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HALF-INVERSE SPECTRAL PROBLEM FOR DIFFERENTIAL PENCILS WITH INTERACTION-POINT AND EIGENVALUE-DEPENDENT BOUNDARY CONDITIONS

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Abstract

The inverse spectral problem of recovering for a quadratic pencil of Sturm-Liouville operators with the interaction point and the eigenvalue parameter linearly contained in the boundary conditions are studied. The uniqueness theorem for the solution of the inverse problem according to the Weyl function is proved and a constructive procedure for finding its solution is obtained.

Keywords: Inverse spectral problem; Quadratic pencil of Sturm-Liouville operators; Eigenvalue-dependent boundary conditions; Interaction point.

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1. Introduction

We consider the boundary value problem (BVP) $L = L(q(x), \alpha, h_j, H_j,\ j = 0, 1)$:

$$ly := y'' + (\lambda^2 - q(x))y = 0, \quad x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right), \tag{1.1}$$

$$U(y) := y'(0) - (h_1\lambda + h_0)y(0) = 0, \tag{1.2}$$

$$V(y) := y'(\pi) + (H_1\lambda + H_0)y(\pi) = 0, \tag{1.3}$$

$$I(y) := \begin{cases} y\left(\frac{\pi}{2} + 0\right) = y\left(\frac{\pi}{2} - 0\right) = y\left(\frac{\pi}{2}\right), \\ y'\left(\frac{\pi}{2} + 0\right) - y'\left(\frac{\pi}{2} - 0\right) = 2\alpha\lambda y\left(\frac{\pi}{2}\right), \end{cases} \tag{1.4}$$

where the potential $q(x) \in L_1(0, \pi)$ is a complex-valued function, $\alpha, h_j, H_j \in \mathbb{C}$, $j = 0, 1$; $h_1H_1 = -1$ and $\alpha(h_1 + H_1) - 2 \neq \pm i(h_1 + H_1 + 2\alpha)$, λ is a spectral parameter.

Notice that, we can understand problem (1.1),(1.4) as one of the treatments of the equation

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(1.5) $\quad y'' + (\lambda^2 - 2\lambda p(x) - q(x))y = 0, \quad x \in (0,\pi),$

when $p(x) = \alpha\delta\left(x - \frac{\pi}{2}\right)$(see[1]), where $\delta(x)$ is the Dirac function.

Differential equations with linear or non-linearly dependence on the spectral parameter arise in various problems of mathematics as well as in applications. In particular, several examples of spectral problems arising in mechanical engineering and having differential equations and boundary conditions depending on the spectral parameter are provided in [18, 21, 22].

Inverse problems for differential pensils are more diffucult to investigate and nowadays there exists only a small number of papers in this direction. In particular, in the case $p(x) \equiv 0$, the inverse problems for equation (1.5) with λ-dependent boundary conditions were investigated in [3, 4, 8, 12]. Such problems play an important role in mathematics and have many applications in natural sciences and engineering (see[11, 16, 17, 19] and the references therein). Some aspects of inverse spectral problems for second order differential pensils ($p(x) \in W_1^1(0,\pi)$, $h_1 = H_1 = 0$) were investigated in [6, 13, 14, 15, 20, 24]. For this inverse problem, which in the case $p(x) \in W_1^1(0,\pi)$ is proved in [7, 23].

In this paper uniqueness theorem is proved and a constructive procedure for solving the half inverse problem is given. As the basic spectral characteristic we introduce and investigate the so-called Weyl function, which is an analogue of the classical Weyl function for the Sturm-Liouville operators (see[12]).

2. Properties Of The Spectral Characteristics

Let $y(x)$ and $z(x)$ be continuously differentiable functions on $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right)$. Denote $\langle y, z\rangle := yz' - y'z$. If $y(x)$ and $z(x)$ satisfy the matching conditions (1.4), then

(2.1) $\quad \langle y, z\rangle_{x=\frac{\pi}{2}-0} = \langle y, z\rangle_{x=\frac{\pi}{2}+0}\,,$

i.e. the function $\langle y, z\rangle$ is continuous on $(0,\pi)$.

