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AN UPPER BOUND ON THE SPECTRAL RADIUS OF WEIGHTED GRAPHS

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Abstract

We consider weighted graphs, where the edge weights are positive definite matrices. The eigenvalues of a graph are the eigenvalues of its adjacency matrix. We obtain another upper bound which is sharp on the spectral radius of the adjacency matrix and compare with some known upper bounds with the help of some examples of graphs. We also characterize graphs for which the bound is attained.

Keywords: Weighted graph, Adjacency matrix, Spectral radius, Upper bound

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1. Introduction

We consider simple graphs, that is, graph which have no loops or parallel edges. Hence a graph $G = (V, E)$ consists of a finite set of vertices, V, and a set of edges, E, each of whose elements are an unordered pair of distinct vertices. Generally, V is taken as $V = \{1, 2, .., n\}$.

A weighted graph is a graph, each edge of which has been assigned a square matrix, called the weight of the edge. All the weight matrices will be assumed to be of same order and will be assumed to be positive matrix. In this paper, by "weighted graph" we will mean "a weighted graph with each of its edges bearing a positive definite matrix as weight", unless otherwise stated.

Now we introduce some notations. Let G be a weighted graph on n vertices. Denote by $w_{i,j}$ the positive definite weight matrix of order p of the edge ij, and assume that $w_{i,j} = w_{j,i}$ .We write $i \sim j$ if vertices i and j are adjacent. Let $w_i = \sum_{j:j\sim i} w_{i,j}$.

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The adjacency matrix of a graph G is a block matrix, denoted and defined as $A(G) = (a_{ij})$ where

$$a_{i,j} = \begin{cases} w_{i,j} & \text{if } i \sim j \\ 0 & \text{otherwise.} \end{cases}$$

Note that in the definition above, the zero denotes the $p \times p$ zero matrix. Thus $A(G)$ is a square matrix of order np. For any symmetric matrix K, let $\rho_1(K)$ denote the largest eigenvalue, in modulus (i.e., the spectral radius), of K.

Let us give some more definitions. Let $G = (V, E)$. If V is the disjoint union of two nonempty sets V_1 and V_2 such that every vertex i in V_1 has the same $\rho_1(w_i)$ and every vertex j in V_2 has the same $\rho_1(w_j)$, then G will be called a weight-semiregular graph. If $\rho_1(w_i) = \rho_1(w_j)$ in weight semiregular graph, then G will be called a weight-regular graph.

Upper and lower bounds for the spectral radius for unweighted graphs have been investigated to a great extent in the literature [1,2,3,4,5,7,8]. The main result of this paper, contained in Section 2, gives a new upper bounds on the spectral radius for weighted graphs, where the edge weights are positive definite matrices. We compare our bound with in [6] and [9].

2. An Upper Bound On The Spectral Radius Of Weighted Graphs

2.1. Theorem (Rayleigh-Ritz [10]). *Let $A \in M_n$ be Hermitian, and let the eigenvalues of A be ordered such that $\rho_n \leq \rho_{n-1} \leq ... \leq \rho_1$. Then*

$$\rho_n x^{\ T} x \leq x^T A x \leq \rho_1 x^T x$$

and

$$\rho_{\max} = \rho_1 = \max_{x \neq 0} \frac{x^T A x}{x^T x} = \max_{x^T x = 1} x^T A x$$

$$\rho_{\min} = \rho_n = \min_{x \neq 0} \frac{x^T A x}{x^T x} = \min_{x^T x = 1} x^T A x$$

for all $x \in \mathbb{C}^n$.

2.2. Proposition. *Let $A \in M_n$ have eigenvalues $\{\rho_i\}$. Even if A is not Hermitian, one has the bounds*

$$\min_{x \neq 0} \left| \frac{x^T A x}{x^T x} \right| \leq |\rho_i| \leq \max_{x \neq 0} \left| \frac{x^T A x}{x^T x} \right| \tag{2.1}$$

for $i = 1, 2, .., n$.

