Special Helices on the Ellipsoid

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Abstract: In this study, we investigate three types of special helices whose axis is a fixed constant Killing vector field on the Ellipsoid $S^{2}_{a_1,a_2,a_3}$ in $\mathbb{R}^{3}_{a_1,a_2,a_3}$. Then, we obtain the curvatures of all special helices on the ellipsoid $S^{2}_{a_1,a_2,a_3}$ and give some characterizations of these curves. Moreover, we present various examples and visualize their images using the Mathematica program.

Keywords: Frame fields, Killing vector field, Special curves and surfaces.

1 Introduction

The spherical curves are the special space curves that lie on the sphere. If the sphere is constructed by using the elliptical inner product, then the elliptical 2-sphere is obtained. This sphere is an ellipsoid according to the Euclidean sense. We summarize some studies about spherical curves: Firstly, Wong proved the condition for a curve to be on a sphere and gave some characterizations for this curve [10, 11]. In [3], Breuer et al. gave an explicit characterization of the spherical curve. In [6], the author investigated the characterization of the dual spherical curve. Then, in [2], the author obtained a differential equation for characterizing of the dual spherical curves. Besides, in [4], İlarslan presented the spherical curve characterization for non-null regular curves in Lorentzian 3-space. Ayyıldız introduced the dual Lorentzian spherical curves [1]. Moreover, Izumiya and Takeuchi defined the slant helices and conical geodesic curve and gave a classification of special developable surfaces under the condition of the existence of such a special helix as a geodesic [5]. Scofield derived a curve of constant precession and proved that this curve is tangent indicatrix of a spherical helix [9].

In the present work, we give some characterizations for the special helices whose axis is the fixed constant Killing vector field on the elliptical 2-sphere. Furthermore, we give various examples and draw their images by using the Mathematica program.

2 Preliminaries

Let we take $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3) \in \mathbb{R}^{3}$ and $a_1, a_2, a_3 \in \mathbb{R}^{+}$ then the elliptical inner product defined as

$$B : \mathbb{R}^{3} \times \mathbb{R}^{3} \rightarrow \mathbb{R}; B(u, v) = a_1 x_1 y_1 + a_2 x_2 y_2 + a_3 x_3 y_3,$$  \hspace{1cm} (1)

The 3-dimensional real vector space $\mathbb{R}^{3}$ equipped with the elliptical inner product will be represented by $\mathbb{R}^{3}_{a_1,a_2,a_3}$. The norm of a vector associated with the scalar product $B$ is defined as

$$\|u\|_B = \sqrt{B(u, u)}.$$  \hspace{1cm} (2)

Two vectors $u$ and $v$ are called elliptically orthogonal vectors if $B(u, v) = 0$. In addition, if $u$ is an elliptically orthonormal vector then $B(u, u) = 1$. The cosine of the angle between two vectors $u$ and $v$ is defined as

$$\cos \theta = \frac{B(u, v)}{\|u\|_B \|v\|_B},$$  \hspace{1cm} (3)

where $\theta$ is compatible with the parameters of the angular parametric equations of ellipse or ellipsoid. The cross product of two vector fields $X, Y \in \mathbb{R}^{3}_{a_1,a_2,a_3}$ is given by

$$X \times_{E} Y = \Delta \begin{vmatrix} \frac{c_1}{a_1} & \frac{c_2}{a_2} & \frac{c_3}{a_3} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix},$$  \hspace{1cm} (4)

where $\Delta = \sqrt{a_1 a_2 a_3}, a_1, a_2, a_3 \in \mathbb{R}^{+}$ [7].
Let us take the ellipsoid denoted by $S^2_{a_1,a_2,a_3}$ in $\mathbb{R}^3_{a_1,a_2,a_3}$. Then, the sectional curvature of the ellipsoid generated by the non-degenerated plane $\{u, v\}$ is defined as

$$K(u, v) = \frac{B(R(u, v)u, v)}{B(u, u)B(v, v) - B(u, v)^2},$$

where $R$ is the Riemannian curvature tensor given by

$$R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X, Y]} Z.$$  \hspace{1cm} (6)

The ellipsoid has the constant sectional curvature. Therefore, the curvature tensor $R$ is written as follows

$$R(X, Y)Z = C\{B(Z, X)Y - B(Z, Y)X\},$$

where $C$ is the constant sectional curvature.

