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A COUPLED FIXED POINT RESULT IN PARTIALLY ORDERED PARTIAL METRIC SPACES THROUGH IMPLICIT FUNCTION

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Abstract

In this manuscript, we discuss the existence of coupled fixed points in the context of partially ordered metric spaces through implicit relations for mappings $F : X \times X \to X$ such that F has the mixed monotone property. Our main theorem improves and extends various results in the literature. We also state an example to illustrate our work.

Keywords: Coupled fixed point; Mixed monotone property; Implicit relation; Ordered partial metric space

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1. Introduction and Preliminaries

The notion of partial metric space, introduced by Matthews [46], is a generalization of metric space defined by Fréchet [20] in 1906. Roughly speaking the most remarkable property in a partial metric space is that the self-distance need not be zero. Nonzero self-distance makes perfect sense in the framework of Computer Sciences, in particular, in the Domain Theory and Semantics (see e.g., [39, 40, 51, 23, 57, 65, 66, 74, 75]). In the paper [46], Matthews proved an analog of the well-known Banach contraction principle in the context of complete partial metric spaces. After this result, many authors have conducted further research on fixed point theorems in the same class of spaces. Furthermore they studied topological properties of partial metric spaces (see e.g.,[2]-[7], [9, 16, 18, 24, 25] [35]-[33],[40],[60]-[64],[67, 73]).

A partial metric is a function $p : X \times X \to [0, \infty)$ which satisfies the following conditions
(P1) $p(x, y) = p(y, x)$,
(P2) If $p(x, x) = p(x, y) = p(y, y)$, then $x = y$,

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(P3) $p(x,x) \leq p(x,y)$,
(P4) $p(x,z) + p(y,y) \leq p(x,y) + p(y,z)$,
for all $x,y,z \in X$. Then the pair (X,p) is called a partial metric space.

1.1. Example. (See [68]) Let (X,d) and (X,p) be a metric space and a partial metric space, respectively. Mappings $\rho_i : X \times X \longrightarrow \mathbb{R}^+$ for $i \in \{1,2,3\}$ defined by

$$\begin{aligned}
\rho_1(x,y) &= d(x,y) + p(x,y)\\
\rho_2(x,y) &= d(x,y) + \max\{\omega(x), \omega(y)\}\\
\rho_3(x,y) &= d(x,y) + a
\end{aligned}$$

induce partial metrics on X, where $\omega : X \longrightarrow \mathbb{R}^+$ is an arbitrary function and $a \geq 0$.

Each partial metric p on X generates a T_0 topology τ_p on X with the family of open p-balls $\{B_p(x,\varepsilon) : x \in X,\ \varepsilon > 0\}$ as a base, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$. Similarly, a closed p-ball is defined as $B_p[x,\varepsilon] = \{y \in X : p(x,y) \leq p(x,x) + \varepsilon\}$.

1.2. Definition. (See [46]) Let (X,p) be a partial metric space.

- (i) A sequence $\{x_n\}$ in X converges to $x \in X$ whenever $\lim\limits_{n\to\infty} p(x,x_n) = p(x,x)$,
- (ii) A sequence $\{x_n\}$ in X is called Cauchy whenever $\lim\limits_{n,m\to\infty} p(x_n,x_m)$ exists (and finite),
- (iii) (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$, such that, $\lim\limits_{n,m\to\infty} p(x_n,x_m) = p(x,x)$.
- (iv) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $f(B(x_0,\delta)) \subset B(f(x_0),\varepsilon)$.

Romaguera (see [60]) defined the notion of 0-Cauchy sequence in partial metric spaces. Furthermore, he introduced the concept of 0-completeness in the same class of spaces.

1.3. Remark. Notice that the limit of a sequence in partial metric space need not to be unique. For more details and examples see e.g. [37].

1.4. Definition. (See [60]) A sequence $\{x_n\}$ in a partial metric space (X,p) is called 0-Cauchy if $\lim\limits_{n,m\to\infty} p(x_n,x_m) = 0$. A partial metric space (X,p) is said to be 0-complete if every 0-Cauchy sequence in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = 0$. In this case, p is said to be a 0-complete partial metric on X.

