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ROUGH IDEAL CONVERGENCE

Sudip Kumar Pal [a †], Debraj Chandra [a ‡], and Sudipta Dutta [a §]

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Abstract

In this paper we extend the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence. We define the set of rough ideal limit points and prove several results associated with this set.

Keywords: Ideal, filter, rough $\mathcal{I}$-convergence, rough $\mathcal{I}$-limit point.

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1. Introduction

Since 1951 when Steinhaus [23] and Fast [8] (also Schoenberg [22]) defined statistical convergence for sequences of real numbers, several generalizations and applications of this notion have been investigated. In particular two interesting generalizations of statistical convergence were introduced by Kostyrko et. al. [11], using the notion of ideals of $\mathbb{N}$, the set of positive integers who named them as $\mathcal{I}$ and $\mathcal{I}^*$-convergence. A lot of work has been done on ideal convergence since then as can be seen from [4, 5, 7, 12, 13, 14] where more references can be found.

The idea of rough convergence was introduced by Phu [16] who also introduced the concepts of rough limit points and roughness degree. The basic properties of this interesting concept were studied by Phu [16, 17] in finite dimensional normed linear spaces who later extended the results into infinite dimensional spaces [18]. It should be mentioned that the idea of rough convergence occurs very naturally in numerical analysis and has interesting applications there. Recently Aytar [1] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. So in view of the recent applications of ideals in the theory of convergence of sequences, it seems very natural

[a]Department of Mathematics, Jadavpur University, Kolkata-700032, West Bengal, India.
[†]Email: `sudipkmpal@yahoo.co.in`
[‡]Email: `debrajchandra1986@gmail.com`
[§]Email: `dutta.sudipta@ymail.com`
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to extend the interesting concept of rough convergence further by using ideals which we mainly do here.

We primarily introduce the notion of rough ideal convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence. We define the set of rough ideal limit points and prove several results regarding this set.

It is to be noted that our results and methods of proofs are ideal analogues of the parallel results of [16] and [2] and present those results in the most general possible form. This also enhances the applicability of these concepts.

2. Preliminaries

A family $\mathcal{I} \subset 2^Y$ subsets of a non-empty set Y is said to be an ideal in Y if (i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ (ii) $A \in \mathcal{I},\ B \subset A$ implies $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$. Let $\mathcal{I}$ be a non-trivial ideal in Y (i.e. $Y \notin \mathcal{I}$), $Y \neq \emptyset$. Then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset Y : \text{there exists } A \in \mathcal{I},\ M = Y \setminus A\}$ is a filter in Y (recall that a non empty class $\mathcal{F}$ of subsets of Y is said to be a filter in Y provided (i) $\emptyset \notin \mathcal{F}$, (ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, (iii) $A \in \mathcal{F}, A \subset B$ implies $B \in \mathcal{F}$). It is called the filter associated with the ideal $\mathcal{I}$.

Throughout the paper $\mathcal{I}$ will stand for a non-trivial admissible ideal of $\mathbb{N}$, set of all positive integers.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a normed linear space $(X, \|.\|)$ is said to be $\mathcal{I}$-convergent to $x \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \varepsilon\}$ belongs to $\mathcal{I}$ [11] (see [14] for the definition in more general form in topological spaces).

Throughout the paper let r be a non-negative real number and $(X, \|.\|)$ be a normed linear space. The sequence $x = \{x_i\}_{i\in\mathbb{N}}$ in X is said to be rough convergent (or in short r-convergent) to x_* [16] and is denoted by $x_i \xrightarrow{r} x_*$, provided that for any $\varepsilon > 0$ there exists $i_\varepsilon \in \mathbb{N}$ such that $i \geq i_\varepsilon$ implies $\|x_i - x_*\| < r + \varepsilon$. The set $LIM_x^r = \{x_* \in X : x_i \xrightarrow{r} x_*\}$ is called the r-limit set of the sequence $x = \{x_i\}_{i\in\mathbb{N}}$. A sequence $\{x_i\}_{i\in\mathbb{N}}$ is said to be r-convergent if $LIM_x^r \neq \emptyset$. In this case r is called the convergence degree of the sequence $\{x_i\}_{i\in\mathbb{N}}$. For $r = 0$ we get the notion of ordinary convergence.