A curve $\gamma$ on the ellipsoid $S^2_{a_1,a_2,a_3}$ defined by $\gamma(s) = \varphi(\alpha(s))$ and a unit normal vector field $Z$ along the surface $S^2_{a_1,a_2,a_3}$ defined

$$Z = \frac{\varphi_u \times E \varphi_v}{\|\varphi_u \times \varphi_v\|}.$$  \hspace{1cm} (8)

Since $S^2_{a_1,a_2,a_3}$ is sphere according to the elliptical inner product, the unit normal vector field $Z$ along the surface $S^2_{a_1,a_2,a_3}$ equal to the position vector of the curve $\gamma$. Then, we found an orthonormal frame $\{t = \gamma', y = \gamma \times E \gamma', \gamma\}$ which is called the elliptical Darboux frame along the curve $\gamma$. The corresponding Darboux formulae of $\gamma$ is written as

$$t' = -\gamma + k_{ge} y,$$

$$\gamma' = t,$$

$$y' = -k_{ge} t,$$

where $k_{ge} = -1, k_{ge} = B(\gamma'', y)$ and $\tau_r = 0$ are geodesic curvature, asymptotic curvature, and principal curvature of $\gamma$ on the surface $S^2_{a_1,a_2,a_3}$, respectively. Moreover, it is found as the following relation

$$y \times E t = \gamma, \ z \times E y = t, \ z \times E t = -y.$$  \hspace{1cm} (10)

[8].

**Lemma 1.** Let $\varphi : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3_{a_1,a_2,a_3}, \varphi(U) = S^2_{a_1,a_2,a_3}$ be an ellipsoid and $\gamma : I \subset \mathbb{R} \rightarrow U$ be a regular curve on the $S^2_{a_1,a_2,a_3}$. Provided that $V$ be a vector field along the curve $\gamma$ then the variation of $\gamma$ defined by $\Gamma : I \times (-\varepsilon, \varepsilon) \rightarrow S^2_{a_1,a_2,a_3}(C)$ with $\gamma(s, 0)$ the initial curve satisfy $\Gamma(s, 0) = \gamma(s)$. The variations of the geodesic curvature function $k_{ge}(s, w)$ and the speed function $v(s, w)$ at $w = 0$ are calculated as follows:

$$V(v) = \left(\frac{\partial v}{\partial w}(s, w)\right)\bigg|_{w=0} = -v\rho,$$

$$V(k_{ge}) = \left(\frac{\partial k_{ge}}{\partial w}(s, w)\right)\bigg|_{w=0} = B(-R(V, t)t + \nabla^2_1 V, y) - \frac{1}{v_E} B(-R(V, t)t + \nabla^2_1 V, \gamma),$$

where $\rho = B(\nabla t, V)$ and $R$ stands for the curvature tensor of $S^2_{a_1,a_2,a_3}$ [8].

**Proposition 1.** If $V(s)$ is the restriction to $\gamma(s)$ of a Killing vector field $V$ of $S^2_{a_1,a_2,a_3}$ then the variations of the elliptical Darboux curvature functions and speed function of $\gamma$ satisfy:

$$V(v) = V(k_{ge}) = 0,$$  \hspace{1cm} (12)

[8].