Observe that each 0-Cauchy sequence is also a Cauchy sequence in a partial metric space. In particular, we note that each complete partial metric is a 0-complete partial metric on X. However the converse is not true and the following example demonstrate that there exists a 0-complete partial metric that is not complete.

1.5. Example. (See [60, 61]) Let $(\mathbb{Q} \cap [0,\infty), p)$ be the partial metric space, where $\mathbb{Q}$ and $p(x,y)$ represent the set of rational numbers and the partial metric $\max\{x,y\}$, respectively.

One of the characterizations of continuity of mappings in partial metric spaces was given by Samet at al. [64] as follows:

1.6. Lemma. *(See [64]) Let (X,p) be a partial metric space. The function $F : X \to X$ is continuous if given a sequence $\{x_n\} \in \mathbb{N}$ and $x \in X$ such that $p(x,x) = \lim\limits_{n\to+\infty} p(x,x_n)$, then*

$$p(Fx,Fx) = \lim_{n\to+\infty} p(Fx,Fx_n).$$

1.7. Example. (See [64]) Consider $X = [0, \infty)$ endowed with the partial metric $p : X \times X \to [0, \infty)$ defined by $p(x, y) = \max\{x, y\}$ for all $x, y \geq 0$. Let $F : X \to X$ be a non-decreasing function. If F is continuous with respect to the standard metric $d(x, y) = |x - y|$ for all $x, y \geq 0$, then F is continuous with respect to the partial metric p.

There is a close relationship between metrics and partial metrics. Indeed, if p is a partial metric on X, then the function $d_p : X \times X \to [0, \infty)$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a metric on X. Moreover,

$$\lim_{n\to\infty} d_p(x, x_n) = 0 \Leftrightarrow \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) = p(x, x). \tag{1.1}$$

1.8. Lemma. *(See e.g.* [46]*) Let (X, p) be a partial metric space.*

(*a*) *A sequence $\{x_n\}$ is Cauchy if and only if $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_p),*

(*b*) *(X, p) is complete if and only if the metric space (X, d_p) is complete.*

The following lemmas will be frequently used in the proof of the main result.

1.9. Lemma. *Let (X, p) be a partial metric space. Then*
(a) If $p(x, y) = 0$, then $x = y$,
(b) If $x \neq y$, then $p(x, y) > 0$.
(c) If $x_n \to z$ with $p(z, z) = 0$, then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for all $y \in X$.

In [26], Guo and Lakshmikantham introduced the notion of coupled fixed point. In 2006, Gnana-Bhaskar and Lakshmikantham[10] defined the notion of mixed monotone mapping and reconsidered coupled fixed point in the context of partially ordered set. In this initial paper, the authors proved some coupled fixed point theorems for the mixed monotone mappings and discussed the existence and uniqueness of solution for a periodic boundary value problem. Next, we recall the necessary definitions and their basic results in this direction.

1.10. Definition. (See [10]) Let $(X, \preceq)$ be a partially ordered set. The mapping $F : X \times X \to X$ is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

1.11. Definition. (See [10]) An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$$x = F(x, y) \text{ and } y = F(y, x).$$

We state now the main results of Gnana-Bhaskar and Lakshmikantham in [10].

1.12. Theorem. *(See* [10]*) Let $(X, \preceq)$ be a partially ordered set. Suppose there exists a metric d on X such that (X, d) is a complete metric space. Assume that there exists a $k \in [0, 1)$ with*

$$d\left(F\left(x, y\right), F\left(u, v\right)\right) \leq \frac{k}{2}\left[d\left(x, u\right) + d\left(y, v\right)\right]$$

for all $x \succeq u$ and $y \preceq v$. Let either

(a) *$F : X \times X \to X$ be a continuous mapping with the mixed monotone property on X, or,*

(b) *X has the following property:*
 (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*
 (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.*

If there exist two elements $x_0, y_0 \in X$ with

$$x_0 \preceq F(x_0, y_0) \quad \text{and} \quad y_0 \succeq F(y_0, x_0)$$

then there exist $x, y \in X$ such that

$$x = F(x, y) \quad \text{and} \quad y = F(y, x).$$

Following Theorem 1.12, several coupled coincidence/fixed point theorems and their applications to integral equations, matrix equations, periodic boundary value problems have been reported (see e.g. [9, 11, 12, 13, 21, 38, 29, 34, 41, 43, 44, 42, 45, 63] and references therein).

