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ON THE CONVERGENCE THEOREMS OF AN IMPLICIT ITERATION PROCESS FOR ASYMPTOTICALLY QUASI I-NONEXPANSIVE MAPPINGS

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Abstract

In this paper, we establish an implicit iterative process for convergence to a common fixed point of two asymptotically quasi I-nonexpansive mappings in Banach spaces, and prove weak and strong convergence of this process to a common fixed point of such mappings. Our results improve and extend corresponding results of [7, 10, 14, 18] to two asymptotically quasi I-nonexpansive mappings.

Keywords: Asymptotically quasi I-nonexpansive mappings; Common fixed point; Uniformly convex Banach space

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1. Introduction

Throughout this paper, $\mathbb{N}$ will denote the set of all positive integers. Let K be a nonempty subset of a real normed linear space E and T be a self-mapping of K. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$ and by $F := F(T) \cap F(S) = \{x \in K : Tx = Sx = x\}$, the set of common fixed points of the mappings T and S. Now let us recall some known definitions.

1.1. Definition. A mapping $T : K \to K$ is said to be

(i) nonexpansive, if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$.

(ii) asymptotically nonexpansive, if there exists a real sequence $\{\lambda_n\} \subset [1, \infty)$, with $\lim_{n\to\infty} \lambda_n = 1$, such that $\|T^n x - T^n y\| \leq \lambda_n \|x - y\|$ for all $x, y \in K$.

(iii) quasi-nonexpansive, if $\|Tx - p\| \leq \|x - p\|$ for all $x \in K$ and $p \in F(T)$ where $F(T)$ is the set of all fixed points of T.

(iv) asymptotically quasi-nonexpansive, if there exists a real sequence $\{\lambda_n\} \subset [1, \infty)$, with $\lim_{n\to\infty} \lambda_n = 1$, such that $\|T^n x - p\| \leq \lambda_n \|x - p\|$ for all $x \in K$ and $p \in F(T)$.

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(v) uniformly L-Lipschitzian, if there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \leq L \|x - y\|$ for all $x, y \in K$.

1.2. Remarks. Note that from the above definitions, it is easy to see that if $F(T)$ is nonempty, a nonexpansive mapping must be quasi-nonexpansive, and an asymtotically nonexpansive mapping must be asmyptotically quasi-nonexpansive. It is obvious that, an asymptotically nonexpansive mapping is also uniformly L-Lipschitzian with $L = \sup\{\lambda_n : n \in \mathbb{N}\}$. However, the converses of these claims are not true in general (see [6]).

If K is a closed nonempty subset of a Banach space and $T : K \to K$ is nonexpansive, then it is known that T may not have a fixed point (unlike the case if T is a strict contraction), and even when it has, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ (the so-called Picard sequence) may fail to converge to such a fixed point.

In [1] Browder studied the iterative construction for fixed points of nonexpansive mappings on closed and convex subsets of a Hilbert space. Note that for the past 30 years or so, the studies of the iterative processes for the approximation of fixed points of nonexpansive mappings and fixed points of some of their generalizations have been flourishing areas of research for many mathematicians (see for more details [2, 6]).

In [3] Diaz and Metcalf studied quasi-nonexpansive mappings in Banach spaces. Ghosh and Debnath [4] established a necessary and sufficient condition for convergence of the Ishikawa iterates of a quasi-nonexpansive mapping on a closed convex subset of a Banach space. The iterative approximation problems for nonexpansive mapping, asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mapping were studied extensively by Goebel and Kirk [5], Liu [9], and Tan and Xu [15], Wittmann [17] in the settings of Hilbert spaces.

There are many methods for approximating fixed points of a nonexpansive mapping. Xu and Ori [18] introduced implicit iteration process to approximate a common fixed point of a finite family of nonexpansive mappings in a Hilbert space. Recently, Sun [14] has extended an implicit iteration process for a finite family of nonexpansive mappings, due to Xu and Ori, to the case of asymptotically quasi-nonexpansive mapping in a setting of Banach spaces.