We now introduce the following definition.

2.1. Definition. A sequence $x = \{x_i\}_{i\in\mathbb{N}}$ in X is said to be r-$\mathcal{I}$-convergent to x_*, denoted by $x_i \xrightarrow{r\mathcal{I}} x_*$, provided that the set $\{i \in \mathbb{N} : \|x_i - x_*\| \geq r + \varepsilon\} \in \mathcal{I}$, for any $\varepsilon > 0$.

Here r is called the roughness degree. If we take $r = 0$, we obtain the notion of $\mathcal{I}$-convergence.

The motivation behind introducing this notion is same as in [16] and [2]. For instance assume that the sequence $y = \{y_i\}_{i\in\mathbb{N}}$ is $\mathcal{I}$-convergent and can not be measured or calculated exactly and one has to do with an approximated (or $\mathcal{I}$-approximated) sequence $x = \{x_i\}_{i\in\mathbb{N}}$ satisfying $\|x_i - y_i\| \leq r$ for all i (or for almost all i that is $\{i \in \mathbb{N} : \|x_i - y_i\| > r\} \in \mathcal{I}$). Then $\mathcal{I}$-convergence of the sequence x is not assured, but as the inclusion $\{i \in \mathbb{N} : \|y_i - y_*\| \geq \varepsilon\} \supseteq \{i \in \mathbb{N} : \|x_i - y_*\| \geq r + \varepsilon\}$ holds so the sequence $\{x_i\}_{i\in\mathbb{N}}$ is r-$\mathcal{I}$-convergent.

In general the rough $\mathcal{I}$-limit of a sequence may not be unique for the roughness degree $r > 0$. We define $\mathcal{I}$-$LIM_x^r = \{x_* \in X : x_i \xrightarrow{r\mathcal{I}} x_*\}$. The sequence x is said to be r-$\mathcal{I}$-convergent provided that $\mathcal{I}$-$LIM_x^r \neq \emptyset$. In [16] (see also [2]) it was observed that for a sequence $x = \{x_i\}_{i\in\mathbb{N}}$ of real numbers, $LIM_x^r = [\limsup x - r, \liminf x + r]$. Similarly here we can have $\mathcal{I}$-$LIM_x^r = [\mathcal{I}\text{-}\limsup x - r, \mathcal{I}\text{-}\liminf x + r]$.

3. Main Results

It is known that there exists unbounded sequences $x = \{x_i\}_{i\in\mathbb{N}}$ for which $LIM_x^r = \emptyset$ (see [16]). But such a sequence might be rough $\mathcal{I}$-convergent. Let $\mathcal{I}$ be a non-trivial admissible ideal of $\mathbb{N}$. Choose an infinite subset A of $\mathbb{N}$ such that $A \in \mathcal{I}$. Define

$$x_i = \begin{cases} (-1)^i & \text{if } i \notin A \\ i & \text{if } i \in A. \end{cases}$$

Then

$$\mathcal{I}\text{-}LIM_x^r = \begin{cases} \emptyset & \text{if } r < 1 \\ \text{[1-r,r-1]} & \text{otherwise.} \end{cases}$$

Clearly $LIM_x^r = \emptyset$ for all $r \geq 0$. The above example shows that $\mathcal{I}\text{-}LIM_x^r \neq \emptyset$ does not imply that $LIM_x^r \neq \emptyset$. But the converse is always true if $\mathcal{I}$ is admissible i.e. $LIM_x^r \subseteq \mathcal{I}\text{-}LIM_x^r$.

