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AN APPLICATION OF HYPERHARMONIC NUMBERS IN MATRICES

Mustafa Bahşi * and Süleyman Solak † ‡

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Abstract

In this study, firstly we defined an $n \times k$ matrix, $G^{(r)}_{n,k}$, whose entries consist of hyperharmonic numbers. Then we obtained relation between Pascal matrices and $G^{r}_{n,k}$. Finally we calculated the determinant of $G^{r}_{n,n}$.

Keywords: Pascal Matrix; Hyperharmonic numbers; Determinant

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1. Introduction

For $n > 0$ and $1 \leq i \leq k$, let define the order-k sequences be as following:

$$g^i_n = \sum_{j=1}^{k} c_j g^i_{n-j} \tag{1.1}$$

with initial values $g^i_{1-k}, g^i_{2-k}, \ldots, g^i_0$, where c_j $(1 \leq j \leq k)$ are constant coefficients, g^i_n is the nth term of ith sequence. Let the $k \times k$ matrix be as following:

$$G_n = \begin{bmatrix} g^1_n & g^2_n & \cdots & g^k_n \\ g^1_{n-1} & g^2_{n-1} & \cdots & g^k_{n-1} \\ \vdots & \vdots & & \vdots \\ g^1_{n-k+1} & g^2_{n-k+1} & \cdots & g^k_{n-k+1} \end{bmatrix} \tag{1.2}$$

There have been many papers related to the sequences as in (1.1) [1, 2, 3, 4, 5]. In [1], Kalman obtained a number of closed-form formulas for the generalized sequence by matrix method.

*Aksaray University, Education Faculty, Aksaray TURKEY. Email: (M. Bahşi) mhvbahsi@yahoo.com

†N. E. Universty, A. K. Education Faculty, Konya TURKEY. Email: (S. Solak) ssolak42@yahoo.com

‡Corresponding author

In [2], Er defined the order-k Fibonacci numbers as a sequence which satisfies the recurrence (1.1) with the boundary conditions for $1-k \leq n \leq 0$

$$g_n^i = \begin{cases} 1, & \text{if } i = 1-n \\ 0, & \text{otherwise} \end{cases}.$$

When $k = 2$ and $c_j = 1\,(1 \leq j \leq k)$, this reduces to the well-known conventional Fibonacci numbers. Also, Er showed that

$$\begin{bmatrix} g_{n+1}^i & g_n^i & \ldots & g_{n-k+2}^i \end{bmatrix}^T = C \begin{bmatrix} g_n^i & g_{n-1}^i & \ldots & g_{n-k+1}^i \end{bmatrix}^T,$$

where

$$C = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_{k-1} & c_k \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Then he obtained

$$G_{n+1} = CG_n,$$

where G_n is $k \times k$ matrix as in (1.2).

In [3], Karaduman showed that

$$G_n = C^n$$

and

$$\det(G_n) = \begin{cases} (-1)^n, & \text{if } k \text{ is even} \\ 1, & \text{if } k \text{ is odd} \end{cases}$$

for $c_j = 1\,(1 \leq j \leq k)$.

In [4], Tasci and Kilic gave a new generalization of the Lucas numbers in matrices. Also, they presented a relation between the generalized order-k Lucas numbers and Fibonacci numbers. In [5], Fu and Zhou obtained some new results on matrices related to Fibonacci and Lucas numbers.

The nth hyperharmonic number of order r, $H_n^{(r)}$, defined as: for $n, r \geq 1$

$$H_n^{(r)} = \sum_{k=1}^{n} H_k^{(r-1)} \tag{1.3}$$

where $H_n^{(0)} = \frac{1}{n}$. From the definition of $H_n^{(r)}$, we have $H_1^{(r)} = 1$, and $H_n^{(1)} = \sum\limits_{k=1}^{n} \frac{1}{k} = H_n$ where H_n is nth ordinary harmonic number. Also, hyperharmonic numbers have the recurrence relation as follows: $H_n^{(r)} = H_n^{(r-1)} + H_{n-1}^{(r)}$.