Let $\varphi(x,\lambda)$, $\psi(x,\lambda)$, $C(x,\lambda)$, $S(x,\lambda)$ be solutions of equation (1.1) under the conditions

$$C(0,\lambda) = S'(0,\lambda) = \varphi(0,\lambda) = \psi(\pi,\lambda) = 1,$$
$$C'(0,\lambda) = S(0,\lambda) = U(\varphi) = V(\psi) = 0.$$

Denote

$$\Delta(\lambda) = \langle \varphi(x,\lambda), \psi(x,\lambda)\rangle\,.$$

By virtue of (2.1) and the Ostrogradskii-Liouville theorem (see[9]), $\Delta(\lambda)$ does not depend on x. The function $\Delta(\lambda)$ is called characteristic function of L. Clearly,

(2.2) $\quad \Delta(\lambda) = -V(\varphi) = U(\psi).$

Let $C_0(x,\lambda)$ and $S_0(x,\lambda)$ be smooth solutioons of (1.1) on the interval $[0,\pi]$ under the initial conditions

$$C_0(x,\lambda) = S_0'(x,\lambda) = 1, C_0'(x,\lambda) = S_0(x,\lambda) = 0.$$

(2.3) $\quad C(x,\lambda) = C_0(x,\lambda), \quad S(x,\lambda) = S_0(x,\lambda), \quad x < \dfrac{\pi}{2},$

(2.4) $$\begin{matrix} C(x,\lambda) = A_1C_0(x,\lambda) + B_1S_0(x,\lambda), \\ S(x,\lambda) = A_2C_0(x,\lambda) + B_2S_0(x,\lambda), \end{matrix} \;,\; x > \frac{\pi}{2},$$

where

(2.5) $$\begin{cases} A_1 = 1 - 2\alpha\lambda C_0\left(\frac{\pi}{2},\lambda\right)S_0\left(\frac{\pi}{2},\lambda\right), \; B_1 = 2\alpha\lambda C_0^2\left(\frac{\pi}{2},\lambda\right) \\ A_2 = -2\alpha\lambda S_0^2\left(\frac{\pi}{2},\lambda\right), \; B_2 = 1 + 2\alpha\lambda C_0\left(\frac{\pi}{2},\lambda\right)S_0\left(\frac{\pi}{2},\lambda\right) \end{cases}$$

It is easy to verify that the function $C_0(x,\lambda)$ satisfies the following integral equation

$$C_0(x,\lambda) = \cos\lambda x + \int_0^x \frac{\sin\lambda(x-t)}{\lambda} q(t) C_0(t,\lambda)dt \tag{2.6}$$

It follows from (2.6) that

$$\begin{gathered} C_0(x,\lambda) = \cos\lambda x + \tfrac{\sin\lambda x}{2\lambda}\int_0^x q(t)dt + \tfrac{1}{2\lambda}\int_0^x q(t)\sin\lambda(x-2t)dt \\ + \bigcirc\left(\tfrac{1}{\lambda^2}\exp(|\tau|\,x)\right), \end{gathered} \tag{2.7}$$

$$\begin{gathered} C_0'(x,\lambda) = -\lambda\sin\lambda x + \tfrac{\cos\lambda x}{2}\int_0^x q(t)dt + \tfrac{1}{2}\int_0^x q(t)\cos\lambda(x-2t)dt \\ + \bigcirc\left(\tfrac{1}{\lambda}\exp(|\tau|\,x)\right), \end{gathered} \tag{2.8}$$

Analogously,

$$\begin{gathered} S_0(x,\lambda) = \tfrac{\sin\lambda x}{\lambda} - \tfrac{\cos\lambda x}{2\lambda^2}\int_0^x q(t)dt + \tfrac{1}{2\lambda^2}\int_0^x q(t)\cos\lambda(x-2t)dt \\ + \bigcirc\left(\tfrac{1}{\lambda^3}\exp(|\tau|\,x)\right), \end{gathered} \tag{2.9}$$

$$\begin{gathered} S_0'(x,\lambda) = \cos\lambda x + \tfrac{\sin\lambda x}{2\lambda}\int_0^x q(t)dt - \tfrac{1}{2\lambda}\int_0^x q(t)\sin\lambda(x-2t)dt \\ + \bigcirc\left(\tfrac{1}{\lambda^2}\exp(|\tau|\,x)\right) \end{gathered} \tag{2.10}$$

where $\tau = \mathrm{Im}\lambda$.