There are many concepts which generalize a notion of nonexpansive mapping. One of such concepts is I-nonexpansivity of a mapping T ([13]). Let us recall some notions.

1.3. Definition. Let $T, I : K \to K$ be two mappings of a nonempty subset K of a real normed linear space X. Then T is said to be

(i) I-nonexpansive, if $\|Tx - Ty\| \leq \|Ix - Iy\|$ for all $x, y \in K$.

(ii) asymptotically I-nonexpansive, if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n \|I^n x - I^n y\|$ for all $x, y \in K$ and all $n \in \mathbb{N}$.

(iii) quasi I-nonexpansive, if $\|Tx - p\| \leq \|Ix - p\|$ for all $x \in K$, $p \in F(T) \cap F(I)$ and all $n \in \mathbb{N}$.

(iv) asymptotically quasi I-nonexpansive, if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - p\| \leq k_n \|I^n x - p\|$ for all $x \in K$, $p \in F(T) \cap F(I)$ and all $n \in \mathbb{N}$.

(v) uniformly L-Lipschitzian I-nonexpansive mapping, if there exists $L > 0$ such that $\|T^n x - T^n y\| \leq L \|I^n x - I^n y\|$ for all $x, y \in K$ and all $n \in \mathbb{N}$.

1.4. Remarks. From above definitions, it is easy to see that if $F(T)$ is nonempty, an I-nonexpansive mapping must be quasi I-nonexpansive, and an asymptotically I-nonexpansive mapping must be asymptotically quasi I-nonexpansive. But the converse does not hold.

Best approximation properties of I-nonexpansive mappings were investigated in [13]. In [16], the weak and strong convergence of implicit iteration process to a common fixed point of a finite family of I-asymptotically nonexpansive mappings were studied.

Recently, Mukhamedov and Saburov [10] extended of the implicit iterative process, defined in [14], to asymptotically quasi I-nonexpansive defined on a uniformly convex Banach space. Namely, assume that K is a nonempty convex subset of a real Banach space X and $T : K \to K$ is an asymptotically quasi I-nonexpansive mapping, and $I : K \to K$ is an asymptotically quasi-nonexpansive mapping. Then for given two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0,1]$ they used the following implicit iteration process:

$$(1.1) \qquad \begin{cases} x_0 \in K, \\ x_n = (1-\alpha_n)\, x_{n-1} + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)\, x_n + \beta_n I^n x_n, \quad n \in \mathbb{N}. \end{cases}$$

They also proved weak and strong convergence of the implicit iterative process (1.1) to a common fixed point of T and I.

Motivated by above works, in this paper, we introduce the following implicit iteration process for approximating the common fixed points of asymptotically quasi I-nonexpansive mappings T_1, T_2 and asymptotically quasi-nonexpansive mapping I:

$$(1.2) \qquad \begin{cases} x_0 \in K, \\ x_n = \alpha_n x_{n-1} + \beta_n I^n x_n + \gamma_n T_1^n x_n + \theta_n T_2^n x_n, \quad n \in \mathbb{N}. \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\theta_n\}$ are four real sequences in $(0,1)$ satisfying $\alpha_n + \beta_n + \gamma_n + \theta_n = 1$.

2. Preliminaries

Recall that a Banach space X is said to satisfy Opial condition [11], if for each sequence $\{x_n\}$ in X, x_n converging weakly to x implies that

$$(2.1) \qquad \liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$. It is well known that (see [8]) inequality (2.1) is equivalent to

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

2.1. Definition. Let K be a closed subset of a real Banach space X and let $T : K \to K$ be a mapping.

(i) A mapping T is said to be semiclosed (demiclosed) at zero, if for each bounded sequence $\{x_n\}$ in K, the condition x_n converges weakly to $x \in K$ and Tx_n converges strongly to 0 imply $Tx = 0$.