In nonlinear analysis, especially in fixed point theory, implicit relations on metric spaces have been investigated heavily in many articles (see, e.g., [6], [19, 22], [53] and references therein). In this paper, by using implicit relations, we examine the existence of a coupled fixed point theorem for mappings $F : X \times X \to X$ satisfying the mixed monotone property in the context of partial metric spaces. A set of implicit relations, denoted by $\mathbb{H}$, is the collection of all continuous functions $H : (\mathbb{R}^+)^5 \to \mathbb{R}$ which satisfy

(H1) $H(t_1, t_2, t_3, t_4, t_5)$ is non-increasing in t_2 and t_5, and
(H2) there exists a function $\varphi \in \Phi$ such that

$$H(u, u + v, v, w, u + v) \leq 0 \text{ implies } u \leq \varphi(\max\{v, w\})$$

where Φ denotes the set of all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with the properties
(i) φ is continuous and non-decreasing,
(ii) $\varphi(t) < t$ for each $t > 0$ and $\varphi(0) = 0$.

1.13. Example. It is easy to check that the following functions are in $\mathbb{H}$

(H_1) $H(t_1, t_2, t_3, t_4, t_5) = t_1 - \alpha t_2 - \beta t_3 - \gamma t_4 - \theta t_5$, where $\alpha, \beta, \gamma, \theta$ are non-negative real numbers satisfying $2\alpha + \beta + \gamma + 2\theta < 1$.
(H_2) $H(t_1, t_2, t_3, t_4, t_5) = t_1 - \alpha \max\{t_2/2, t_3, t_4, t_5/2\}$, where $\alpha \in (0, 1)$.
(H_3) $H(t_1, t_2, t_3, t_4, t_5) = t_1 - \varphi(\max\{t_3, t_4\}$, where $\varphi \in \Phi$.

In this paper, we prove a coupled fixed point theorem for mappings satisfying certain implicit relations in the framework of partial metric spaces.

2. Coupled fixed point theorem

We start this section with our main result.

2.1. Theorem. *Let $(X, p, \preceq)$ be a partially ordered complete partial metric space. Suppose $F : X \times X \to X$ be a mapping such that F has the mixed monotone property. Assume that there exists $H \in \mathbb{H}$ such that*

$$H \begin{pmatrix} p(F(x,y), F(u,v)), p(F(x,y), x) + p(F(u,v), u), \\ p(x,u), p(y,v), p(F(x,y), u) + p(F(u,v), x) \end{pmatrix} \leq 0 \tag{2.1}$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose that either

(a) *F is continuous or*
(b) *X has the following property*
 (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*
 (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.*

If there exist two elements $x_0, y_0 \in X$ *with*

$$x_0 \preceq F(x_0, y_0) \quad \textit{and} \quad y_0 \succeq F(y_0, x_0),$$

then F *has a coupled fixed point in* X.

Proof. Let $x_0, y_0 \in X$ be such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. We construct the iterative sequences $\{x_n\}$ and $\{y_n\}$ in X as follows

$$x_{n+1} = F(x_n, y_n) \text{ and } y_{n+1} = F(y_n, x_n) \quad \text{for all } n \geq 0. \tag{2.2}$$

By using the mathematical induction and the mixed monotone property of F, we can show that

$$x_n \preceq x_{n+1} \text{ and } y_n \succeq y_{n+1} \quad \text{for all } n \geq 0. \tag{2.3}$$

If there is some $n_0 \in \mathbb{N}^*$ such that $x_{n_0} = x_{n_0+1}$ and $y_{n_0} = y_{n_0+1}$ then

$$x_{n_0} = x_{n_0+1} = F(x_{n_0}, y_{n_0}) \text{ and } y_{n_0} = y_{n_0+1} = F(y_{n_0}, x_{n_0})$$

which concludes that (x_{n_0}, y_{n_0}) is a coupled fixed point of F. Thus we assume that $x_{n_0} \neq x_{n_0+1}$ or $y_{n_0} \neq y_{n_0+1}$ for all n. By Lemma 1.9, we have $\max\{p(x_{n+1}, x_n), p(y_{n+1}, y_n)\} > 0$ for all n.