Proof. Let $A \in M_n$ be and $\{\rho_i\}$ be eigenvalues of A for $i = 1, 2, .., n$. Since $x^T x \geq 0$ for any $x \in \mathbb{C}^n$, we get

$$\frac{x^T A x}{x^T x} \leq \left| \frac{x^T A x}{x^T x} \right| = \frac{\left| x^T A x \right|}{x^T x}$$

i.e.,

$$\max_{x \neq o} \frac{x^T A x}{x^T x} \leq \max_{x \neq o} \frac{\left| x^T A x \right|}{x^T x} \tag{2.2}$$

On the other hand, from Cauchy-Schwarz inequality, we have

$$\|A\| = \max_{x \neq o} \frac{x^T A x}{x^T x} \tag{2.3}$$

Then we get

$$(2.4)\qquad |\rho_i| \leq \|A\| \leq \max_{x\neq o}\frac{\left|x^TAx\right|}{x^Tx} = \max_{x\neq o}\left|\frac{x^TAx}{x^Tx}\right|$$

from (2.2) and (2.3) such that $\{\rho_i\}$ is eigenvalue of A for $i = 1, 2, .., n$. Now let x be eigenvector corresponding to eigenvalue ρ_n of A. Then we get

$$\left|x^TAx\right| = |\rho_n|\, x^Tx$$

i.e.,

$$(2.5)\qquad \min_{x\neq o}\frac{\left|x^TAx\right|}{x^Tx} = |\rho_n| \leq |\rho_i|$$

Hence, we have

$$\min_{x\neq 0}\left|\frac{x^TAx}{x^Tx}\right| \leq |\rho_i| \leq \max_{x\neq 0}\left|\frac{x^TAx}{x^Tx}\right|$$

from inequalities in (2.4) and (2.5). □

2.3. Corollary. *Let $A \in M_n$ have eigenvalues $\{\rho_i\}$. Even if A is not Hermitian, one has the bounds*

$$(2.6)\qquad \min_{x\neq 0,\ y\neq 0}\left|\frac{x^TAy}{x^Ty}\right| \leq |\rho_i| \leq \max_{x\neq 0,\ y\neq 0}\left|\frac{x^TAy}{x^Ty}\right|$$

for any $\bar{x} \in R^n$ $(\bar{x} \neq \bar{0})$, $\bar{y} \in R^n$ $(\bar{y} \neq \bar{0})$ and for $i = 1, 2, .., n$.

Proof. If y is taken as eigenvector corresponding to eigenvalue ρ_n of A or eigenvector corresponding to ρ_1 eigenvalue of A, we can see inequality in (2.6) as similar to the proof of Proposition 2.2. □

2.4. Lemma (Horn and Johnson [10]). *Let B be a Hermitian $n \times n$ matrix with ρ_1 as its largest eigenvalue, in modulus. then for any $\bar{x} \in R^n$ $(\bar{x} \neq \bar{0})$, $\bar{y} \in R^n$ $(\bar{y} \neq \bar{0})$, the spectral radius $|\rho_1|$ satisfies*

$$(2.7)\qquad \left|\bar{x}^TB\bar{y}\right| \leq |\rho_1|\sqrt{\bar{x}^T\bar{x}}\sqrt{\bar{y}^T\bar{y}}$$

Equality holds if and only if $\bar{x}$ is an eigenvector of B corresponding to ρ_1 and $\bar{y} = \alpha\bar{x}$ for some $\alpha \in R$.

2.5. Theorem (Das et al. [6]). *Let G be a weighted graph which is simple, connected and let ρ_1 be the largest eigenvalue (in modulus) of G so that $|\rho_1|$ is the spectral radius of G. Then*

$$(2.8)\qquad |\rho_1| \leq \max_{i\sim j}\left\{\sqrt{\sum_{k:k\sim i}\rho_1(w_{i,k})\sum_{k:k\sim j}\rho_1(w_{j,k})}\right\}$$

where $w_{i,j}$ is the positive definite weight matrix of order p of the edge ij. Moreover equality holds in (2.8) if and only if

(i) G is a weighted-regular graph or G is a weight-semiregular bipartite graph;

(ii) $w_{i,j}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{i,j})$ for all i, j.