3.1. Theorem. *For a sequence $x = \{x_i\}_{i\in\mathbb{N}}$ we have $diam(\mathcal{I}\text{-}LIM_x^r) \leq 2r$. In general $diam(\mathcal{I}\text{-}LIM_x^r)$ has no smaller bound.*

Proof. Assume that $\text{diam}(\mathcal{I}\text{-}LIM_x^r) > 2r$. Then there exists $y, z \in \mathcal{I}\text{-}LIM_x^r$ such that $\|y - z\| > 2r$. Choose $0 < \varepsilon < \frac{\|y-z\|}{2} - r$. Put $A_1 = \{i \in \mathbb{N} : \|x_i - y\| \geq r + \varepsilon\}$ and $A_2 = \{i \in \mathbb{N} : \|x_i - z\| \geq r + \varepsilon\}$. Then $A_1, A_2 \in \mathcal{I}$ and hence $M = \mathbb{N} \setminus (A_1 \cup A_2) \in \mathcal{F}(\mathcal{I})$ and so $M \neq \emptyset$. Let $i \in M$. Now $\|y - z\| \leq \|x_i - y\| + \|x_i - z\| < 2(r + \varepsilon) < \|y - z\|$, which is a contradiction. Thus $\text{diam}(\mathcal{I}\text{-}LIM_x^r) \leq 2r$.

To prove the converse part, consider a sequence $x = \{x_i\}_{i\in\mathbb{N}}$ such that $\mathcal{I}\text{-lim } x = x_*$. Let $\varepsilon > 0$ be given. Then $A = \{i \in \mathbb{N} : \|x_i - x_*\| \geq \varepsilon\} \in \mathcal{I}$. Now for each $y \in \bar{B}_r(x_*) = \{y \in X : \|y - x_*\| \leq r\}$ we have

$$\|x_i - y\| \leq \|x_i - x_*\| + \|x_* - y\| < r + \varepsilon, \quad \text{whenever } i \notin A.$$

Which shows that $y \in \mathcal{I}\text{-}LIM_x^r$ and consequently we can write $\mathcal{I}\text{-}LIM_x^r = \bar{B}_r(x_*)$. This shows that in general upper bound $2r$ of the diameter of the set $\mathcal{I}\text{-}LIM_x^r$ can not be decreased anymore. □

In [16] it was established that there exists a non-negative real number r such that $LIM_x^r \neq \emptyset$ for a bounded sequence x. As $LIM_x^r \neq \emptyset$ implies $\mathcal{I}\text{-}LIM_x^r \neq \emptyset$, this result is also true for $\mathcal{I}\text{-}LIM_x^r$ for any admissible ideal $\mathcal{I}$. The converse implication is not generally true. For this we have the following result.

3.2. Theorem. *A sequence $x = \{x_i\}_{i\in\mathbb{N}}$ is $\mathcal{I}$-bounded if and only if there exists a non-negative real number r such that $\mathcal{I}\text{-}LIM_x^r \neq \emptyset$. (A sequence $\{x_i\}_{i\in\mathbb{N}}$ is said to be $\mathcal{I}$-bounded if there exists a positive real number G such that the set $A = \{i \in \mathbb{N} : \|x_i\| \geq G\} \in \mathcal{I}$)[7].*

Proof. Let $x = \{x_i\}_{i\in\mathbb{N}}$ be an $\mathcal{I}$-bounded sequence. Then there exists a positive real number G such that $A = \{i \in \mathbb{N} : \|x_i\| \geq G\} \in \mathcal{I}$. Let $\bar{r} = \sup\{\|x_i\| : i \in M = \mathbb{N} \setminus A\}$. The set $\mathcal{I}\text{-}LIM_x^{\bar{r}}$ contains the origin of X and so $\mathcal{I}\text{-}LIM_x^r \neq \emptyset$.

Conversely suppose that $\mathcal{I}\text{-}LIM_x^r \neq \emptyset$ for some $r > 0$, then there exists $x_* \in \mathcal{I}\text{-}LIM_x^r$ i.e. $\{i \in \mathbb{N} : \|x_i - x_*\| \geq r + \varepsilon\} \in \mathcal{I}$ for each $\varepsilon > 0$. This in turn implies that $\{x_i\}_{i\in\mathbb{N}}$ is $\mathcal{I}$-bounded. □

Next we present some topological and geometrical properties of the r-$\mathcal{I}$-limit set of a sequence.