In [6], Conway and Guy gave an equality as follows:

$$H_n^{(r)} = \binom{n+r-1}{r-1} (H_{n+r-1} - H_{r-1})$$

and in [7], Benjamin and et all. gave

$$H_n^{(r)} = \sum_{s=1}^{n} \binom{n+r-s-1}{r-1} \frac{1}{s}. \tag{1.4}$$

Let the $n \times k$ matrix be as following:

$$G_{n,k}^{(r)} = \begin{bmatrix} H_n^{(r)} & H_n^{(r+1)} & \cdots & H_n^{(r+k-1)} \\ H_{n-1}^{(r)} & H_{n-1}^{(r+1)} & \cdots & H_{n-1}^{(r+k-1)} \\ \vdots & \vdots & & \vdots \\ H_1^{(r)} & H_1^{(r+1)} & \cdots & H_1^{(r+k-1)} \end{bmatrix} \tag{1.5}$$

where $H_n^{(r)}$ is nth hyperharmonic number of order r defined as in (1.3). In Section 2, we derive the relation between Pascal matrices and $G_{n,k}^{(r)}$. Also, we calculate the determinant of $G_{n,n}^{(r)}$.

Now we give some preliminaries related to our study. The $n \times k$ Pascal and $n \times n$ lower triangular Pascal matrices are respectively defined as

$$P = (p_{ij}) = \binom{i+j-2}{j-1}, \tag{1.6}$$

$$P_L = (q_{ij}) = \begin{cases} \binom{i-1}{j-1}, & \text{if } i \geq j \\ 0, & \text{otherwise} \end{cases} . \tag{1.7}$$

For example, the matrices P and P_L of order 5 are

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix}, \quad P_L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}.$$

The matrices P and P_L in (1.6) and (1.7) have the following properties [8, 9] :

(1) $P = P_L P_L^T$, where P_L^T is transpose of P_L.

(2) $Det(P) = 1$.

(3) $P_L^{-1} = diag\,[-1, 1, -1, \ldots, (-1)^n]\, P_L diag\,[-1, 1, -1, \ldots, (-1)^n]$.

(4) $P^{-1} = diag\,[-1, 1, -1, \ldots, (-1)^n]\; P_L^T P_L\; diag\,[-1, 1, -1, \ldots, (-1)^n]$.

Let the $n \times n$ matrices H and A be as

$$H = \begin{bmatrix} \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & \frac{1}{2} & 1 \\ \frac{1}{n-1} & \frac{1}{n-2} & \frac{1}{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ \frac{1}{2} & 1 & 0 & \cdots & & 0 \\ 1 & 0 & 0 & \cdots & & 0 \end{bmatrix} \tag{1.8}$$

and

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{1.9}$$

Then from the principle of mathematical induction on r , we have

$$A^r = B_r = (b_{ij}) = \begin{cases} \binom{j-i+r-1}{r-1}, & \text{if} \quad i \le j \\ 0, & \text{otherwise} \end{cases} . \tag{1.10}$$

Also, the determinants of A and H have the forms:

$$\det(A) = 1$$

and

$$\det(H) = \begin{cases} 1, & \text{if} \quad n \equiv 0,1 \pmod 4 \\ -1, & \text{if} \quad n \equiv 2,3 \pmod 4 \end{cases} .$$