(ii) A mapping T is said to be semicompact, if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^* \in K$ strongly.

We restate the following lemmas which play key roles in our proofs.

2.2. Lemma. [15]*Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers with $\sum_{n=1}^{\infty} b_n < \infty$. If one of the following conditions is satisfied:*

(i) $a_{n+1} \leq a_n + b_n, n \in \mathbb{N}$,

(ii) $a_{n+1} \leq (1 + b_n)\, a_n, n \in \mathbb{N}$,

then the limit $\lim_{n\to\infty} a_n$ exists.

2.3. Lemma. [12] *Let X be a uniformly convex Banach space and let a, b be two constants with $0 < a < b < 1$. Suppose that $\{t_n\} \subset [a, b]$ is a real sequence and $\{x_n\}, \{y_n\}$ are two sequences in X. Then the conditions*

$$\lim_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = d, \quad \limsup_{n\to\infty} \|x_n\| \leq d, \quad \limsup_{n\to\infty} \|y_n\| \leq d$$

imply that $\lim_{n\to\infty} \|x_n - y_n\| = 0$, where $d \geq 0$ is a constant.

3. Main Results

Before proving our main results, we would like to remark as follows. Let $T_1, T_2 : K \to K$ be two asymptotically quasi I-nonexpansive mappings with $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\|T_1^n x - p\| \leq k_n \|I^n x - p\|$, $\|T_2^n x - p\| \leq l_n \|I^n x - p\|$. Also, $I : K \to K$ be an asymptotically quasi-nonexpansive mapping with a sequence $\{t_n\} \subset [1, \infty)$. Throughout this paper, we assume that $h_n = \max_{n\in\mathbb{N}} \{k_n, l_n, t_n\}$ and $F := F(T_1) \cap F(T_2) \cap F(I) \neq \emptyset$.

3.1. Lemma. *Let X be a real Banach space and K be a nonempty closed convex subset of X, Let $T_1, T_2 : K \to K$ be two asymptotically quasi I-nonexpansive mappings with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ and $I : K \to K$ be an asymptotically quasi-nonexpansive mapping with a sequence $\{t_n\} \subset [1, \infty)$ such that $F := F(T_1) \cap F(T_2) \cap F(I) \neq \emptyset$. Suppose that $\delta = \sup_n (1 - \alpha_n)$, $M = \sup_n h_n^2 \geq 1$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\theta_n\}$ are four real sequences in $(0, 1)$ which satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n + \theta_n = 1$,

(ii) $\delta M < 1$,

(iii) $\sum_{n=1}^{\infty} (1 - \alpha_n)(h_n^2 - 1) < \infty$.

If $\{x_n\}$ is the implicit iterative sequence defined by (1.2), then

(1) $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F$.

(2) The sequence $\{x_n\}$ generated by (1.2) converges strongly to a common fixed point in F if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. For any $p \in F \neq \emptyset$, it follows from (1.2) that

$$\begin{aligned}
\|x_n - p\| &= \|\alpha_n x_{n-1} + \beta_n I^n x_n + \gamma_n T_1^n x_n + \theta_n T_2^n x_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + \beta_n \|I^n x_n - p\| + \gamma_n \|T_1^n x_n - p\| + \theta_n \|T_2^n x_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + \beta_n h_n \|x_n - p\| + \gamma_n h_n \|I^n x_n - p\| + \theta_n h_n \|I^n x_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + \beta_n h_n \|x_n - p\| + \gamma_n h_n^2 \|x_n - p\| + \theta_n h_n^2 \|x_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + (\beta_n + \gamma_n + \theta_n) h_n^2 \|x_n - p\| \\
&= \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n) h_n^2 \|x_n - p\|
\end{aligned} \tag{3.1}$$

which means

$$\left(1 - h_n^2 + \alpha_n h_n^2\right) \|x_n - p\| \leq \alpha_n \|x_{n-1} - p\|. \tag{3.2}$$