Since $x_{n+1} \succeq x_n$ and $y_{n+1} \preceq y_n$, from (2.1), we have

$$H\begin{pmatrix} p(F(x_{n+1}, y_{n+1}), F(x_n, y_n)), p(F(x_{n+1}, y_{n+1}), x_{n+1}) + p(F(x_n, y_n), x_n), \\ p(x_{n+1}, x_n), p(y_{n+1}, y_n), p(F(x_{n+1}, y_{n+1}), x_n) + p(F(x_n, y_n), x_{n+1}) \end{pmatrix} \leq 0$$

or

$$H\begin{pmatrix} p(x_{n+2}, x_{n+1}), p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n), \\ p(x_{n+1}, x_n),\ p(y_{n+1}, y_n),\ p(x_{n+2}, x_n) + p(x_{n+1}, x_{n+1}) \end{pmatrix} \leq 0. \tag{2.4}$$

Due to $(P4)$ we have,

$$\begin{aligned} p(x_{n+2}, x_n) &\leq p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n) - p(x_{n+1}, x_{n+1}) \\ &\leq p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n). \end{aligned} \tag{2.5}$$

By the properties of H and (2.5), the inequality in (2.4) turns into

$$H\begin{pmatrix} p(x_{n+2}, x_{n+1}), p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n), \\ p(x_{n+1}, x_n), p(y_{n+1}, y_n), p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n) \end{pmatrix} \leq 0$$

which yields that

$$p(x_{n+2}, x_{n+1}) \leq \varphi(\max\{p(x_{n+1}, x_n), p(y_{n+1}, y_n)\}). \tag{2.6}$$

Similarly, one can show that

$$p(y_{n+2}, y_{n+1}) \leq \varphi(\max\{p(x_{n+1}, x_n), p(y_{n+1}, y_n)\}). \tag{2.7}$$

From (2.6) and (2.7), we have

$$\max\{p(x_{n+2}, x_{n+1}), p(y_{n+2}, y_{n+1})\} \leq \varphi(\max\{p(x_{n+1}, x_n), p(y_{n+1}, y_n)\}) \tag{2.8}$$

which implies

$$\max\{p(x_{n+2}, x_{n+1}), p(y_{n+2}, y_{n+1})\} < \max\{p(x_{n+1}, x_n), p(y_{n+1}, y_n)\}$$

by the property of φ. This means that $\{p_n := \max\{p(x_{n+1}, x_n), p(y_{n+1}, y_n)\}\}$ is a decreasing sequence of positive real numbers. So there is an $L \geq 0$ such that

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} \max\{p(x_{n+1}, x_n), p(y_{n+1}, y_n)\} = L. \tag{2.9}$$

We shall show that $L = 0$. Assume, to the contrary, that $L > 0$. Taking $n \to \infty$ in (2.8), we have

$$L \leq \lim_{n\to\infty} \varphi(p_n) = \varphi(L) < L$$

which is a contradiction. Thus $L = 0$. Hence,

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = 0 \text{ and } \lim_{n\to\infty} p(y_{n+1}, y_n) = 0. \tag{2.10}$$

Next, we show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{x_n\}$ or $\{y_n\}$ is not a Cauchy sequence. This means that there exists an $\varepsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}, \{x_{m(k)}\}$ of $\{x_n\}$ and $\{y_{n(k)}\}, \{y_{m(k)}\}$ of $\{y_n\}$ with $n(k) > m(k) \geq k$ such that

$$\max\left\{p\left(x_{n(k)}, x_{m(k)}\right), p\left(y_{n(k)}, y_{m(k)}\right)\right\} \geq \varepsilon. \tag{2.11}$$