2. The Main Results

2.1. Lemma. *Let the $n\times k$, $n\times n$, $n\times k$ matrices $G_{n,k}^{(r)}, H, P$ be as in* (1.5), (1.8) *and* (1.6), *respectively. Then,*

$$G_{n,k}^{(1)} = HP.$$

Proof. From matrix multiplication, we have

$$HP = \begin{bmatrix} \sum\limits_{s=1}^{n} \frac{1}{s} & \sum\limits_{s=1}^{n} (n-s+1)\frac{1}{s} & \sum\limits_{s=1}^{n} \begin{pmatrix} n-s+2 \\ 2 \end{pmatrix} \frac{1}{s} & \cdots & \sum\limits_{s=1}^{n} \begin{pmatrix} n-s+k-1 \\ k-1 \end{pmatrix} \frac{1}{s} \\ \sum\limits_{s=1}^{n-1} \frac{1}{s} & \sum\limits_{s=1}^{n-1} (n-s)\frac{1}{s} & \sum\limits_{s=1}^{n-1} \begin{pmatrix} n-s+1 \\ 2 \end{pmatrix} \frac{1}{s} & \cdots & \sum\limits_{s=1}^{n-1} \begin{pmatrix} n-s+k-2 \\ k-1 \end{pmatrix} \frac{1}{s} \\ \vdots & \vdots & \vdots & & \vdots \\ \sum\limits_{s=1}^{2} \frac{1}{s} & \sum\limits_{s=1}^{2} (3-s)\frac{1}{s} & \sum\limits_{s=1}^{2} \begin{pmatrix} 4-s \\ 2 \end{pmatrix} \frac{1}{s} & \cdots & \sum\limits_{s=1}^{2} \begin{pmatrix} k-s+1 \\ k-1 \end{pmatrix} \frac{1}{s} \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

From (1.4) and since $H_1^{(r)} = 1$,

$$HP = \begin{bmatrix} H_n^{(1)} & H_n^{(2)} & \cdots & H_n^{(k)} \\ H_{n-1}^{(1)} & H_{n-1}^{(2)} & \cdots & H_{n-1}^{(k)} \\ \vdots & \vdots & & \vdots \\ H_1^{(1)} & H_1^{(2)} & \cdots & H_1^{(k)} \end{bmatrix} = G_{n,k}^{(1)}$$

Thus, the proof is completed. □

2.2. Lemma. *Let the $n\times k$ matrices $G_{n,k}^{(r)}$, P and $n\times n$ matrices H, A be as in* (1.4), (1.6), (1.8) *and* (1.9), *respectively. Then,*

$$G_{n,k}^{(r+1)} = A^r HP.$$

Proof. From matrix multiplication and (1.3), we have

$$\begin{bmatrix} H_n^{(r+1)} & H_{n-1}^{(r+1)} & \dots & H_1^{(r+1)} \end{bmatrix}^T = A \begin{bmatrix} H_n^{(r)} & H_{n-1}^{(r)} & \dots & H_1^{(r)} \end{bmatrix}^T . \tag{2.1}$$

Generalizing (2.1), we derive

$$G_{n,k}^{(r+1)} = AG_{n,k}^{(r)}.$$

By using the principle of mathematical induction, we write

$$G_{n,k}^{(r+1)} = A^r G_{n,k}^{(1)}. \tag{2.2}$$

From Lemma 2.1, the Eq. (2.2) is rewritten as

$$G_{n,k}^{(r+1)} = A^r HP.$$

□

2.3. Corollary. *Let the $n \times k$ matrices $G_{n,k}^{(r)}$, P and $n \times n$ matrices H, B_r be as in* (1.5), (1.6), (1.8) *and* (1.10), *respectively. Then,*

$$G_{n,k}^{(r+1)} = B_r HP.$$

For example, taking $n = 4, k = 3$ and $r = 5$ in Corollary 2.3, we have

$$G_{4,3}^{(6)} = \begin{bmatrix} \frac{275}{4} & \frac{1207}{12} & \frac{1691}{12} \\ \frac{73}{3} & \frac{191}{6} & \frac{121}{3} \\ \frac{13}{2} & \frac{15}{2} & \frac{17}{2} \\ 1 & 1 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 5 & 15 & 35 \\ 0 & 1 & 5 & 15 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{bmatrix} = B_5 HP.$$

2.4. Theorem. *Let the nth hyperharmonic number of order r, $H_n^{(r)}$, be as in* (1.3). *Then, we have*

$$H_n^{(r+s)} = \sum_{t=1}^{n} \binom{n+r-t-1}{r-1} H_t^{(s)}$$

where $r \geq 1$ and $s \geq 0$.