By condition (ii) we have $h_n^2 - \alpha_n h_n^2 = h_n^2 (1 - \alpha_n) \leq M\delta < 1$, and therefore

$$1 - h_n^2 + \alpha_n h_n^2 \geq 1 - M\delta > 0.$$

Hence from (3.2) we obtain

$$\begin{aligned}\|x_n - p\| &\le \frac{\alpha_n}{1 - h_n^2 + \alpha_n h_n^2} \|x_{n-1} - p\| \\ &= \left(1 + \frac{h_n^2 - \alpha_n h_n^2 + \alpha_n - 1}{1 - h_n^2 + \alpha_n h_n^2}\right) \|x_{n-1} - p\| \\ &= \left(1 + \frac{(1-\alpha_n)\left(h_n^2 - 1\right)}{1 - h_n^2 + \alpha_n h_n^2}\right) \|x_{n-1} - p\| \\ &\le \left(1 + \frac{(1-\alpha_n)\left(h_n^2 - 1\right)}{1 - M\delta}\right) \|x_{n-1} - p\|.\end{aligned}$$

By putting $b_n = (1-\alpha_n)\left(h_n^2 - 1\right)/(1 - M\delta)$ the last inequality can be rewritten as follows:

(3.3) $$\|x_n - p\| \le (1 + b_n)\|x_{n-1} - p\|.$$

From condition (iii) we get

$$\sum_{n=1}^{\infty} b_n = \frac{1}{1 - M\delta} \sum_{n=1}^{\infty} (1-\alpha_n)\left(h_n^2 - 1\right) < \infty.$$

Denoting $a_n = \|x_{n-1} - p\|$ in (3.3) one gets

$$a_{n+1} \le (1 + b_n)\, a_n$$

and Lemma 2.2 implies the existence of the limit $\lim_{n\to\infty} a_n$. This means the limit

$$\lim_{n\to\infty} \|x_n - p\| = d$$

exists, where $d \ge 0$ is a constant.

It follows from (3.3) that

$$d(x_n, F) \le (1 + b_n)\, d(x_{n-1}, F).$$

So from Lemma 2.2, we obtain $\lim_{n\to\infty} d(x_n, F)$ exists. Furthermore, since $\liminf_{n\to\infty} d(x_n, F) = 0$, then $\lim_{n\to\infty} d(x_n, F) = 0$. Next we prove that $\{x_n\}$ is a Cauchy sequence in K. Let $\epsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, F) = 0$, there exists a positive integer n_0 such that

$$d(x_n, F) < \frac{\epsilon}{4}, \quad \forall n \ge n_0.$$

In particular, $\inf\{\|x_{n_0} - p\| : p \in F\} < \frac{\epsilon}{4}$. Thus there must exist $p^* \in F$ such that

$$\|x_{n_0} - p^*\| < \frac{\epsilon}{2}.$$

Now, for all $m, n \ge n_0$, we have

$$\begin{aligned}\|x_{n+m} - x_n\| &\le \|x_{n+m} - p^*\| + \|x_n - p^*\| \\ &\le 2\|x_{n_0} - p^*\| \\ &< 2\left(\frac{\epsilon}{2}\right) = \epsilon.\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset K of a Banach space E and so it must converge to a point q in K. And $\lim_{n\to\infty} d(x_n, F) = 0$ gives that $d(q, F) = 0$. By the routine proof, we know F is a closed subset of K. Thus $q \in F$. This completes the proof. □

3.2. Theorem. *Let X be a real uniformly convex Banach space and let K be a nonempty closed convex subset of X, Let $T_1, T_2 : K \to K$ be two uniformly L_1 and L_2-Lipschitzian asymptotically quasi-I-nonexpansive mappings with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ and let $I : K \to K$ be a be uniformly L_3-Lipschitzian asymptotically quasi-nonexpansive mapping with a sequence $\{t_n\} \subset [1, \infty)$ such that $F := F(T_1) \cap F(T_2) \cap F(I) \neq \emptyset$. Suppose that $\delta = \sup_n (1 - \alpha_n), M = \sup_n h_n^2 \geq 1$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\theta_n\}$ are four real sequences in $(0, 1)$ which satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n + \theta_n = 1$,

(ii) $\delta M < 1$,

(iii) $\sum_{n=1}^{\infty} (1 - \alpha_n)\left(h_n^2 - 1\right) < \infty$.