2.6. Theorem ([9]). *Let G be a weighted graph which is simple, connected and let ρ_1 be the largest eigenvalue (in modulus) of G so that $|\rho_1|$ is the spectral radius of G. Then*

$$|\rho_1| \leq \max_i \left\{ \sqrt{\sum_{k:k\sim i} \rho_1(w_{i,k}^2) + \sum_j \sum_{k' \in N_i \cap N_j} \rho_1(w_{i,k'} w_{k',j})} \right\} \tag{2.9}$$

where $w_{i,j}$ is the positive definite weight matrix of order p of the edge ij and $N_i \cap N_j$ is the set of common neighbors of i and j. Moreover equality holds in (2.9) if and only if

(i) G is a weighted-regular graph or G is a weight-semiregular bipartite graph;

(ii) $w_{i,j}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{i,j})$ for all i, j.

2.7. Theorem. *Let G be a weighted graph which is simple, connected and let ρ_1 be the largest eigenvalue (in modulus) of G ,so that $|\rho_1|$ is the spectral radius of G. Then*

(2.10)

$$|\rho_1| \leq \max_{i\sim j} \left[\left(\begin{array}{c} \sum\limits_{k:k\sim i} \rho_1(w_{i,k}^2) \\ +\sum\limits_t \sum\limits_{k' \in N_i \cap N_t} \rho_1(w_{i,k'} w_{k',t}) \end{array} \right) \left(\begin{array}{c} \sum\limits_{k:k\sim j} \rho_1(w_{j,k}^2) \\ +\sum\limits_t \sum\limits_{k' \in N_j \cap N_t} \rho_1(w_{j,k'} w_{k',t}) \end{array} \right) \right]^{\frac{1}{4}}$$

where $w_{i,j}$ is the positive definite weight matrix of order p of the edge ij and $N_i \cap N_j$ is the set of common neighbors of i and j.

Proof. Let consider matrix $A^2(G)$ such that $A(G)$ is the adjacency matrix of graph G and $|\rho_1|$ the spectral radius of $A(G)$ adjacency matrix. So, ρ_1^2 is also the spectral radius of $A^2(G)$. Let $\bar{X} = (\bar{x}_1^T, \bar{x}_2^T, ..., \bar{x}_n^T)^T$ be an eigenvector corresponding to the spectral radius ρ_1^2 for $A^2(G)$. We assume that $\bar{x}_i$ is the vector component of $\bar{X}$ such that

$$\bar{x}_i^T \bar{x}_i = \max_{k\in V} \left\{ \bar{x}_k^T \bar{x}_k \right\} \tag{2.11}$$

and for every $k \in V$ we get

$$\bar{x}_j^T \bar{x}_j \geq \bar{x}_k^T \bar{x}_k \tag{2.12}$$

such that $i \sim j$. Since $\bar{X}$ is nonzero, so is $\bar{x}_i$ .The (i, j) th block of A^2 is

$$A^2(G) = \begin{cases} \sum\limits_{k:k\sim i} w_{i,k}^2 & \text{if } i \sim j \\ \sum\limits_{k:k\in N_i \cap N_j} w_{i,k} w_{k,j} & \text{otherwise.} \end{cases}$$

We have

$$A^2(G)\bar{X} = \rho_1^2 \bar{X} \tag{2.13}$$

From the i th equation of (2.13), we have

$$\rho_1^2 \bar{x}_i = \sum_{k:k\sim i} w_{i,k}^2 \bar{x}_i + \sum_t \sum_{k:k\in N_i \cap N_t} w_{i,k} w_{k,t} \bar{x}_t$$

i.e.,

$$\rho_1^2 \bar{x}_i^T \bar{x}_i = \sum_{k:k\sim i} \bar{x}_i^T w_{i,k}^2 \bar{x}_i + \sum_t \sum_{k:k\in N_i \cap N_t} \bar{x}_i^T \left(w_{i,k} w_{k,t} \right) \bar{x}_t \tag{2.14}$$