By virtue of (2.5) and (2.7)-(2.10)

$$A_1 = \quad 1 - \alpha\sin\lambda\pi + \bigcirc\left(\frac{1}{\lambda}\right),\ B_1 = \alpha\lambda(1+\cos\lambda\pi) + \alpha\sin\lambda\pi\int_0^{\frac{\pi}{2}} q(t)dt + \bigcirc\left(\frac{1}{\lambda}\right),$$

$$A_2 = \quad \alpha\frac{\cos\lambda\pi - 1}{\lambda} + \bigcirc\left(\frac{1}{\lambda^2}\right),\ B_2 = 1 + \alpha\sin\lambda\pi - \alpha\frac{\cos\lambda\pi}{\lambda}\int_0^{\frac{\pi}{2}} q(t)dt + \bigcirc\left(\frac{1}{\lambda^2}\right).$$

Since $\varphi(x,\lambda) = C(x,\lambda) + (h_1\lambda + h_0)S(x,\lambda)$, we calculate using (2.3)-(2.10)

$$\varphi(x,\lambda) = \cos\lambda x + h_1\sin\lambda x + \bigcirc\left(\frac{1}{\lambda}\exp(|\tau|\,x)\right),\quad x < \frac{\pi}{2}, \tag{2.11}$$

$$\left.\begin{gathered} \varphi(x,\lambda) = (1-h_1\alpha)\cos\lambda x + (h_1+\alpha)\sin\lambda x + h_1\alpha\cos\lambda(\pi - x) \\ -\alpha\sin\lambda(\pi-x) + \bigcirc\left(\tfrac{1}{\lambda}\exp(|\tau|\,x)\right) \end{gathered}\right.,\ x > \frac{\pi}{2}, \tag{2.12}$$

$$\varphi'(x,\lambda) = \lambda(-\sin\lambda x + h_1\cos\lambda x) + \bigcirc\left(\exp(|\tau|\,x)\right),\quad x < \frac{\pi}{2}, \tag{2.13}$$

$$\left.\begin{gathered} \varphi'(x,\lambda) = \lambda((h_1\alpha - 1)\sin\lambda x + (h_1+\alpha)\cos\lambda x + h_1\alpha\sin\lambda(\pi-x) \\ +\alpha\cos\lambda(\pi-x)) + \bigcirc\left(\exp(|\tau|\,x)\right) \end{gathered}\right.,\ x > \frac{\pi}{2}, \tag{2.14}$$

Can be obtained analogously

$$\psi(x,\lambda) = \cos\lambda(\pi-x) + H_1\sin\lambda(\pi-x) + \bigcirc\left(\frac{1}{\lambda}\exp(|\tau|\,(\pi-x))\right),\quad x > \frac{\pi}{2}, \tag{2.15}$$

$$\left.\begin{gathered} \psi(x,\lambda) = (1-H_1\alpha)\cos\lambda(\pi-x) + (H_1+\alpha)\sin\lambda(\pi-x) \\ +H_1\alpha\cos\lambda x - \alpha\sin\lambda x + \bigcirc\left(\tfrac{1}{\lambda}\exp(|\tau|\,(\pi-x))\right) \end{gathered}\right.,\quad x < \frac{\pi}{2}, \tag{2.16}$$

$$\psi'(x,\lambda) = \lambda(\sin\lambda(\pi-x) - H_1\cos\lambda(\pi-x)) + \bigcirc\left(\exp(|\tau|\,(\pi-x))\right),\quad x > \frac{\pi}{2}, \tag{2.17}$$

$$\left.\begin{gathered} \psi'(x,\lambda) = \lambda((1-H_1\alpha)\sin\lambda(\pi-x) - (H_1+\alpha)\cos\lambda(\pi-x) \\ -H_1\alpha\sin\lambda x - \alpha\cos\lambda x) + \bigcirc\left(\exp(|\tau|\,(\pi-x))\right). \end{gathered}\right.,\quad x < \frac{\pi}{2}, \tag{2.18}$$

It follows from (2.2), (2.12) and (2.14) that

$$\Delta(\lambda) = \lambda D \sin(\lambda - w)\pi + \bigcirc\left(\exp(|\tau|\,\pi)\right), \tag{2.19}$$

where

$$D = \sqrt{(h_1 + H_1 + 2\alpha)^2 + (2 - \alpha h_1 - \alpha H_1)^2}$$

$$w = \frac{1}{2\pi i} \ln \frac{i(h_1 + H_1 + 2\alpha) + \alpha(h_1 + H_1) - 2}{i(h_1 + H_1 + 2\alpha) - \alpha(h_1 + H_1) + 2} \tag{2.20}$$