Then the implicitly iterative sequence $\{x_n\}$ defined by (1.2) satisfies the following:

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = \lim_{n\to\infty} \|x_n - T_2 x_n\| = \lim_{n\to\infty} \|x_n - I x_n\| = 0.$$

Proof. From Lemma 3.1, we know that $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in F$. We suppose that $\lim_{n\to\infty} \|x_n - p\| = d$. That is,

$$\begin{aligned}(3.4)\qquad \lim_{n\to\infty} \|x_n - p\| &= \lim_{n\to\infty} \|\alpha_n x_{n-1} + \beta_n I^n x_n + \gamma_n T_1^n x_n + \theta_n T_2^n x_n - p\| \\ &= \lim_{n\to\infty} \|\alpha_n (x_{n-1} - p) + \beta_n (I^n x_n - p) + \gamma_n (T_1^n x_n - p) \\ &\quad + \theta_n (T_2^n x_n - p)\| \\ &= \lim_{n\to\infty} \left\|\alpha_n (x_{n-1} - p) + (1 - \alpha_n)\left[\frac{\beta_n}{1 - \alpha_n}(I^n x_n - p)\right.\right. \\ &\quad \left.\left. + \frac{\gamma_n}{1 - \alpha_n}(T_1^n x_n - p) + \frac{\theta_n}{1 - \alpha_n}(T_2^n x_n - p)\right]\right\| \\ &= d.\end{aligned}$$

It follows from $\lim_{n\to\infty} \|x_n - p\| = d$ that $\lim_{n\to\infty} \|x_{n-1} - p\| = d$. Taking lim sup on both sides, we obtain

$$(3.5)\qquad \limsup_{n\to\infty} \|x_{n-1} - p\| = d.$$

In addition, from (3.4), we have

$$\begin{aligned}(3.6)\qquad &\limsup_{n\to\infty} \left\|\frac{\beta_n}{1 - \alpha_n}(I^n x_n - p) + \frac{\gamma_n}{1 - \alpha_n}(T_1^n x_n - p) + \frac{\theta_n}{1 - \alpha_n}(T_2^n x_n - p)\right\| \\ &\leq \limsup_{n\to\infty} \left(\frac{\beta_n}{1 - \alpha_n}\|I^n x_n - p\| + \frac{\gamma_n}{1 - \alpha_n}\|T_1^n x_n - p\| + \frac{\theta_n}{1 - \alpha_n}\|T_2^n x_n - p\|\right) \\ &\leq \limsup_{n\to\infty} \left(\frac{\beta_n}{1 - \alpha_n} h_n \|x_n - p\| + \frac{\gamma_n}{1 - \alpha_n} h_n \|I^n x_n - p\| + \frac{\theta_n}{1 - \alpha_n} h_n \|I^n x_n - p\|\right) \\ &\leq \limsup_{n\to\infty} \left(\frac{\beta_n}{1 - \alpha_n} h_n \|x_n - p\| + \frac{\gamma_n}{1 - \alpha_n} h_n^2 \|x_n - p\| + \frac{\theta_n}{1 - \alpha_n} h_n^2 \|x_n - p\|\right) \\ &\leq \limsup_{n\to\infty} \left[\frac{h_n^2}{1 - \alpha_n}(\beta_n + \gamma_n + \theta_n)\|x_n - p\|\right] \\ &= \limsup_{n\to\infty} h_n^2 \|x_n - p\| = d.\end{aligned}$$