Furthermore, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ and satisfies (2.11). Then

$$\max\left\{p\left(x_{n(k)-1}, x_{m(k)}\right), p\left(y_{n(k)-1}, y_{m(k)}\right)\right\} < \varepsilon. \tag{2.12}$$

Using the triangle inequality and (2.12), we have

$$\begin{aligned} p\left(x_{n(k)}, x_{m(k)}\right) &\leq p\left(x_{n(k)}, x_{n(k)-1}\right) + p\left(x_{n(k)-1}, x_{m(k)}\right) - p(x_{n(k)-1}, x_{n(k)-1}) \\ &\leq p\left(x_{n(k)}, x_{n(k)-1}\right) + p\left(x_{n(k)-1}, x_{m(k)}\right) \\ &< p\left(x_{n(k)}, x_{n(k)-1}\right) + \varepsilon \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} p\left(y_{n(k)}, y_{m(k)}\right) &\leq p\left(y_{n(k)}, y_{n(k)-1}\right) + p\left(y_{n(k)-1}, y_{m(k)}\right) - p(y_{n(k)-1}, y_{n(k)-1} \\ &\leq p\left(y_{n(k)}, y_{n(k)-1}\right) + p\left(y_{n(k)-1}, y_{m(k)}\right)) \\ &< p\left(y_{n(k)}, y_{n(k)-1}\right) + \varepsilon. \end{aligned} \tag{2.14}$$

From (2.11), (2.13) and (2.14), we have

$$\begin{aligned} \varepsilon &\leq \max\left\{p\left(x_{n(k)}, x_{m(k)}\right), p\left(y_{n(k)}, y_{m(k)}\right)\right\} \\ &< \max\left\{p\left(x_{n(k)}, x_{n(k)-1}\right), p\left(y_{n(k)}, y_{n(k)-1}\right)\right\} + \varepsilon. \end{aligned}$$

Letting $k \to \infty$ in the inequality above and using (2.9) we get

$$\lim_{k\to\infty} \max\left\{p\left(x_{n(k)}, x_{m(k)}\right), p\left(y_{n(k)}, y_{m(k)}\right)\right\} = \varepsilon. \tag{2.15}$$

By the triangle inequality

$$\begin{aligned} p\left(x_{n(k)}, x_{m(k)}\right) &\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)-1}) + p\left(x_{m(k)-1}, x_{m(k)}\right) \\ &\qquad - p(x_{n(k)-1}, x_{n(k)-1}) - p(x_{m(k)-1}, x_{m(k)-1}) \\ &\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)-1}) + p\left(x_{m(k)-1}, x_{m(k)}\right) \end{aligned}$$

and

$$\begin{aligned} p\left(y_{n(k)}, y_{m(k)}\right) &\leq p\left(y_{n(k)}, y_{n(k)-1}\right) + p\left(y_{n(k)-1}, y_{m(k)-1}\right) + p\left(gy_{m(k-1}, y_{m(k)}\right) \\ &\qquad - p(y_{n(k)-1}, y_{n(k)-1}) - p(y_{m(k)-1}, y_{m(k)-1}) \\ &\leq p\left(y_{n(k)}, y_{n(k)-1}\right) + p\left(y_{n(k)-1}, y_{m(k)-1}\right) + p\left(y_{m(k-1}, y_{m(k)}\right). \end{aligned}$$

From the last two inequalities and (2.11), we have

$$\begin{aligned} \varepsilon &\leq \max\left\{p\left(x_{n(k)}, x_{m(k)}\right), p\left(y_{n(k)}, y_{m(k)}\right)\right\} \\ &\leq \max\left\{p\left(x_{n(k)}, x_{n(k)-1}\right), p\left(y_{n(k)}, y_{n(k)-1}\right)\right\} \\ &\quad + \max\left\{p\left(x_{m(k)-1}, x_{m(k)}\right), p\left(y_{m(k)-1}, y_{m(k)}\right)\right\} \\ &\quad + \max\left\{p\left(x_{n(k)-1}, x_{m(k)-1}\right), p\left(y_{n(k)-1}, y_{m(k)-1}\right)\right\}. \end{aligned} \tag{2.16}$$