### 3 Special helices on the ellipsoid $S^2_{a_1,a_2,a_3}$

**Definition 1.** Let $\varphi : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3_{a_1,a_2,a_3}, \varphi(U) = S^2_{a_1,a_2,a_3}$ be an ellipsoid and $\gamma : I \subset \mathbb{R} \rightarrow U$ be a regular curve on the $S^2_{a_1,a_2,a_3}$. Then we say that $\gamma$ is a type-1 special helix, type-2 special helix, or type-3 special helix if $B(V, t) = \text{const.}, B(V, \gamma) = \text{const.},$ and $B(V, y) = \text{const.}$, respectively.

**Theorem 1.** Let $\varphi : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3, \varphi(U) = S^2_{a_1,a_2,a_3}$ be an ellipsoid and $\gamma : I \subset \mathbb{R} \rightarrow U$ be a regular curve on $S^2_{a_1,a_2,a_3}$ and $V$ be a Killing vector field along the curve $\gamma$. Then $\gamma$ is a type-1 special helix with the axis $V$ if and only if the geodesic curvature of the curve $\gamma$ satisfy the following equation:

$$k_{ge} = \cot \theta,$$

where $\theta$ satisfy

$$\theta'' \sin^2 \theta - \omega \theta' \cos \theta = 0,$$

[8].
Now, we can give the following corollary without proof. The proof of the corollary similar to Scofield’s work [9].

Corollary 1. Let \( \varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3, \varphi(U) = S^2_{a_1,a_2,a_3} \) be an ellipsoid and \( \gamma : I \subset \mathbb{R} \to U \) be a type-1 special helix with the Killing axis \( V \) on \( S^2_{a_1,a_2,a_3} \). Then, the integral curve of \( \gamma \) is an elliptical constant processation curve on the elliptical hyperboloid.

Theorem 2. Let \( \varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3_{a_1,a_2,a_3}, \varphi(U) = S^2_{a_1,a_2,a_3} \) be an ellipsoid and \( \gamma : I \subset \mathbb{R} \to U \) be a regular curve on the \( S^2_{a_1,a_2,a_3} \). Then \( \gamma \) is a type-2 special helix with the axis \( V \) if and only if the geodesic curvature of the curve \( \gamma \) satisfy the following equation:

\[
k_{ge} = \frac{c_1}{\sin \theta} - \theta',
\]

where \( c \) is a constant.

Proof: If \( \gamma \) is a type-2 special helix with the Killing axis \( V \) then \( V \) is written as

\[
V = \cos \theta t + c_1 \gamma + \sin \theta y, \quad c_1 = \text{const}.
\]

Differentiating eq.(14) with respect to \( s \), we found the following equation

\[
\nabla_T V = ((- \theta' - k_{ge}) \sin \theta + c_1) t + (- \cos \theta) \gamma + (\cos \theta k_{ge} + \theta' \cos \theta) y.
\]

Using the equation \( V = 0 \) in Lemma 1, we found

\[
k_{ge} = \frac{c_1}{\sin \theta} - \theta'
\]

The differentiation of eq.(15) is obtained as

\[
\nabla^2_T V = (-1 - k_{ge}^2 + k_{ge} \theta') \cos \theta t + \theta' \sin \theta y + ((k_{ge} + \theta') \cos \theta) y.
\]

Moreover, we have the following equation

\[
R(V, t) t = C(B(t, V) t - B(t, V)).
\]

Using the Darboux frame equations and eq.(14), we deduce

\[
R(V, t) t = -C(c_1 \gamma + \sin \theta y).
\]

Considering the eq.(17) and eq.(19) with the second equation in Lemma 1 and the Proposition 1, we reach the following equations

\[
\theta = \text{const. or } (C + 1) \sin^2 \theta - c_1 \theta' \sin \theta + c_1 = 0.
\]

\( \square \)