3.3. Theorem. *The r-$\mathcal{I}$-limit set $\mathcal{I}$-LIM_x^r of a sequence $x = \{x_i\}_{i\in\mathbb{N}}$ is a closed set.*

Proof. If $\mathcal{I}$-$LIM_x^r = \emptyset$, then there is nothing to prove. So assume that $\mathcal{I}$-$LIM_x^r \neq \emptyset$. Suppose that $\{y_i\}_{i\in\mathbb{N}} \subseteq \mathcal{I}$-$LIM_x^r$ and $y_i \to y_*$ as $i \to \infty$. Let $\varepsilon > 0$ be given. Then there exists $i_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that $\|y_i - y_*\| < \frac{\varepsilon}{2}$ for all $i > i_{\frac{\varepsilon}{2}}$. Let $i_0 \in \mathbb{N}$ such that $y_{i_0} \in \{y_i\}_{i\in\mathbb{N}} \subseteq \mathcal{I}$-$LIM_x^r$. Consequently we have $A = \{i \in \mathbb{N} : \|x_i - y_{i_0}\| \geq r + \frac{\varepsilon}{2}\} \in \mathcal{I}$. Clearly $M = \mathbb{N} \setminus A \in \mathcal{F}(I)$ and so $M \neq \emptyset$. Choose $k \in M$. Choose an $i_0 > i_{\frac{\varepsilon}{2}}$. We have $\|x_k - y_*\| \leq \|x_k - y_{i_0}\| + \|y_{i_0} - y_*\| < r + \varepsilon$. Hence

$$M = \{i \in \mathbb{N} : \|x_i - y_{i_0}\| < r + \frac{\varepsilon}{2}\} \subseteq \{i \in \mathbb{N} : \|x_i - y_*\| < r + \varepsilon\}$$

where $\{i \in \mathbb{N} : \|x_i - y_{i_0}\| < r + \frac{\varepsilon}{2}\} \in \mathcal{F}(\mathcal{I})$. Consequently $\{i \in \mathbb{N} : \|x_i - y_*\| < r + \varepsilon\} \in \mathcal{F}(\mathcal{I})$ and so $\{i \in \mathbb{N} : \|x_i - y_*\| \geq r + \varepsilon\} \in \mathcal{I}$. This completes the proof.

□

3.4. Theorem. *The r-$\mathcal{I}$-limit set $\mathcal{I}$-LIM_x^r of a sequence $x = \{x_i\}_{i\in\mathbb{N}}$ is convex.*

Proof. Consider $y_0, y_1 \in \mathcal{I}$-LIM_x^r. Let $\varepsilon > 0$ be given. Put $A_1 = \{i \in \mathbb{N} : \|x_i - y_0\| \geq r + \varepsilon\}$ and $A_2 = \{i \in \mathbb{N} : \|x_i - y_1\| \geq r + \varepsilon\}$. Then $A_1, A_2 \in \mathcal{I}$ which implies $M = \mathbb{N} \setminus (A_1 \cup A_2) \in \mathcal{F}(\mathcal{I})$ and so M is non-empty. Now for each $i \in M$ and $\lambda \in [0, 1]$ we have

$$\|x_i - [(1-\lambda)y_0 + \lambda y_1]\| = \|(1-\lambda)(x_i - y_0) + \lambda(x_i - y_1)\| < r + \varepsilon.$$

Consequently $\{i \in \mathbb{N} : \|x_i - [(1-\lambda)y_0 + \lambda y_1]\| \geq r + \varepsilon\} \in \mathcal{I}$, which shows that $(1-\lambda)y_0 + \lambda y_1 \in \mathcal{I}$-$LIM_x^r$.

□

3.5. Theorem. *Let $r > 0$. Then a sequence $x = \{x_i\}_{i\in\mathbb{N}}$ is r-$\mathcal{I}$-convergent to x_* if and only if there exists a sequence $y = \{y_i\}_{i\in\mathbb{N}}$ such that $\mathcal{I}$-lim $y = x_*$ and $\|x_i - y_i\| \leq r$ for $i \in \mathbb{N}$.*

Proof. Let $x_i \xrightarrow{rI} x_*$. Then we have

$\mathcal{I}$-lim sup$\|x_i - x_*\| \leq r$......(1).