It follows from (3.4), (3.5), (3.6) and Lemma 2.3 that

$$\begin{aligned}
&\lim_{n\to\infty}\left\|(x_{n-1}-p)-\left[\frac{\beta_n}{1-\alpha_n}\left(I^n x_n-p\right)+\frac{\gamma_n}{1-\alpha_n}\left(T_1^n x_n-p\right)+\frac{\theta_n}{1-\alpha_n}\left(T_2^n x_n-p\right)\right]\right\| \\
&=\lim_{n\to\infty}\left(\frac{1}{1-\alpha_n}\right)\left\|(1-\alpha_n)\left(x_{n-1}-p\right)-\beta_n\left(I^n x_n-p\right)-\gamma_n\left(T_1^n x_n-p\right)-\theta_n\left(T_2^n x_n-p\right)\right\| \\
&=\lim_{n\to\infty}\left(\frac{1}{1-\alpha_n}\right)\left\|x_n-x_{n-1}\right\| \\
&=0.
\end{aligned}$$

Since the sequence $\{\alpha_n\}$ in $(0,1)$, there are some constants $a, b \in (0,1)$ such that $0 < a \leq \alpha_n \leq b < 1$. So, we have

$$\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0. \tag{3.7}$$

On the other hand, from (3.4), we have

$$\begin{aligned}
\lim_{n\to\infty}\|x_n-p\| = \lim_{n\to\infty}&\left\|\beta_n\left(I^n x_n-p\right)+(1-\beta_n)\left[\frac{\alpha_n}{1-\beta_n}\left(x_{n-1}-p\right)\right.\right. \\
&\left.\left.+\frac{\gamma_n}{1-\beta_n}\left(T_1^n x_n-p\right)+\frac{\theta_n}{1-\beta_n}\left(T_2^n x_n-p\right)\right]\right\| \\
&=d.
\end{aligned} \tag{3.8}$$

Since I is a uniformly L_3-Lipschitzian asymptotically quasi-nonexpansive mapping, we have

$$\|I^n x_n - p\| \leq h_n \|x_n - p\|.$$

Taking lim sup on both sides, we get that

$$\limsup_{n\to\infty}\|I^n x_n - p\| \leq d \tag{3.9}$$

and

$$\begin{aligned}
&\limsup_{n\to\infty}\left\|\frac{\alpha_n}{1-\beta_n}\left(x_n-p\right)+\frac{\gamma_n}{1-\beta_n}\left(T_1^n x_n-p\right)+\frac{\theta_n}{1-\beta_n}\left(T_2^n x_n-p\right)\right\| \\
&\leq\limsup_{n\to\infty}\left(\frac{\alpha_n}{1-\beta_n}\|x_n-p\|+\frac{\gamma_n}{1-\beta_n}\|T_1^n x_n-p\|+\frac{\theta_n}{1-\beta_n}\|T_2^n x_n-p\|\right) \\
&\leq\limsup_{n\to\infty}\left(\frac{\alpha_n}{1-\beta_n}h_n\|x_n-p\|+\frac{\gamma_n}{1-\beta_n}h_n^2\|x_n-p\|+\frac{\theta_n}{1-\beta_n}h_n^2\|x_n-p\|\right) \\
&\leq\limsup_{n\to\infty}\left[\frac{h_n^2}{1-\beta_n}\left(\alpha_n+\gamma_n+\theta_n\right)\|x_n-p\|\right] \\
&=\limsup_{n\to\infty}h_n^2\|x_n-p\|=d.
\end{aligned} \tag{3.10}$$