Again, by the triangle inequality,

$$\begin{aligned} p\left(x_{n(k)-1}, x_{m(k)-1}\right) &\leq p\left(x_{n(k)-1}, gx_{m(k)}\right) + p\left(x_{m(k)}, x_{m(k)-1}\right) - p(x_{m(k)}, x_{m(k)}) \\ &\leq p\left(x_{n(k)-1}, x_{m(k)}\right) + p\left(x_{m(k)}, x_{m(k)-1}\right) \\ &< p\left(x_{m(k)}, x_{m(k)-1}\right) + \varepsilon \end{aligned}$$

and

$$\begin{aligned} p\left(y_{n(k)-1}, y_{m(k)-1}\right) &\leq p\left(y_{n(k)-1}, gy_{m(k)}\right) + p\left(y_{m(k)}, y_{m(k)-1}\right) - p(y_{m(k)}, y_{m(k)}) \\ &\leq p\left(y_{n(k)-1}, y_{m(k)}\right) + p\left(y_{m(k)}, y_{m(k)-1}\right) \\ &< p\left(y_{m(k)}, y_{m(k)-1}\right) + \varepsilon. \end{aligned}$$

Therefore,

$$\begin{aligned} &\max\left\{p\left(x_{n(k)-1}, x_{m(k)-1}\right), p\left(y_{n(k)-1}, y_{m(k)-1}\right)\right\} \\ &< \max\left\{p\left(x_{m(k)}, x_{m(k)-1}\right), p\left(y_{m(k)}, y_{m(k)-1}\right)\right\} + \varepsilon. \end{aligned} \tag{2.17}$$

From (2.16) and (2.17), we have

$$\begin{aligned} \varepsilon &- \max\left\{p\left(x_{n(k)}, x_{n(k)-1}\right), p\left(y_{n(k)}, y_{n(k)-1}\right)\right\} \\ &\quad - \max\left\{p\left(x_{m(k)-1}, x_{m(k)}\right), p\left(y_{m(k)-1}, y_{m(k)}\right)\right\} \\ &\leq \max\left\{p\left(x_{n(k)-1}, x_{m(k)-1}\right), p\left(y_{n(k)-1}, y_{m(k)-1}\right)\right\} \\ &< \max\left\{p\left(x_{m(k)}, x_{m(k)-1}\right), p\left(y_{m(k)}, y_{m(k)-1}\right)\right\} + \varepsilon. \end{aligned}$$

Taking the limit as $k \to \infty$ in the inequalities above and using (2.9), we get

$$\lim_{k\to\infty} \max\left\{p\left(x_{n(k)-1}, x_{m(k)-1}\right), p\left(y_{n(k)-1}, y_{m(k)-1}\right)\right\} = \varepsilon. \tag{2.18}$$

From (2.15) and (2.18), the sequences $\{p(x_{n(k)}, x_{m(k)})\}$, $\{p\left(y_{n(k)}, y_{m(k)}\right)\}$, $\{p\left(x_{n(k)-1}, x_{m(k)-1}\right)\}$ and $\{p\left(y_{n(k)-1}, y_{m(k)-1}\right)\}$ have subsequences converging to ε_1, ε_2, ε_3 and ε_4, respectively, and $\max\{\varepsilon_1, \varepsilon_2\} = \max\{\varepsilon_3, \varepsilon_4\} = \varepsilon > 0$. We may assume that

$$\lim_{k\to\infty} p(x_{n(k)}, x_{m(k)}) = \varepsilon_1, \qquad \lim_{k\to\infty} p\left(y_{n(k)}, y_{m(k)}\right) = \varepsilon_2,$$

$$\lim_{k\to\infty} p\left(x_{n(k)-1}, x_{m(k)-1}\right) = \varepsilon_3 \text{ and } \lim_{k\to\infty} p\left(y_{n(k)-1}, gy_{m(k)-1}\right) = \varepsilon_4.$$