By the standard method using (2.19) and Rouche's theorem (see, for example,[2]) one can show that the eigenvalues λ_n, $n \in \mathbb{Z} := \{n : n = 0, \pm 1, \pm 2, ...\}$ have the form

$$\lambda_n = n + w + \bigcirc\left(\frac{1}{n}\right), \quad |n| \to \infty \tag{2.21}$$

According to (2.21) for sufficiently large $|n|$ the eigenvalues λ_n are simple. By virtue of (2.20), (2.21) the specification of the spectrum determines the value w up to an integer summand.

Using Hadamard's factorization theorem [10, p.289] and the asymptotics (2.19) one can expand the characteristic function $\Delta(\lambda)$ into an infinite product. We have

$$\Delta(\lambda) = c\exp(c_1\lambda)\lambda^m \prod_{\lambda_n \neq 0} \left(1 - \frac{\lambda}{\lambda_n}\right) \exp\left(\frac{\lambda}{\lambda_n}\right), \; c, c_1 - \text{const}, \tag{2.22}$$

where $m \geq 0$ is the multiplicity of the zero eigenvalue. Consider the function $\Delta_0(\lambda) = \lambda \sin(\lambda - w)\pi$, whose expansionhas the form

$$\Delta_0(\lambda) = c^0 \exp(c_1^0\lambda) \prod_{n=-\infty}^{\infty} \left(1 - \frac{\lambda}{\lambda_{n0}}\right) \exp\left(\frac{\lambda}{\lambda_{n0}}\right), \tag{2.23}$$

where

$$\begin{gathered} c^0 = \lim_{\lambda\to 0} \tfrac{\Delta_0(\lambda)}{\lambda} = -\sin w\pi, \\ c_1^0 = \lim_{\lambda\to 0} \tfrac{d}{d\lambda} \ln \tfrac{\Delta_0(\lambda)}{\lambda} = -\pi \cot w\pi, \;\; \lambda_{n0} = n + w \end{gathered} \tag{2.24}$$

According to (2.19) we have

$$\frac{\Delta(\lambda)}{\Delta_0(\lambda)} = D + \bigcirc\left(\frac{1}{\lambda}\right), \quad \lambda \in G_\delta^w := \{\lambda : |\lambda - n - w| \geq \delta > 0, \; n \in \mathbb{Z}\}, |\lambda| \to \infty.$$

3. Half Inverse Problem

Let $\Phi(x, \lambda)$ be the solution of (1.1) under the conditions $U(\Phi) = 1$, $V(\Phi) = 0$ and under the matching conditions (1.4). We set $M(\lambda) := \Phi(0, \lambda)$. The functions $\Phi(x, \lambda)$ and $M(\lambda)$ are called the Weyl solution and the Weyl function for the BVP L, respectively. Clearly,

$$\Phi(x, \lambda) = \frac{\psi(x, \lambda)}{\Delta(\lambda)} = S(x, \lambda) + M(\lambda)\varphi(x, \lambda), \tag{3.1}$$

$$\langle \varphi(x, \lambda), \Phi(x, \lambda) \rangle = 1, \tag{3.2}$$

$$M(\lambda) = \frac{\Delta^0(\lambda)}{\Delta(\lambda)}, \tag{3.3}$$

where $\Delta^0(\lambda) := \psi(0, \lambda)$ is the characteristic function of the BVP L_1 for equation (1.1) with the boundary conditions $y(0) = 0, V(y) = 0$ and with the matching conditions (1.4). Let $\{\lambda_{n,1}\}$ be zeros of $\Delta^0(\lambda)$, i.e. the eigenvalues of L_1. Clearly, $\{\lambda_n\} \cap \{\lambda_{n,1}\} = \varnothing$.

Consider the following inverse problem

Problem: Let the spectrum $\{\lambda_n\}_{n\in\mathbb{Z}}$ of L be given. Find L, provided the numbers h_0, h_1 and the function $q(x)$ on $\left(0, \frac{\pi}{2}\right)$ are known a priori.