Corollary 2. Let \( \gamma \) be a type-2 special helix on the ellipsoid with the axis

\[
V = \cos \theta t + c_1 \gamma + \sin \theta y, \quad \theta = \text{const.},
\]

then \( \gamma \) has the following parametric representation

\[
\gamma(s) = A_1 + \frac{A_2}{\sqrt{1 + \frac{c_1^2}{\sin^2 \theta}}} \cos \left( \sqrt{1 + \frac{c_1^2}{\sin^2 \theta}} \right) + \frac{A_3}{\sqrt{1 + \frac{c_1^2}{\sin^2 \theta}}} \sin \left( \sqrt{1 + \frac{c_1^2}{\sin^2 \theta}} s \right),
\]

where \( A_1, A_2, A_3 \in \mathbb{R}^3_{a_1,a_2,a_3} \) and \( c_1 \in \mathbb{R} \).

Proof: Let \( \gamma \) be a type-2 special helix on the ellipsoid with the axis

\[
V = \cos \theta t + c_1 \gamma + \sin \theta y, \quad \theta = \text{const.},
\]

then the elliptical curvature of \( \gamma \) calculated as

\[
k_{ge} = \frac{c_1}{\sin \theta}.
\]

On the other hand, from the Darboux frame equations \( \gamma \) satisfy the following third order differential equation

\[
k_{ge} \gamma''' - k_{ge}^2 \gamma'' + (k_{ge}^3 + k_{ge}) \gamma' - k_{ge}' \gamma = 0.
\]

If \( k_{ge} \) is written in the eq.(22) and the differential equation is solved then it is obtained that \( \gamma \) has the following parametric representation

\[
\gamma(s) = A_1 + \frac{A_2}{\sqrt{1 + \frac{c_1^2}{\sin^2 \theta}}} \cos \left( \sqrt{1 + \frac{c_1^2}{\sin^2 \theta}} \right) + \frac{A_3}{\sqrt{1 + \frac{c_1^2}{\sin^2 \theta}}} \sin \left( \sqrt{1 + \frac{c_1^2}{\sin^2 \theta}} s \right),
\]

where \( A_1, A_2, A_3 \in \mathbb{R}^3_{a_1,a_2,a_3} \) and \( c_1 \in \mathbb{R} \).
Theorem 3. Let \( \varphi : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3_{a_1,a_2,a_3}, \varphi(U) = \mathbb{S}^2_{a_1,a_2,a_3} \) be an ellipsoid and \( \gamma : I \subset \mathbb{R} \rightarrow U \) be a regular curve on the \( \mathbb{S}^2_{a_1,a_2,a_3} \). Then \( \gamma \) is type-3 special helix with the axis \( V \) if and only if the geodesic curvature of the curve \( \gamma \) satisfy the following equation:

\[
k_{ge} = \frac{(1 - \theta') \sin \theta}{c_2},
\]

here \( \theta \) satisfies

\[
(1 - \theta') \sin \theta (- c_2^2 \theta'' - \theta' \sin \theta \cos \theta + (1 - \theta') \theta' \cos \theta) = \theta' \theta'' \cos \theta = 0,
\]

where \( c_2 \) is a constant.

Proof: If \( \gamma \) is a type-3 special helix with the Killing axis \( V \) then \( V \) is written as

\[
V = \cos \theta t + \sin \theta y + c_2 y.
\]

By differentiating eq.(25), we get

\[
\nabla_T V = ((1 - \theta') \sin \theta - c_2 k_{ge}) t + (1 - \theta') \cos \theta \gamma + k_{ge} \cos \theta y.
\]

By using the equation \( V(v) = 0 \) in Lemma 1, we reach

\[
k_{ge} = \frac{(1 - \theta') \sin \theta}{c_2}.
\]

If we take the differentiation of eq.(26), we obtain

\[
\nabla^2_T V = ((1 - \theta') \cos \theta - k_{ge} \cos \theta) t + (\theta'' \cos \theta - (1 - \theta') \theta' \sin \theta \gamma) + \frac{k_{ge} \cos \theta - k_{ge} \theta' \sin \theta \gamma) y}.
\]