We first suppose that $\varepsilon_1 = \max\{\varepsilon_1, \varepsilon_2\} = \varepsilon$. Since $n(k) > m(k)$, $x_{n(k)-1} \succeq x_{m(k)-1}$ and $y_{n(k)-1} \preceq y_{m(k)-1}$. From (2.1), we have

$$H\begin{pmatrix} p(F(x_{n(k)-1}, y_{n(k)-1}), F(x_{m(k)-1}, y_{m(k)-1})), p(F(x_{n(k)-1}, y_{n(k)-1}), x_{n(k)-1}) + \\ p(F(x_{m(k)-1}, y_{m(k)-1}), x_{m(k)-1}), p(x_{n(k)-1}, x_{m(k)-1}), p(y_{n(k)-1}, y_{m(k)-1}), \\ p(F(x_{n(k)-1}, y_{n(k)-1}), x_{m(k)-1}) + p(F(x_{m(k)-1}, y_{m(k)-1}), x_{n(k)-1}) \end{pmatrix} \leq 0$$

or

$$H\begin{pmatrix} p(x_{n(k)}, x_{m(k)}),\ \ p(x_{n(k)}, x_{n(k)-1}) + p(x_{m(k)}, x_{m(k)-1}), \\ p(x_{n(k)-1}, x_{m(k)-1}), p(y_{n(k)-1}, y_{m(k)-1}), p(x_{n(k)}, x_{m(k)-1}) + p(x_{m(k)}, x_{n(k)-1}) \end{pmatrix} \leq 0$$

or

$$H\begin{pmatrix} p(x_{n(k)}, x_{m(k)}),\ p(x_{n(k)}, x_{n(k)-1}) + p(x_{m(k)}, x_{m(k)-1}), \\ p(x_{n(k)-1}, x_{m(k)-1}), p(y_{n(k)-1}, y_{m(k)-1}), p(x_{n(k)}, x_{m(k)}) + \\ p(x_{m(k)}, x_{m(k)-1}) + p(x_{m(k)}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{n(k)-1}) \end{pmatrix} \leq 0.$$

Letting $k \to \infty$ in the last inequality together with (2.10), we derive

$$H\left(\varepsilon_1, 0, \varepsilon_3, \varepsilon_4, \varepsilon_1 + \varepsilon_3\right) \leq 0.$$

Hence, we get

$$H\left(\varepsilon_1, \varepsilon_1 + \varepsilon_3, \varepsilon_3, \varepsilon_4, \varepsilon_1 + \varepsilon_3\right) \leq 0$$

which implies $\varepsilon = \varepsilon_1 \leq \varphi(\max\{\varepsilon_3, \varepsilon_4\}) = \varphi(\varepsilon) < \varepsilon$. This is a contradiction.

Using the same argument as above for the case $\varepsilon_2 = \max\{\varepsilon_1, \varepsilon_2\} = \varepsilon$, we also get a contradiction. Thus $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since X is complete, there exist $x, y \in X$ such that

$$\begin{array}{c} \lim\limits_{n,m\to\infty} p(x_n, x_m) = \lim\limits_{n\to\infty} p(x_n, x) = p(x,x) \\ \text{and} \\ \lim\limits_{n,m\to\infty} p(y_n, y_m) = \lim\limits_{n\to\infty} p(y_n, y) = p(y,y). \end{array} \tag{2.19}$$

We want to show that $p(x,x) = 0 = p(y,y)$. Suppose, on the contrary, that

$$p(x,x) = \delta > 0 \text{ and } p(y,y) = \gamma > 0. \tag{2.20}$$

Then we see that

$$H\begin{pmatrix} p(F(x_{n-1}, y_{n-1}), F(x_{m-1}, y_{m-1})), p(F(x_{n-1}, y_{n-1}), x_{n-1})+ \\ p(F(x_{m-1}, y_{m-1}), x_{m-1}), p(x_{n-1}, x_{m-1}), p(y_{n-1}, y_{m-1}), \\ p(F(x_{n-1}, y_{n-1}), x_{m-1}) + p(F(x_{m-1}, y_{m-1}), x_{n-1}) \end{pmatrix} \leq 0$$

or

$$H\begin{pmatrix} p(x_n, x_m), \ \ p(x_n, x_{n-1}) + p(x_m, x_{m-1}), \\ p(x_{n-1}, x_{m-1}), p(y_{n-1}, y_{m-1}), p(x_n, x_{m-1}) + p(x_m, x_{n-1}) \end{pmatrix} \leq 0.$$