3.1. Theorem. *The specification of the spectrum $\{\lambda_n\}_{n\in\mathbb{Z}}$ determines L uniquely, provided the numbers h_0, h_1 and the function $q(x)$ on $\left(0, \frac{\pi}{2}\right)$ are known a priori.*

Proof. The formula (3.1) give

(3.4) $$\psi(x,\lambda) = \Delta(\lambda)S(x,\lambda) + \Delta^0(\lambda)\varphi(x,\lambda).$$

Denote

(3.5) $$\Delta_1(\lambda) := -\psi'(\frac{\pi}{2},\lambda),\ \Delta_1^0(\lambda) := \psi(\frac{\pi}{2},\lambda),$$

(3.6) $$\Delta_2(\lambda) := \varphi'(\frac{\pi}{2},\lambda),\ \Delta_3(\lambda) := -\varphi(\frac{\pi}{2},\lambda),$$

We note that $\Delta_1(\lambda)$ is a characteristic function of the BVP

(3.7) $$ly = 0,\ \frac{\pi}{2} < x < \pi,\ y'\left(\frac{\pi}{2}\right) = V(y) = 0$$

The function

(3.8) $$M_1(\lambda) = \frac{\Delta_1^0(\lambda)}{\Delta_1(\lambda)},$$

is the Weyl function for (3.7).

The functions $\Delta_2(\lambda)$ and $\Delta_3(\lambda)$ are the characteristic functions of the BVPs

(3.9) $$ly = 0,\ 0 < x < \frac{\pi}{2},\ U(y) = y'\left(\frac{\pi}{2}\right) = 0$$

(3.10) $$ly = 0,\ 0 < x < \frac{\pi}{2},\ U(y) = y\left(\frac{\pi}{2}\right) = 0$$

respectively. According to (3.4) - (3.6) we have

(3.11) $$\left\{\begin{array}{l} \Delta_1^0(\lambda) := \Delta(\lambda)S(\frac{\pi}{2},\lambda) - \ \Delta^0(\lambda)\Delta_3(\lambda), \\ -\Delta_1(\lambda) := \Delta(\lambda)S'(\frac{\pi}{2}+0,\lambda) + \ \Delta^0(\lambda)\Delta_2(\lambda). \end{array}\right.$$

According to (2.15) - (2.17) we get

(3.12) $$\left\{\begin{array}{l} b_0(\lambda) := \Delta_1^0(\lambda) - \sqrt{1+H_1^2}\cos\frac{\pi}{2}(\lambda - 2\omega^0) = \bigcirc\left(\exp(|\tau|\,\frac{\pi}{2})\right), \\ b_1(\lambda) := \Delta_1(\lambda) + \lambda\sqrt{1+H_1^2}\sin\frac{\pi}{2}(\lambda - 2\omega^0) = \bigcirc\left(\exp(|\tau|\,\frac{\pi}{2})\right), \end{array}\right.$$

where

$$\omega^0 := \frac{1}{2\pi i}\ln\frac{i - H_1}{i + H_1}.$$

Let $\{\lambda_{n,2}\}_{n\in\mathbb{Z}}$ and $\{\lambda_{n,3}\}_{n\in\mathbb{Z}}$ be the spectra of the BVPs (3.9) and (3.10). Thus, we have

(3.13) $$\begin{array}{l} \lambda_{n,2} = 2n + \omega_1 + \bigcirc\left(\frac{1}{n}\right), \\ \lambda_{n,3} = 2n + \omega_2 + \bigcirc\left(\frac{1}{n}\right), \end{array},\ n \to \infty$$

where

$$\omega_1 := \frac{1}{2\pi i}\ln\frac{i - h_1}{i + h_1},\ \omega_2 := \omega_1 - \frac{1}{2}$$

Let $m_n^{(1)}$ and $m_n^{(2)}$ be the multiplicities of the zeros $\lambda_{n,2}$ and $\lambda_{n,3}$ respectively. By virtue of (3.13) we have $m_n^{(1)} = m_n^{(2)} = 1$ for sufficiently large $|n|$. Using the known method (see[12]) one can prove the following estimates for sufficiently large $|\lambda|$:

$$\begin{array}{l} |\Delta_2(\lambda)| \geq c_\delta \exp(|\tau|\,\frac{\pi}{2}),\ \lambda \in \{\lambda : |\lambda - 2n - \omega_1| \geq \delta > 0,\ n \in \mathbb{Z}\} \\ |\Delta_3(\lambda)| \geq c_\delta \exp(|\tau|\,\frac{\pi}{2}),\ \lambda \in \{\lambda : |\lambda - 2n - \omega_2| \geq \delta > 0,\ n \in \mathbb{Z}\} \end{array}$$