Taking modulus on the both sides of equality in (2.14), we get

$$\begin{aligned} \rho_1^2 \bar{x}_i^T \bar{x}_i &= \left| \sum_{k:k\sim i} \bar{x}_i^T w_{i,k}^2 \bar{x}_i + \sum_t \sum_{k:k\in N_i\cap N_t} \bar{x}_i^T \left(w_{i,k} w_{k,t}\right) \bar{x}_t \right| \\ &\leq \sum_{k:k\sim i} \left| \bar{x}_i^T w_{i,k}^2 \bar{x}_i \right| + \sum_t \sum_{k:k\in N_i\cap N_t} \left| \bar{x}_i^T \left(w_{i,k} w_{k,t}\right) \bar{x}_t \right| \end{aligned} \tag{2.15}$$

Since $w_{i,k}$ is the positive definite matrix for every i, k, $w_{i,k}^2$ matrices are also positive definite. So, using inequality in (2.7),we have

$$\begin{aligned} \rho_1^2 \bar{x}_i^T \bar{x}_i &\leq \sum_{k:k\sim i} \left| \bar{x}_i^T w_{i,k}^2 \bar{x}_i \right| + \sum_t \sum_{k:k\in N_i\cap N_t} \left| \bar{x}_i^T \left(w_{i,k} w_{k,t}\right) \bar{x}_t \right| \\ &\leq \sqrt{\bar{x}_i^T \bar{x}_i}\sqrt{\bar{x}_i^T \bar{x}_i} \sum_{k:k\sim i} \rho_1\left(w_{i,k}^2\right) + \sum_t \sum_{k:k\in N_i\cap N_t} \left| \bar{x}_i^T \left(w_{i,k} w_{k,t}\right) \bar{x}_t \right| \end{aligned} \tag{2.16}$$

Now we will discuss the modulus $\left| \bar{x}_i^T \left(w_{i,k} w_{k,t}\right) \bar{x}_t \right|$ at the two cases for $k \sim i$ and $k \sim t$ such that $1 \leq t \leq n$.

Case 1: Let $w_{i,k} w_{k,t}$ be Hermitian matrix for $k \sim i$ and $k \sim t$ such that $1 \leq t \leq n$. Then, we have

$$\left| \bar{x}_i^T \left(w_{i,k} w_{k,t}\right) \bar{x}_t \right| \leq \rho_1\left(w_{i,k} w_{k,t}\right) \sqrt{\bar{x}_i^T \bar{x}_i}\sqrt{\bar{x}_t^T \bar{x}_t}$$

from inequality in (2.7) and we have

$$\rho_1^2 \bar{x}_i^T \bar{x}_i \leq \sqrt{\bar{x}_i^T \bar{x}_i}\sqrt{\bar{x}_i^T \bar{x}_i} \sum_{k:k\sim i} \rho_1\left(w_{i,k}^2\right) + \sqrt{\bar{x}_i^T \bar{x}_i}\sqrt{\bar{x}_t^T \bar{x}_t} \sum_t \sum_{k:k\in N_i\cap N_t} \rho_1\left(w_{i,k} w_{k,t}\right) \tag{2.17}$$

from inequality in (2.16).

Case 2: Let $w_{i,k} w_{k,t}$ not be Hermitian matrix for $k \sim i$ and $k \sim t$ such that $1 \leq t \leq n$. From equality in (2.11) and inequality in (2.12), the ratio of

$$\frac{\left| \bar{x}_i^T \left(w_{i,k} w_{k,t}\right) \bar{x}_t \right|}{\sqrt{\bar{x}_i^T \bar{x}_i}\sqrt{\bar{x}_t^T \bar{x}_t}}$$

is minimum. So, from (2.6) and inequality in (2.16), we have

$$\begin{aligned} \rho_1^2 \bar{x}_i^T \bar{x}_i \leq{}& \sqrt{\bar{x}_i^T \bar{x}_i}\sqrt{\bar{x}_i^T \bar{x}_i} \sum_{k:k\sim i} \rho_1\left(w_{i,k}^2\right) \\ &+ \sqrt{\bar{x}_i^T \bar{x}_i}\sqrt{\bar{x}_t^T \bar{x}_t} \sum_t \sum_{k:k\in N_i\cap N_t} \left|\rho_m\right| \left(w_{i,k} w_{k,t}\right) \end{aligned} \tag{2.18}$$