Now we define

$$y_i = \begin{cases} x_* & \text{if } \|x_i - x_*\| \leq r \\ x_i + r\frac{x_* - x_i}{\|x_i - x_*\|} & \text{otherwise.} \end{cases}$$

Then

$$\|y_i - x_*\| = \begin{cases} 0 & \text{if } \|x_i - x_*\| \leq r \\ \|x_i - x_*\| - r & \text{otherwise.} \end{cases}$$

Thus $\|x_i - y_i\| \leq \|x_i - x_*\| + \|x_* - y_i\| \leq r$ for all $i \in \mathcal{F}(\mathcal{I})$. Also from (1) $\mathcal{I}$-lim sup $\|y_i - x_*\| = 0$. Thus $\mathcal{I}$-lim $y = x_*$.

Conversely suppose that the given condition holds. For any $\varepsilon > 0$, the set $A = \{i \in \mathbb{N} : \|y_i - x_*\| \geq \varepsilon\} \in \mathcal{I}$. Here we note that

$$\|x_i - x_*\| \leq \|x_i - y_i\| + \|y_i - x_*\| < r + \varepsilon, \text{ if } i \in M = \mathbb{N} \setminus A.$$

This shows that

$$\{i \in \mathbb{N} : \|x_i - x_*\| \geq r + \varepsilon\} \subseteq \{i \in \mathbb{N} : \|y_i - x_*\| \geq \varepsilon\}$$

and so $x_i \xrightarrow{rI} x_*$.

□

We now recall the following definitions from ([11, 14]).

3.6. Definition. A point $\xi \in X$ is called an $\mathcal{I}$-limit point of a sequence $x = \{x_i\}_{i\in\mathbb{N}}$ if there exists a set $K = \{k_1, k_2, ...\} \in \mathcal{F}(\mathcal{I})$ such that $\lim_{n\to\infty} x_{k_n} = \xi$. The set of all $\mathcal{I}$-limit points of the sequence x will be denoted by $\lambda_x(\mathcal{I})$.

3.7. Definition. A point $\xi \in X$ is called an $\mathcal{I}$-cluster point of a sequence $x = \{x_i\}_{i\in\mathbb{N}}$ if for any $\varepsilon > 0$, $\{n \in \mathbb{N} : \|x_n - \xi\| < \varepsilon\} \notin \mathcal{I}$. The set of all $\mathcal{I}$-cluster points of x will be denoted by $\Lambda_x(\mathcal{I})$.

It was known that $\lambda_x(\mathcal{I}) \subseteq \Lambda_x(\mathcal{I})$ (see [11]).

3.1. Proposition. For an arbitrary $c \in \Lambda_x(\mathcal{I})$ where $x = \{x_i\}_{i\in\mathbb{N}}$, we have $\|x_* - c\| \leq r$ for all $x_* \in \mathcal{I}\text{-}LIM_x^r$.

Proof. If possible suppose that there exists $c \in \Lambda_x(\mathcal{I})$ and $x_* \in \mathcal{I}\text{-}LIM_x^r$ such that $\|x_* - c\| > r$. Put $\varepsilon = \frac{\|x_*-c\|-r}{2}$. We have $A = \{i \in \mathbb{N} : \|x_i - c\| < \varepsilon\} \notin \mathcal{I}$. Let us write $B = \{i \in \mathbb{N} : \|x_i - x_*\| \geq r + \varepsilon\}$. Now for $i \in A$, $\|x_* - x_i\| \geq \|x_* - c\| - \|x_i - c\| > 2\varepsilon + r - \varepsilon = r + \varepsilon$. Then $A \subset B$ implies that $B \notin \mathcal{I}$ which contradicts the fact that $x_* \in \mathcal{I}\text{-}LIM_x^r$. Thus $\|x_* - c\| \leq r$ for all $x_* \in \mathcal{I}\text{-}LIM_x^r$ and $c \in \Lambda_x(\mathcal{I})$. □

3.8. Theorem. *A sequence $x = \{x_i\}_{i\in\mathbb{N}}$ is $\mathcal{I}$-convergent to x_* if and only if $\mathcal{I}\text{-}LIM_x^r = \bar{B}_r(x_*)$.*