The specification of the numbers $\{\lambda_{n,2}\}_{n\in\mathbb{Z}}$, $\{\lambda_{n,3}\}_{n\in\mathbb{Z}}$, $\left\{b_1^{(\upsilon)}(\lambda_{n,3})\right\}_{\upsilon=\overline{0,m_n^{(2)}-1},n\in\mathbb{Z}}$, $\left\{b_0^{(\upsilon)}(\lambda_{n,2})\right\}_{\upsilon=\overline{0,m_n^{(1)}-1},n\in\mathbb{Z}}$ and the value ω^0 uniquely determines the functions $\Delta_1(\lambda)$, $\Delta_1^0(\lambda)$ by the formula

$$\Delta_1(\lambda) = -\lambda\sqrt{1+H_1^2}\sin\frac{\pi}{2}(\lambda-2\omega^0)+b_1(\lambda), \tag{3.14}$$

$$\Delta_1^0(\lambda) = \sqrt{1+H_1^2}\cos\frac{\pi}{2}(\lambda-2\omega^0)+b_0(\lambda), \tag{3.15}$$

where

$$b_1(\lambda) = \sum_{n\in\mathbb{Z}} b_1(\lambda_{n,3})\frac{\Delta_2(\lambda)}{(\lambda-\lambda_{n,3})\Delta_2'(\lambda_{n,3})},$$
$$b_0(\lambda) = \sum_{n\in\mathbb{Z}} b_0(\lambda_{n,2})\frac{\Delta_3(\lambda)}{(\lambda-\lambda_{n,2})\Delta_3'(\lambda_{n,2})},$$

if all zeros of the functions $\Delta_2(\lambda), \Delta_3(\lambda)$ are simple $(\Delta_2'(\lambda_{n,3})\neq 0, \Delta_3'(\lambda_{n,2})\neq 0)$. The case of multiple zeros requires minor modifications.

Using the given numbers h_0, h_1 and the function $q(x)$, $x\in\left(0,\frac{\pi}{2}\right)$, we find the number α, the functions $S(\frac{\pi}{2}+0,\lambda)=S(\frac{\pi}{2}-0,\lambda)$, $S'(\frac{\pi}{2}+0,\lambda)=S'(\frac{\pi}{2}-0,\lambda)+2\alpha\lambda S(\frac{\pi}{2}-0,\lambda)$ and the functions $\Delta_2(\lambda), \Delta_3(\lambda)$ by formula (3.6). Then using the given spectrum $\{\lambda_n\}_{n\in\mathbb{Z}}$ we find the number ω from the asymptotics (2.21) by the formula $\omega=\lim_{n\to\infty}(\lambda_n-\omega)$ and construct the function $\Delta(\lambda)$ by formula (2.22). Find the number α by formula (2.20). According to (3.11), (3.12)we have

$$b_1^{(\upsilon)}(\lambda_{n,3}) = \frac{d^\upsilon}{d\lambda^\upsilon}\left(-\Delta(\lambda)S'(\frac{\pi}{2}+0,\lambda)+\lambda\sqrt{1+H_1^2}\sin\frac{\pi}{2}(\lambda-2\omega^0)\right)|_{\lambda=\lambda_{n,3}},$$
$$b_0^{(\upsilon)}(\lambda_{n,2}) = \frac{d^\upsilon}{d\lambda^\upsilon}\left(\Delta(\lambda)S(\frac{\pi}{2},\lambda)-\lambda\sqrt{1+H_1^2}\cos\frac{\pi}{2}(\lambda-2\omega^0)\right)|_{\lambda=\lambda_{n,2}},$$

According to formula (3.8) the function $M_1(\lambda)$ is uniquely determined by specifying the given data. According to the uniqueness theorem in [5] the specification of $M_1(\lambda)$ uniquely determines the number H_0 and the function $q(x)$ on $\left(\frac{\pi}{2},\pi\right)$. □

The proof of Theorem is constructive for solving the half inverse problem.

References

[1] Albeverio, S., Gesztesy, F., Hoegh-Krohn and R., Holden, H. with an appendix by P. Exner. *Solvable Models in Quantum Mechanics (second edition)*, AMS Chelsea Publ., 2005.