It follows from (3.8), (3.9), (3.10) and Lemma 2.3 that

$$\lim_{n\to\infty}\|x_n - I^n x_n\| = 0. \tag{3.11}$$

In a similar way, we have

$$\lim_{n\to\infty}\|x_n - T_1^n x_n\| = 0, \tag{3.12}$$

and

$$\lim_{n\to\infty}\|x_n - T_2^n x_n\| = 0. \tag{3.13}$$

Finally, from

$$\begin{aligned}\|x_n - T_1 x_n\| &\le \|x_n - T_1^n x_n\| + \|T_1^n x_n - T_1 x_n\| \\ &\le \|x_n - T_1^n x_n\| + L_1 \left\|T_1^{n-1} x_n - x_n\right\| \\ &\le \|x_n - T_1^n x_n\| + L_1 \left(\left\|T_1^{n-1} x_n - T_1^{n-1} x_{n-1}\right\|\right. \\ &\left. + \left\|T_1^{n-1} x_{n-1} - x_{n-1}\right\| + \|x_{n-1} - x_n\|\right) \\ &\le \|x_n - T_1^n x_n\| + L_1 \left(L_1 \|x_n - x_{n-1}\|\right. \\ &\left. + \left\|T_1^{n-1} x_{n-1} - x_{n-1}\right\| + \|x_{n-1} - x_n\|\right) \\ &= \|x_n - T_1^n x_n\| + L_1 (L_1 + 1) \|x_n - x_{n-1}\| \\ &+ L_1 \left\|T_1^{n-1} x_{n-1} - x_{n-1}\right\|\end{aligned}$$

with (3.7) and (3.12) we obtain

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0. \tag{3.14}$$

Analogously, we have

$$\|x_n - T_2 x_n\| \le \|x_n - T_2^n x_n\| + L_2 (L_2 + 1) \|x_n - x_{n-1}\| + L_2 \left\|T_2^{n-1} x_{n-1} - x_{n-1}\right\|$$

and

$$\|x_n - I x_n\| \le \|x_n - I^n x_n\| + L_3 (L_3 + 1) \|x_n - x_{n-1}\| + L_3 \left\|I^{n-1} x_{n-1} - x_{n-1}\right\|$$

which with $(3.7), (3.11)$ and (3.13) imply

$$\lim_{n\to\infty} \|x_n - T_2 x_n\| = 0 \tag{3.15}$$

and

$$\lim_{n\to\infty} \|x_n - I x_n\| = 0. \tag{3.16}$$

This completes the proof. □

3.3. Theorem. *Let X be a real uniformly convex Banach space satisfying Opial condition and let $K, T_1, T_2, I, \{x_n\}$ be same as in Theorem 3.2. Suppose that $E : X \to X$ is an identity mapping and is satisfied the conditions in Theorem 3.2. If the mappings $E - T_1, E - T_2, E - I$ are semiclosed at zero, then the implicitly iterative sequence $\{x_n\}$ defined by (1.2) converges weakly to a common fixed point of T_1, T_2 and I.*

Proof. Let $p \in F$, then according to Lemma 3.1 the sequence $\{\|x_n - p\|\}$ converges. This provides that $\{x_n\}$ is a bounded sequence. Since X is uniformly convex, then every bounded subset of X is weakly compact. Since $\{x_n\}$ is a bounded sequence in K, then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q \in K$. Hence from $(3.14), (3.15)$ and (3.16), we have

$$\lim_{n\to\infty} \|x_{n_k} - T_1 x_{n_k}\| = 0, \quad \lim_{n\to\infty} \|x_{n_k} - T_2 x_{n_k}\| = 0, \quad \lim_{n\to\infty} \|x_{n_k} - I x_{n_k}\| = 0.$$

Since the mappings $E - T_1, E - T_2, E - I$ are semiclosed at zero, therefore, we find $T_1 q = q, T_2 q = q, Iq = q$, which means $q \in F = F(T_1) \cap F(T_2) \cap F(I)$.

Finally, let us prove that $\{x_n\}$ converges weakly to q. In fact, suppose the contrary, that is, there exists some subsequence $\left\{x_{n_j}\right\} \subset \{x_n\}$ such that $\left\{x_{n_j}\right\}$ converges weakly to $q_1 \in K$ and $q_1 \neq q$. Then by the same method as given above, we can also prove that $q_1 \in F = F(T_1) \cap F(T_2) \cap F(I)$.