Proof. In Theorem 3.1 we have already proved the necessity part. For the sufficiency, let $\mathcal{I}\text{-}LIM_x^r = \bar{B}_r(x_*)(\neq \emptyset)$. Thus the sequence $x = \{x_i\}_{i\in\mathbb{N}}$ is $\mathcal{I}$-bounded. Suppose that x has another $\mathcal{I}$-cluster point x'_* different from x_*. The point $\bar{x_*} = x_* + \frac{r}{\|x_*-x'_*\|}(x_* - x'_*)$ satisfies $\|\bar{x_*} - x'_*\| = (\frac{r}{\|x_*-x'_*\|} + 1)(\|x_* - x'_*\|) = r + \|x_* - x'_*\| > r$. Since $x'_* \in \Lambda_x(\mathcal{I})$, by Proposition 3.1 $\bar{x_*} \notin \mathcal{I}\text{-}LIM_x^r$. But this is impossible as $\|\bar{x_*} - x_*\| = r$ and $\mathcal{I}\text{-}LIM_x^r = \bar{B}_r(x_*)$. Since x_* is the unique $\mathcal{I}$-cluster point of x, it follows that x is $\mathcal{I}$-convergent to x_*. □

3.1. Corollary. If $(X, \|.\|)$ is a strictly convex space and $x = \{x_i\}_{i\in\mathbb{N}}$ is a sequence in X. Also if there exists $y_1, y_2 \in \mathcal{I}\text{-}LIM_x^r$ such that $\|y_1 - y_2\| = 2r$, then this sequence is $\mathcal{I}$-convergent to $\frac{y_1+y_2}{2}$.

The proof is straightforward and so is omitted.

3.9. Theorem. (i) *If $c \in \Lambda_x(\mathcal{I})$ then $\mathcal{I}\text{-}LIM_x^r \subseteq \bar{B}_r(c)$.*

(ii) $\mathcal{I}\text{-}LIM_x^r = \bigcap_{c\in\Lambda_x(\mathcal{I})} \bar{B}_r(c) = \{x_* \in \mathbb{R}^n : \Lambda_x(\mathcal{I}) \subseteq \bar{B}_r(x_*)\}$.

Proof. (i) If $x_* \in \mathcal{I}\text{-}LIM_x^r$ and $c \in \Lambda_x(\mathcal{I})$ then $\|x_* - c\| \leq r$. Hence the result follows.

(ii) By (i) we can write $\mathcal{I}\text{-}LIM_x^r \subseteq \bigcap_{c\in\Lambda_x(\mathcal{I})} \bar{B}_r(c)$. Assume that $y \in \bigcap_{c\in\Lambda_x(\mathcal{I})} \bar{B}_r(c)$. We have $\|y - c\| \leq r$ for all $c \in \Lambda_x(\mathcal{I})$ and so $\Lambda_x(\mathcal{I}) \subseteq \bar{B}_r(y)$. Then clearly $\bigcap_{c\in\Lambda_x(\mathcal{I})} \bar{B}_r(c) \subseteq \{x_* \in X : \Lambda_x(\mathcal{I}) \subseteq \bar{B}_r(x_*)\}$. If possible let $y \notin \mathcal{I}\text{-}LIM_x^r$. Then there exists an $\varepsilon > 0$ such that the set $A = \{i \in \mathbb{N} : \|x_i - y\| \geq r + \varepsilon\} \notin \mathcal{I}$, which implies the existence of an $\mathcal{I}$-cluster point c of the sequence x with $\|y - c\| \geq r + \varepsilon$. Hence $\Lambda_x(\mathcal{I}) \subseteq \bar{B}_r(y)$ and $y \notin \{x_* \in X : \Lambda_x(\mathcal{I}) \subseteq \bar{B}_r(x_*)\}$. Finally the fact that $y \in \mathcal{I}\text{-}LIM_x^r$ follows from the observation that $y \in \{x_* \in X : \Lambda_x(\mathcal{I}) \subseteq \bar{B}_r(x_*)\}$. □

3.2. Proposition. Let $x = \{x_i\}_{i\in\mathbb{N}}$ be an $\mathcal{I}$-bounded sequence. If $r = \text{diam}(\Lambda_x(\mathcal{I}))$, then we have $\Lambda_x(\mathcal{I}) \subseteq \mathcal{I}\text{-}LIM_x^r$.