such that $m = 1, 2, .., p$. Let $\rho_1\left(w_{i,k} w_{k,t}\right)$ the largest eigenvalue of $w_{i,k} w_{k,t}$. As we expand inequality in (2.18), we get again in inequality (2.16). Hence, we have

$$\rho_1^2 \bar{x}_i^T \bar{x}_i \leq \sqrt{\bar{x}_i^T \bar{x}_i}\sqrt{\bar{x}_i^T \bar{x}_i} \sum_{k:k\sim i} \rho_1\left(w_{i,k}^2\right) + \sqrt{\bar{x}_i^T \bar{x}_i}\sqrt{\bar{x}_t^T \bar{x}_t} \sum_t \sum_{k:k\in N_i\cap N_t} \rho_1\left(w_{i,k} w_{k,t}\right) \tag{2.19}$$

from Case 1 and Case 2.

From the j th equation of (2.12), we have

$$\rho_1^2 \bar{x}_j = \sum_{k:k\sim j} w_{j,k}^2 \bar{x}_j + \sum_t \sum_{k:k\in N_{ji}\cap N_t} w_{j,k} w_{k,t} \bar{x}_t$$

i.e.,

$$\rho_1^2 \bar{x}_j^T \bar{x}_j = \sum_{k:k\sim j} \bar{x}_j^T w_{j,k}^2 \bar{x}_j + \sum_t \sum_{k:k\in N_j \cap N_t} \bar{x}_j^T \left(w_{j,k} w_{k,t}\right) \bar{x}_t \tag{2.20}$$

Taking modulus on both sides and using of inequality in (2.7), we get

$$\begin{aligned}\rho_1^2 \bar{x}_j^T \bar{x}_j &= \left| \sum_{k:k\sim j} \bar{x}_j^T w_{j,k}^2 \bar{x}_j + \sum_t \sum_{k:k\in N_j \cap N_t} \bar{x}_j^T \left(w_{j,k} w_{k,t}\right) \bar{x}_t \right| \\ &\le \sum_{k:k\sim j} \left| \bar{x}_j^T w_{j,k}^2 \bar{x}_j \right| + \sum_t \sum_{k:k\in N_j \cap N_t} \left| \bar{x}_j^T \left(w_{j,k} w_{k,t}\right) \bar{x}_t \right| \\ &\le \sqrt{\bar{x}_j^T \bar{x}_j} \sqrt{\bar{x}_j^T \bar{x}_j} \sum_{k:k\sim j} \rho_1 \left(w_{j,k}^2\right) + \sum_t \sum_{k:k\in N_j \cap N_t} \left| \bar{x}_j^T \left(w_{j,k} w_{k,t}\right) \bar{x}_t \right| \qquad (2.21)\end{aligned}$$

Similarly, if we consider $w_{j,k}w_{k,t}$ matrix is Hermitian matrix or not Hermitian for $k \sim j$ and $k \sim t$ such that $1 \le t \le n$, we have

$$\rho_1^2 \bar{x}_j^T \bar{x}_j \le \sqrt{\bar{x}_j^T \bar{x}_j} \sqrt{\bar{x}_j^T \bar{x}_j} \sum_{k:k\sim j} \rho_1 \left(w_{j,k}^2\right) + \sqrt{\bar{x}_j^T \bar{x}_j} \sqrt{\bar{x}_t^T \bar{x}_t} \sum_t \sum_{k:k\in N_j \cap N_t} \rho_1 \left(w_{j,k} w_{k,t}\right) \tag{2.22}$$

Multiplying with $\bar{x}_j^T \bar{x}_j$ on right sides of inequality in (2.19) and from equality in (2.11), we get

$$\begin{aligned}\rho_1^2 \bar{x}_i^T \bar{x}_i &\le \left(\bar{x}_j^T \bar{x}_j\right)\left(\bar{x}_i^T \bar{x}_i\right) \sum_{k:k\sim i} \rho_1 \left(w_{i,k}^2\right) \\ &\quad + \left(\bar{x}_j^T \bar{x}_j\right)\left(\bar{x}_i^T \bar{x}_i\right) \sum_t \sum_{k:k\in N_i \cap N_t} \rho_1 \left(w_{i,k} w_{k,t}\right) \qquad (2.23)\end{aligned}$$