By using the triangle inequality (P4), we get

$$H\begin{pmatrix} p(x_n, x_m), \ p(x_n, x_{n-1}) + p(x_m, x_{m-1}), \\ p(x_{n-1}, x_{m-1}), p(y_{n-1}, y_{m-1}), p(x_n, x_m)+ \\ p(x_m, x_{m-1}) + p(x_m, x_{m-1}) + p(x_{m-1}, x_{n-1}) \end{pmatrix} \leq 0.$$

Letting $k \to \infty$, we derive

$$H\left(\delta, 0, \delta, 0, \delta\right) \leq 0$$

by (2.10) and (2.20). Hence, we find

$$H\left(\delta, 0+\delta, \delta, 0, \delta+0\right) \leq H(\delta, 0, \delta, 0, \delta) \leq 0$$

which implies $\delta \leq \varphi(\max\{\delta, 0\}) = \varphi(\delta) < \delta$. This is a contradiction. Hence $\delta = 0$. Analogously we find that $\gamma = 0$.

Now, suppose that the assumption (a) holds. We have

$$p\left(x, F\left(\left(x_n\right), \left(y_n\right)\right)\right) \leq p\left(x, F\left(x_n, y_n\right)\right) + p\left(F\left(x_n, y_n\right), F\left(x_n, y_n\right)\right). \tag{2.21}$$

Taking the limit as $n \to \infty$ in (2.21) and by (2.19), and the continuity of F we get $p(x, F(x,y)) = 0$.

Similarly, we can show that $p(y, F(y,x)) = 0$. Therefore, $x = F(x,y)$ and $y = F(y,x)$.

Finally, suppose that the assumption (b) holds. Since $\{x_n\}$ is a non-decreasing sequence and $x_n \to x$ and $\{y_n\}$ is a non-increasing sequence and $y_n \to y$, by the assumption, we have $x_n \preceq x$ and $y_n \succeq y$ for all n. Regarding (2.2) and (2.19), we have

$$\lim_{n\to\infty} p(x_n, x) = p(x,x) = \lim_{n\to\infty} p(F\left(x_n, y_n\right), x) \tag{2.22}$$

and

$$\lim_{n\to\infty} p(y_n, y) = p(y,y) = \lim_{n\to\infty} p(F\left(y_n, x_n\right), y). \tag{2.23}$$

We also have

$$H\begin{pmatrix} p(F(x_n, y_n), F(x,y)), p(F(x_n, y_n) + p(F(x,y), x) \\ p(x_n, x), p(y_n, y), p(F(x_n, y_n), x) + p(F(x,y), x_n) \end{pmatrix} \leq 0.$$

Letting $n \to \infty$ and using (2.22) and (2.23), we have

$$H(p(x,F(x,y)),p(x,F(x,y)),0,0,p(x,F(x,y))) \le 0$$

which implies that $p(x,F(x,y)) \le \varphi(\max\{0,0\}) = 0$. Hence $x = F(x,y)$. Similarly, one can show that $y = F(y,x)$.

Thus we proved that F has a coupled fixed point in X. □

2.2. Example. (See, e.g., [45]) Let $X = [0,\infty)$ with usual order $\le$. Then, $(X,p,\le)$ be a partially ordered partial metric space where $p(x,y) = \max\{x,y\}$. Suppose $F(x,y) = \begin{cases} \frac{x-y}{2} & \text{if } x \ge y, \\ 0 & \text{otherwise.} \end{cases}$ and $H(t_1,t_2,t_3,t_4,t_5) = t_1 - \frac{1}{2}\max\{t_3,t_4\}$. It is clear that all conditions of Theorem 2.1 are satisfied. Notice that $(0,0)$ is the coupled fixed point of the operator F.

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