\) vanishes. One of the interesting example of this case is that the ruled surface of type \(C\).

**Example 2.3** Assume that the ruled surface of type \(C\) is parametrized by
\[
\Phi(u, v) = (u, u^2 + v \cos u, v \sin u)
\]
where \(r(u) = (u, u^2, 0)\) is the directrix and \(a(u) = (0, \cos u, \sin u)\) is the generator.
Observe that \(x_u = 1, x_v = 0\) and \(\Phi_{vv} = 0\). Thus this surface is a minimal surface shown in Fig.3.

- If \(x_u = 0\) and \(x_v = \lambda\) then, similar to the previous case, the surface is minimal if and only if \(\varphi_{uu} = 0\) holds. One of the exciting example of this case is the helicoid parametrized by
  \[
  \varphi(u, v) = (v, u \cos v, u \sin v).
  \]
- If \(x_u = x_v = 0\) then, the surface is a part of a plane with vanishing mean curvature.
3 References