Similarly, multiplying with $\bar{x}_i^T \bar{x}_i$ on right sides of (2.22) and from inequality in (2.12), we have

$$\begin{aligned}\rho_1^2 \bar{x}_j^T \bar{x}_j &\le \left(\bar{x}_j^T \bar{x}_j\right)\left(\bar{x}_i^T \bar{x}_i\right) \sum_{k:k\sim j} \rho_1 \left(w_{j,k}^2\right) \\ &\quad + \left(\bar{x}_j^T \bar{x}_j\right)\left(\bar{x}_i^T \bar{x}_i\right) \sum_t \sum_{k:k\in N_j \cap N_t} \rho_1 \left(w_{j,k} w_{k,t}\right) \qquad (2.24)\end{aligned}$$

From inequalities in (2.23) and (2.24), we get

$$\begin{aligned}|\rho_1| &\le \left\{ \left(\begin{array}{c} \sum\limits_{k:k\sim i} \rho_1 \left(w_{i,k}^2\right) \\ + \sum\limits_t \sum\limits_{k:k\in N_i \cap N_t} \rho_1 \left(w_{i,k} w_{k,t}\right) \end{array} \right) \left(\begin{array}{c} \sum\limits_{k:k\sim j} \rho_1 \left(w_{j,k}^2\right) \\ + \sum\limits_t \sum\limits_{k:k\in N_j \cap N_t} \rho_1 \left(w_{j,k} w_{k,t}\right) \end{array} \right) \right\}^{\frac{1}{4}} \\ &\le \max_{i\sim j} \left\{ \left(\begin{array}{c} \sum\limits_{k:k\sim i} \rho_1 \left(w_{i,k}^2\right) \\ + \sum\limits_t \sum\limits_{k:k\in N_i \cap N_t} \rho_1 \left(w_{i,k} w_{k,t}\right) \end{array} \right) \left(\begin{array}{c} \sum\limits_{k:k\sim j} \rho_1 \left(w_{j,k}^2\right) \\ + \sum\limits_t \sum\limits_{k:k\in N_j \cap N_t} \rho_1 \left(w_{j,k} w_{k,t}\right) \end{array} \right) \right\}^{\frac{1}{4}}\end{aligned}$$

$\square$

2.8. Corollary. *Let G be a simple connected weighted graph where each edge weight $w_{i,j}$ is a positive number. Then*

$$|\rho_1| \le \max_{i\sim j} \left\{ \left\{ \left(\begin{array}{c} \sum\limits_{k:k\sim i} w_{i,k}^2 \\ + \sum\limits_t \sum\limits_{k:k\in N_i \cap N_t} w_{i,k} w_{k,t} \end{array} \right) \left(\begin{array}{c} \sum\limits_{k:k\sim j} w_{j,k}^2 \\ + \sum\limits_t \sum\limits_{k:k\in N_j \cap N_t} w_{j,k} w_{k,t} \end{array} \right) \right\}^{\frac{1}{4}} \right\}$$

Moreover equality if and only if G bipartite regular graph or a bipartite semiregular graph.

2.9. Corollary. *Let G be a simple connected unweighted graph Then*

$$|\rho_1| \le \max_{i\sim j}\left\{\left\{\left(d_i+\sum_t |N_i\cap N_t|\right)\left(d_j+\sum_t |N_j\cap N_t|\right)\right\}^{\frac{1}{4}}\right\}$$

where d_i and d_j are the degrees of the vertices i and j respectively. Moreover equality if and only if G bipartite regular graph or a bipartite semiregular graph.