From Lemma 3.1, we can prove that the limits $\lim_{n\to\infty} \|x_n - q\|$ and $\lim_{n\to\infty} \|x_n - q_1\|$ exist, and we have

$$\lim_{n\to\infty} \|x_n - q\| = d, \qquad \lim_{n\to\infty} \|x_n - q_1\| = d_1$$

where d and d_1 are two nonnegative numbers. By virtue of the Opial condition of X, we obtain

$$\begin{aligned} d &= \limsup_{n_k \to \infty} \|x_{n_k} - q\| < \limsup_{n_k \to \infty} \|x_{n_k} - q_1\| = d_1 \\ &= \limsup_{n_j \to \infty} \left\|x_{n_j} - q_1\right\| < \limsup_{n_j \to \infty} \left\|x_{n_j} - q\right\| = d. \end{aligned}$$

This is a contradiction. Hence $q_1 = q$. This implies that $\{x_n\}$ converges weakly to q. This completes the proof. □

3.4. Theorem. *Let X be a real uniformly convex Banach space and let $K, T_1, T_2, I, \{x_n\}$ be same as in Theorem* 3.2. *Suppose that the conditions in Theorem* 3.2 *is satisfied. If at least one mapping of the mappings T_1, T_2 and I is semicompact, then the implicitly iterative sequence $\{x_n\}$ defined by* (1.2) *converges strongly to a common fixed point of T_1, T_2 and I.*

Proof. We suppose that at least one mapping of the mappings T_1, T_2 and I is semicompact. Then from (3.14), (3.15) and (3.16), we have

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0, \lim_{n\to\infty} \|x_n - T_2 x_n\| = 0 \text{ and } \lim_{n\to\infty} \|x_n - I x_n\| = 0.$$

From the semicompactness T_1, T_2 and I, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to a $q \in K$. Again, using (3.14), (3.15) and (3.16), we obtain

$$\lim_{n_j\to\infty} \left\|x_{n_j} - T_1 x_{n_j}\right\| = \|q - T_1 q\| = 0, \lim_{n_j\to\infty} \left\|x_{n_j} - T_2 x_{n_j}\right\| = \|q - T_2 q\| = 0$$

and

$$\lim_{n_j\to\infty} \left\|x_{n_j} - I x_{n_j}\right\| = \|q - Iq\| = 0.$$

This shows that $q \in F = F(T_i) \cap F(S_i) \cap F(I_i)$. Since $\lim_{n_j\to\infty} \left\|x_{n_j} - q\right\| = 0$ and $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in F$ by Lemma 3.1, therefore

$$\lim_{n\to\infty} \|x_n - q\| = 0.$$

That is, $\{x_n\}$ converges strongly to q. This completes the proof. □

3.5. Remarks. The main results of this paper can be extended to a finite family of asymptotically quasi I-nonexpansive mappings $\{T_i : 1 \le i \le m\}$, where m is a fixed positive integer and an asymptotically quasi-nonexpansive mapping I, by introducing the following implicit iterative algorithm:

$$\begin{cases} x_0 \in K, \\ x_n = \alpha_{n1} x_{n-1} + \alpha_{n2} I^n x_n + \alpha_{n3} T_1^n x_n + \alpha_{n4} T_2^n x_n + \cdots + \alpha_{n(m+1)} T_m^n x_n, \quad n \ge 1, \end{cases}$$

where $\{\alpha_{n1}\}, \{\alpha_{n2}\}, \ldots,$ and $\{\alpha_{n(m+1)}\}$ are $m+1$ real sequences in $(0,1)$ satisfying $\alpha_{n1} + \alpha_{n2} + \cdots + \alpha_{n(m+1)} = 1$.

We close this section with the following open question.

How to devise an implicit iterative algorithm for approximating common fixed points of an infinite family of asymptotically quasi I-nonexpansive mappings?

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