[2] Bellman, R., Cooke, K. *Differential-Difference Equations*, Academic Press, New York, 1963.

[3] Binding, P. A., Browne, P. J., Watson, B. A. *Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter II*, J. Comp. Appl. Math. **148**(1), 147–168, 2002.

[4] Browne, P. J., Sleeman, B. D. *A uniqueness theorem for inverse eigenparameter dependt Sturm-Liouville problems*, Inverse Problems **13**(6), 1453–1462, 1997.

[5] Buterin, S.A., Yurko, V.A. *Inverse spectral problem for pensils of differential operators on a finite interval*, Vestnik Bashkir. Univ. **4**, 8–12, 2006.

[6] Buterin, S. A. *On inverse spectral problem for non-selfadjoint Sturm-Liouville operator on a finite interval*, J. Math. Anal. Appl. **335**, 739–749, 2007.

[7] Buterin, S. A. *On half inverse problem for differential pensils with the spectral parameter in boundary conditions*, Tamkang J. of Math. **42**(3), 355–364, 2011.

[8] Chernozhukova, A.Yu. and Freiling, G. *A uniqueness theorem for inverse spectral problems depending nonlinearly on the spectral parameter*, Inverse Problems in Science and Engineering **17**(6), 777–785, 2009.

[9] Coddington, E.A., Levinson, N., Theory of ordinary differential equations, McGraw-Hill, New York, USA, 1955.

[10] Conway, J. B., Functions oof one complex variable, vol. 1, Springer, New York, NY, USA, 2nd edition, 1995.

[11] Freiling, G., Yurko V. A. *Inverse Sturm-Liouville Problems and Their Applications*, NOVA Science Publishers, New York, 2001.

[12] Freiling, G., Yurko V. A. *Inverse problems for Sturm-Liouville equations with boundary conditions polynomially dependent on the spectral parameter*, Inverse Problems **26**(5), 055003, 17 pp, 2010.

[13] Gasymov, M. G., Guseinov, G. Sh. *Determination of diffusion operator from spectral data*, Dokl, Akad. Nauk Azerb. SSR **37**(2), 19–23, 1981.

[14] Guseinov, I. M., Nabiev, I. M. *The inverse spectral problem for pencils of differential operators*, Matem. Sbornik **198**(11), 47–66, 2007; English Transl., Sbornik: Mathematics **198**(11), 1579–1598, 2007.

[15] Koyunbakan, H., Panakhov, E. S. *Half-inverse problem for diffusion operators on the finite interval*, J. Math. Anal. Appl. **326**, 1024–1030, 2007.

[16] Levitan, B. M. *Inverse Sturm-Liouville Problems*, Nauka, Mockow, 1984; English Transl., VNU Sci. Press, Utrecht, 1987.

[17] Marchenko, V. A. *Sturm-Liouville Operators and Their Applications*, Naukova Dumka, Kiev, 1977; English Transl., Birkhöuser, 1986.

[18] Mennicken, R. and Möller , M. *Non-self -adjoint boundary value problems*, North-Holland Mathematic Studies **192**, Amsterdam, 2003.

[19] Pöschel, J. and Truowitz, E. *Inverse Spectral Theory*, Academic Press, New York, 1987.

[20] Shieh, C. T. and Yurko, V. A. *Inverse nodal and inverse spectral problem for discontinuous bounday value problems*, J. Math. Anal. Appl. **374**, 266–272, 2008.

[21] Shkalikov, A. A. *Boundary value problems for ordinary differential equations with a parameter in the boundary conditions*, J. Sov. Math. **33**, 1311–1342, 1986; English Transl. from Tr. Semin. im. I.G. Petrovskogo **9**, 190–229, 1983.

[22] Tretter, Ch. *Boundary eigenvalue problems with differential equations $N\eta = \lambda P\eta$ with λ-polynomial boundary conditions*, J. DifferentialEquations **170**(2), 408–471, 2001.

[23] Wang Y. P. *The inverse problem for differential pensils with eigenparameter dependent boundary conditions from interior spectral data*, Appl. Math. Letters **25**(7), 1061–1067, 2012.

[24] Yurko, V. A. *An inverse problem for pencils of differential operators*, Matem. Sbornik **191**, 137–160, 2000; English Tranl., Sbornik: Mathematics **191**, 1561–1586, 2000.