Proof. For unweighted graph, $w_{i,j} = 1$ for $i \sim j$. Therefore $w_{i,k}w_{k,t} = N_i \cap N_t$. Using Theorem 2.7 we get the required result. $\square$

1. Exercise. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ weighted graphs where $V_1 = \{1,2,3,4,5\}$, $E_1 = \{\{1,2\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{4,5\}\}$ and each weights are the positive definite matrix of three order. Let

$$V_2 = \{1,2,3,4,5,6\}, E_2 = \left\{\begin{array}{c}\{1,2\},\{2,3\},\{2,4\},\\ \{2,5\},\{5,6\}\end{array}\right\}$$

such that each weights are the positive definite matrix of order two. Assume that the following adjacency matrices of G_1 and G_2 we give

$$A(G_1) = \begin{bmatrix}
0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & -1 & 0 & 0 & 0 & 0 & 5 & 0 & 2 & 3 & 1 & -1 & 11 & -3 & 1\\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 5 & 2 & 1 & 3 & -1 & -3 & 11 & 1\\
0 & 2 & -1 & 0 & 0 & 0 & 2 & 2 & 5 & -1 & -1 & 5 & 1 & 1 & 8\\
0 & 0 & 0 & 5 & 0 & 2 & 0 & 0 & 0 & 10 & 7 & 5 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 5 & 2 & 0 & 0 & 0 & 7 & 10 & 5 & 0 & 0 & 0\\
0 & 0 & 0 & 2 & 2 & 5 & 0 & 0 & 0 & 5 & 5 & 17 & 0 & 0 & 0\\
0 & 0 & 0 & 3 & 1 & -1 & 10 & 7 & 5 & 0 & 0 & 0 & 6 & 3 & -1\\
0 & 0 & 0 & 1 & 3 & -1 & 7 & 10 & 5 & 0 & 0 & 0 & 3 & 6 & -1\\
0 & 0 & 0 & -1 & -1 & 5 & 5 & 5 & 17 & 0 & 0 & 0 & -1 & -1 & 9\\
0 & 0 & 0 & 11 & -3 & 1 & 0 & 0 & 0 & 6 & 3 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & -3 & 11 & 1 & 0 & 0 & 0 & 3 & 6 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 1 & 8 & 0 & 0 & 0 & -1 & -1 & 9 & 0 & 0 & 0
\end{bmatrix}$$

and

$$A(G_2) = \begin{bmatrix}
0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 1 & 0 & 0 & 3 & 1 & 5 & -3 & 11 & 1 & 0 & 0\\
1 & 3 & 0 & 0 & 1 & 4 & -3 & 4 & 1 & 2 & 0 & 0\\
0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 5 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 11 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & -1\\
0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 6\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 6 & 0 & 0
\end{bmatrix}$$

The spectral radius of G_1 is $\rho_1 = 28,61$ and the spectral radius of G_2 is $\rho_1 = 14.43$ where rounded two decimal places and the above mentioned bounds give the following results.

(2.25)

	ρ_1	(2.8)	(2.9)	(2.10)
G_1	28.61	35.92	31.56	31.12
G_2	14.43	22.41	17.21	16.08

References

[1] Berman, A., Zhang,X.D. *On the spectral radius of graphs with cut vertices*, J. Combin. Theory Ser. B **83**, 233-240, 2001.

[2] Brualdi, R.A., Hoffman, A.J. *On the spectral radius of a (0,1)-matrix*, Linear Algebra Appl. **65**, 133-146, 1985.

[3] Cvetkovic, D., Rowlinson, P. *The largest eigenvalue of a graph:a survey*, Linear Multilinear Algebra **28**, 3-33, 1990.

[4] Das, K.C., Kumar, P. *Bounds on the greatest eigenvalue of graphs*, Indian J. Pure Appl. Math. **123**, 65-74, 1993.

[5] Das, K.C., Kumar, P. *Some new bounds on the spectral radius of graphs*, Discrete Mathematics **281**, 149-161, 2004.

[6] Das, K.C., Bapat, R.B. *A sharp upper bound on the spectral radius of weighted graphs*, Discrete Math. **308** (15), 3180-3186, 2008.

[7] Hong, Y. *Bounds of eigenvalues of graphs*, Discrete Math. **123**, 65-74, 1993.

[8] Stanley, R.P. *A bound on the spectral radius of graphs with e edges*, Lin. Alg. Appl. **67**, 267-269, 1987.

[9] Büyükköse, S., Sorgun, S. *A bound on the Spectral radius of a weighted graph*, G. U. Journal Of Science **22**, 263-266, 2009.

[10] Horn, R.A., Johnson, C.R. *Matrix Analysis*, Cambridge University Press, New York, 1985.