Proof. Let $c \notin \mathcal{I}\text{-}LIM_x^r$. Then there exists an $\varepsilon' > 0$ such that the set $A = \{i \in \mathbb{N} : \|x_i - c\| \geq r + \varepsilon'\} \notin \mathcal{I}$. Since the sequence is $\mathcal{I}$-bounded, there exists an $\mathcal{I}$-cluster point c' such that $\|c - c'\| > r + \frac{\varepsilon'}{2}$. Consequently $c \notin \Lambda_x(\mathcal{I})$ and the result follows. □

We now recall the definitions of $\mathcal{I}$-lim sup x and $\mathcal{I}$-lim inf x and $\mathcal{I}$-core$\{x\}$ from [7] (see also [12]).

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. We put for $t \in \mathbb{R}$, $M_t = \{n : x_n > t\}$, $M^t = \{n : x_n < t\}$.

3.10. Definition. (a) If there is a $t \in \mathbb{R}$ such that $M_t \notin \mathcal{I}$, we put $\mathcal{I}$-lim sup $x = \sup\{t \in \mathbb{R} : M_t \notin \mathcal{I}\}$.

If $M_t \in \mathcal{I}$ holds for each $t \in \mathbb{R}$ then we put $\mathcal{I}$-lim sup $x = -\infty$.

(b) If there is a $t \in \mathbb{R}$ such that $M^t \notin \mathcal{I}$, we put $\mathcal{I}$-lim inf $x = \inf\{t \in \mathbb{R} : M^t \notin \mathcal{I}\}$.

If $M^t \in \mathcal{I}$ holds for each $t \in \mathbb{R}$ then we put $\mathcal{I}$-lim inf $x = +\infty$.

3.11. Definition. For a sequence $x = \{x_i\}_{i\in\mathbb{N}}$ of real numbers, $\mathcal{I}$-core$\{x\}$ is defined to be the closed interval [$\mathcal{I}$-lim inf x, $\mathcal{I}$-lim sup x].

3.12. Theorem. *If $\mathcal{I}\text{-}LIM_x^r \neq \emptyset$, then $\mathcal{I}$-lim sup x and $\mathcal{I}$-lim inf x belong to the set $\mathcal{I}\text{-}LIM_x^{2r}$.*

Proof. Since $\mathcal{I}\text{-}LIM_x^r \neq \emptyset$, $x = \{x_i\}_{i\in\mathbb{N}}$ is $\mathcal{I}$-bounded. The number $\mathcal{I}$-lim inf x is an $\mathcal{I}$-cluster point of x and consequently we have $\|x_* - \mathcal{I}\text{-lim inf } x\| \leq r$ for all $x_* \in \mathcal{I}\text{-}LIM_x^r$. Put $A = \{i \in \mathbb{N} : \|x_* - x_i\| \geq r + \varepsilon\}$. Now if $i \notin A$, then $\|x_i - (\mathcal{I}\text{-lim inf } x)\| \leq \|x_i - x_*\| + \|x_* - (\mathcal{I}\text{-lim inf } x)\| < 2r + \varepsilon$. Thus $\mathcal{I}$-lim inf $x \in \mathcal{I}\text{-}LIM_x^{2r}$. Similarly it can be shown that $\mathcal{I}$-lim sup $x \in \mathcal{I}\text{-}LIM_x^{2r}$. □

3.2. Corollary. Let $x = \{x_i\}_{i\in\mathbb{N}}$ be a sequence of real numbers. If $\mathcal{I}\text{-}LIM_x^r \neq \emptyset$, then $\mathcal{I}$-core$\{x\} \subseteq \mathcal{I}\text{-}LIM_x^{2r}$.

Proof. We have $\mathcal{I}\text{-}LIM_x^r = [\mathcal{I}\text{-lim sup } x - 2r, \mathcal{I}\text{-lim inf } x + 2r]$. Then the result follows from Theorem 3.12. □

3.3. Proposition. Let $x = \{x_i\}_{i\in\mathbb{N}}$ be a sequence of real numbers. Then the diameter diam($\mathcal{I}$-core$\{x\}$) of the set $\mathcal{I}$-core$\{x\}$ is equal to the number r if and only if $\mathcal{I}$-core$\{x\} = \mathcal{I}\text{-}LIM_x^r$.