Proof. Let the matrices $G_{n,k}^{(r)}$ and B_r be as in (1.5) and (1.10), respectively. Then

$$g_{11} = H_n^{(r+s)}, \ b_{1j} = \binom{j+r-2}{r-1} \text{ and } q_{j1} = H_{n-j+1}^{(s)}$$

where $G_{n,k}^{(r+s)} = (g_{ij})$, $B_r = (b_{ij})$ and $G_{n,k}^{(s)} = (q_{ij})$. From Lemma 2.1 and Corollary 2.3, we have $G_{n,k}^{(r+s)} = B_r G_{n,k}^{(s)}$. Then

$$\begin{aligned} g_{11} &= \sum_{j=1}^{n} b_{1j} q_{j1} \\ &= \sum_{j=1}^{n} \binom{j+r-2}{r-1} H_{n-j+1}^{(s)} \\ &= \sum_{t=1}^{n} \binom{n+r-t-1}{r-1} H_t^{(s)}. \end{aligned}$$

Since $g_{11} = H_n^{(r+s)}$, the proof is completed. □

Taking $r = s$ in Theorem 2.4, we can write

$$H_n^{(2r)} = \sum_{t=1}^{n} \binom{n+r-t-1}{r-1} H_t^{(r)}.$$

Also, taking $r+s=2$ in Theorem 2.4, we have

$$H_n^{(2)} = H_n^{(1+1)} = \sum_{t=1}^{n} H_t^{(1)} = \sum_{t=1}^{n} H_t \tag{2.3}$$

and

$$H_n^{(2)} = H_n^{(2+0)} = \sum_{t=1}^{n} (n-t+1)\frac{1}{t} = (n+1)\left(H_{n+1}-1\right). \tag{2.4}$$

Therefore, from (2.3) and (2.4), for sum of the first n ordinary harmonic numbers, we obtain

$$\sum_{t=1}^{n} H_t = (n+1)\left(H_{n+1}-1\right).$$

2.5. Theorem. *Let the matrix $G_{n,k}^{(r)}$ be as in* (1.5)*. Then*

$$\det\left(G_{n,n}^{(r)}\right) = \begin{cases} 1, & \textit{if} \quad n \equiv 0,1 \pmod 4 \\ -1, & \textit{if} \quad n \equiv 2,3 \pmod 4 \end{cases}.$$

Proof. From Lemma 2.2, for $k=n$, we write

$$G_{n,n}^{(r)} = A^{r-1}HP.$$

Then

$$\det\left(G_{n,n}^{(r)}\right) = \left[\det(A)\right]^{r-1}\det(H)\det(P).$$

Since

$$\det(H) = \begin{cases} 1, & \text{if} \quad n \equiv 0,1 \pmod 4 \\ -1, & \text{if} \quad n \equiv 2,3 \pmod 4 \end{cases}, \det(A) = 1 \text{ and } \det(P) = 1,$$

we have

$$\det\left(G_{n,n}^{(r)}\right) = \begin{cases} 1, & \text{if} \quad n \equiv 0,1 \pmod 4 \\ -1, & \text{if} \quad n \equiv 2,3 \pmod 4 \end{cases}.$$

□

Taking $n=2$ in Theorem 2.5, we have

$$\det\left(G_{2,2}^{(r)}\right) = \begin{vmatrix} H_2^{(r)} & H_2^{(r+1)} \\ H_1^{(r)} & H_1^{(r+1)} \end{vmatrix} = -1$$

and

$$H_2^{(r+1)} - H_2^{(r)} = 1$$

where $H_1^{(r)} = 1$. Since $H_2^{(1)} = \frac{3}{2}$, thus we have

$$H_2^{(r)} = \frac{1+2r}{2}.$$

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