Furthermore, we have the following equation

\[
R(V, t) t = C(B(t, V)t - B(t, t)V).
\]

By using the Darboux frame equations and eq.(25), we obtain

\[
R(V, T) T = C(- \sin \theta \gamma - c_2 y).
\]

If we consider the eq.(28) and eq.(30) with the second equation in Lemma 1 and the Proposition 1, we deduce

\[
\theta = \text{const.}
\]

or satisfy the following equation

\[
(1 - \theta') \sin \theta (- c_2^2 \theta'' - \theta' \sin \theta \cos \theta + (1 - \theta') \theta' \cos \theta) = \theta' \theta'' \cos \theta = 0.
\]

\[\square\]

Corollary 3. Let \( \gamma \) be a type-3 special helix on the ellipsoid with the axis

\[
V = \cos \theta t + \sin \theta y + c_2 y, \quad \theta = \text{const.},
\]

then \( \gamma \) has the following parametric representation

\[
\gamma(s) = B_1 + \frac{B_2}{\sqrt{1 + \sin^2 \theta}} \cos(\sqrt{1 + \frac{\sin^2 \theta}{c_2^2}}) + \frac{B_3}{\sqrt{1 + \sin^2 \theta}} \sin(\sqrt{1 + \frac{\sin^2 \theta}{c_2^2}}) s,
\]

where \( B_1, B_2, B_3 \in \mathbb{R}^3_{a_1,a_2,a_3} \) and \( c_2 \in \mathbb{R} \).

Proof: Let \( \gamma \) be a type-3 special helix on the ellipsoid with the axis

\[
V = \cos \theta t + \sin \theta y + c_2 y, \quad \theta = \text{const.},
\]

then the elliptical curvature of \( \gamma \) calculated as

\[
k_{ge} = \frac{\sin \theta}{c_2}.
\]

On the other hand, from the Darboux frame equations, \( \gamma \) satisfy the following third order differential equation

\[
k_{ge} \gamma'''' - k_{ge} \gamma'' + (k_{ge}^3 + k_{ge}) \gamma' - k_{ge}^3 \gamma = 0.
\]

If \( k_{ge} \) is written in the eq.(34) and the differential equation is solved then it is obtained that \( \gamma \) has the following parametric representation

\[
\gamma(s) = B_1 + \frac{B_2}{\sqrt{1 + \sin^2 \theta}} \cos(\sqrt{1 + \frac{\sin^2 \theta}{c_2^2}}) + \frac{B_3}{\sqrt{1 + \sin^2 \theta}} \sin(\sqrt{1 + \frac{\sin^2 \theta}{c_2^2}}) s,
\]

where \( B_1, B_2, B_3 \in \mathbb{R}^3_{a_1,a_2,a_3} \) and \( c_2 \in \mathbb{R} \).

\[\square\]
In the following examples we give various special helices on the ellipsoid.

Examle 1. Let us take the curve parameterized as
\[
\gamma(s) = \frac{1}{2} (1 + k) \cos(1 - k)t - (1 - k) \cos(1 + k)t \frac{1}{2} (1 + k) \sin(1 - k)t - (1 - k) \sin(1 + k)t \sqrt{1 - k^2} \cos kt.
\]

The elliptical curvature of the helix calculated as
\[
k_{gE}(s) = \cot(ks).
\]

Thus, we can easily see that \(\gamma\) is a type-1 special helix. It is illustrated in Figure 1.

![Figure 1. Type-1 special Helices on the Ellipsoid \(S^2_{2,4,9}\), \(k = 0.505\).](image)

Examle 2. Type-2 (type-3) special helices corresponding to different values of the \(A_i, B_i, i = 1, 2, 3\). are illustrated in Figure 2.

![Figure 2. Type-2 (type-3) Special Helices on the Ellipsoid \(S^2_{2,4,9}\).](image)

4 References