Proof. diam($\mathcal{I}-$core$\{x\}$) $= r \Leftrightarrow$ ($\mathcal{I}$-lim sup x) $-$ ($\mathcal{I}$-lim inf x) $= r \Leftrightarrow \mathcal{I}$-core$\{x\} = [\mathcal{I}\text{-lim inf } x, \mathcal{I}\text{-lim sup } x] = [\mathcal{I}\text{-lim sup } x - r, \mathcal{I}\text{-lim inf } x + r] = \mathcal{I}\text{-}LIM_x^r$.

Also it is easy to see that

(a) $r >$ diam($\mathcal{I}$-core$\{x\}$) if and only if $\mathcal{I}$-core$\{x\} \subset \mathcal{I}\text{-}LIM_x^r$,

(b) $r <$ diam($\mathcal{I}$-core$\{x\}$) if and only if $\mathcal{I}\text{-}LIM_x^r \subset \mathcal{I}$-core$\{x\}$. □

3.4. Proposition. If $\bar{r} = \inf\{r \geq 0 : \mathcal{I}\text{-}LIM_x^r \neq \emptyset\}$, then $\bar{r} =$ radius($\mathcal{I}$-core$\{x\}$).

Proof. If the set $\mathcal{I}$-core$\{x\}$ is a singleton, then radius($\mathcal{I}$-core$\{x\}$) $= 0$ and the sequence is $\mathcal{I}$-convergent i.e. $\mathcal{I}\text{-}LIM_x^0 \neq \emptyset$. Hence we get $\bar{r} =$ radius($\mathcal{I}$-core$\{x\}$) $= 0$.

Now assume that the set $\mathcal{I}$-core$\{x\}$ is not a singleton. We can write $\mathcal{I}$-core$\{x\} = [a, b]$, where $a = \mathcal{I}$-lim inf x and $b = \mathcal{I}$-lim sup x. Now let us assume that $\bar{r} \neq$ radius($\mathcal{I}$-core$\{x\}$).

If $\bar{r} <$ radius($\mathcal{I}$-core$\{x\}$), then define $\bar{\varepsilon} = \frac{\frac{b-a}{2} - \bar{r}}{3}$. Now from the definition of $\bar{r}$ it easily follows that $\mathcal{I}\text{-}LIM_x^{\bar{r}+\bar{\varepsilon}} \neq \emptyset$. Then there exists $x_* \in \mathbb{R}$ such that for all $\varepsilon > 0$ the set $A = \{i \in \mathbb{N} : \|x_i - x_*\| \geq (\bar{r} + \bar{\varepsilon}) + \varepsilon\} \in \mathcal{I}$. Since $\bar{r} + \bar{\varepsilon} < \frac{b-a}{2}$ this in turn contradicts the definitions of a and b.

If $\bar{r} >$ radius($\mathcal{I}$-core$\{x\}$), then define $\bar{\varepsilon} = \frac{\bar{r} - \frac{b-a}{2}}{3}$ and $r' = \bar{r} - 2\bar{\varepsilon}$. It is clear that $0 \leq r' < \bar{r}$, and by definitions of a and b, the number $\frac{b-a}{2}$ is in $\mathcal{I}$-$LIM_x^{r'}$. Then we get $r' \in \{r \geq 0 : \mathcal{I}\text{-}LIM_x^r \neq \emptyset\}$, which contradicts the equality $\bar{r} = \inf\{r \geq 0 : \mathcal{I}\text{-}LIM_x^r \neq \emptyset\}$ as $r' < \bar{r}$. □

From Proposition 3.3 and Proposition 3.4 we can conclude that,

3.3. Corollary. For a sequence $x = \{x_i\}_{i \in \mathbb{N}}$ we have $\mathcal{I}$-core$\{x\} = \mathcal{I}$-$LIM_x^{2\bar{r}}$.

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