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APPROXIMATION PROPERTIES OF STANCU TYPE MEYER- KÖNIG AND ZELLER OPERATORS

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Abstract

In this paper, we introduce a Stancu type modification of the q- Meyer-König and Zeller operators and investigate the Korovkin type statistical approximation properties of this modification via $A-$statistical convergence. We also compute rate of convergence of the defined operators by means of modulus of continuity. Furthermore, we give an rth order generalization of our operators and obtain approximation results of them.

Keywords: $q-$ Meyer-König and Zeller operators, Stancu type operators, rth order generalization, statistical convergence, modulus of continuity

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1. Introduction

The classical Meyer-König and Zeller (MKZ) operators defined on $C\,[0,1]$ were introduced in 1960 (see [17]). In order to give the monotonicity properties, Cheney and Sharma [5] modified these operators as follows:

$$M_n(f;x) = \begin{cases} (1-x)^{n+1} \sum\limits_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) \binom{n+k}{k} x^k, & x \in [0,1) \\ f\,(1)\,, & x = 1. \end{cases}$$

During the last decade, $q-$Calculus was intensively used for the construction of generalizations of many classical approximation processes of positive type. The first researches have been achieved by Lupaş [15] and Phillips [20]. Phillips introduced the $q-$type generalization of the classical Bernstein operator and obtained the rate of convergence and the Voronoskaja type asymptotic formula for these operators. Later, Trif [22] defined the

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MKZ operators based on the $q-$integers. Also, in order to give some explicit formulae for second moment of the MKZ operators based on the $q-$integers, Doğru and Duman [6] have presented the following $q-$MKZ operators for $q \in (0,1]$:

$$M_n(f;x) = \begin{cases} \prod_{s=0}^{n}(1-q^s x)\sum_{k=0}^{\infty} f\left(\frac{q^n [k]_q}{[n+k]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q x^k, & x \in [0,1) \\ f(1), & x = 1. \end{cases}$$

Now we recall some definitions about $q-$Calculus. Let $q > 0$. For any $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the $q-$integer of the number n and the $q-$factorial are respectively defined by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q} & \text{if} \quad q \neq 1 \\ n & \text{if} \quad q = 1 \end{cases}, \ [n]_q! = \begin{cases} [1]_q[2]_q...[n]_q & \text{if} \quad n = 1,2,... \\ 1 & \text{if} \quad n = 0. \end{cases}$$

The $q-$binomial coefficients are defined as $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$, $k = 0,1,...,n$. It is obvious that for $q = 1$ one has $[n]_1 = n$, $[n]_1! = n!$ and $\begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}$, the ordinary binomial coefficients. Details of $q-$Calculus can be found in [4].

Recently, Nowak [18] introduced a $q-$type generalization of Stancu's operators [21]. Beside, an other two $q-$analogues of Stancu operators earlier introduced by Lupaş [16]. Agratini [1] presented approximation properties of the mentioned class of operators. In [23], a Stancu type generalization of Baskakov-Durrmeyer operators was constructed and some approximation properties were obtained by Verma, Gupta and Agrawal. In this paper, we introduce a Stancu type modification of the $q-$MKZ operators and investigate the Korovkin type statistical approximation properties of this modification via $A-$statistical convergence.

Now we recall the concepts of regularity of a summability matrix and $A-$statistical convergence. Let $A := (a_{nk})$ be an infinite summability matrix. For a given sequence $x := (x_k)$, the $A-$transform of x, denoted by $Ax := ((Ax)_n)$, is defined as $(Ax)_n := \sum_{k=1}^{\infty} a_{nk}x_k$ provided the series converges for each n. A is said to be regular if $\lim_n (Ax)_n = L$ whenever $\lim x = L$ [11]. Suppose that A is non-negative regular summability matrix. Then x is $A-$statistically convergent to L if for every $\varepsilon > 0$, $\lim_n \sum_{k:|x_k - L| \geq \varepsilon} a_{nk} = 0$ and we write $st_A - \lim x = L$ [9, 12]. Actually, x is $A-$statistically convergent to L if and only if, for every $\varepsilon > 0$, $\delta_A(k \in \mathbb{N} : |x_k - L| \geq \varepsilon) = 0$, where $\delta_A(K)$ denotes the $A-$ density of subset K of the natural numbers and is given by $\delta_A(K) := \lim_n \sum_{k=1}^{\infty} a_{nk}\chi_K(k)$ provided the limit exists, where χ_K is the charateristic function of K. If $A = C_1$, the Cesáro matrix of order one, then $A-$statistically convergence reduces to the statistical convergence [8]. Also, taking $A = I$, the identity matrix, $A-$statistical convergence coincides with the ordinary convergence.

The rest of this paper is organized as follows. In Section 2, we present a Stancu type generalization of $q-$MKZ operators. Furthermore, we obtain statistical Korovkin-type approximation result of the defined operators and compute their rate of convergence by means of modulus of continuity. In Section 3, we give an rth order generalization of our operators and get approximation results of them.

2. Construction of the Operators

In this section, we present a Stancu type generalization of $q-$MKZ operators and obtain statistical Korovkin-type approximation result.

For $f \in C[0,1]$, $n \in \mathbb{N}$ and $q \in (0,1]$

$$M_n^{q,\alpha}(f;x) = \begin{cases} \sum\limits_{k=0}^{\infty} m_{n,k}^{q,\alpha}(x) f\left(\frac{q^n [k]_q}{[n+k]_q}\right), & x \in [0,1) \\ f(1), & x = 1, \end{cases}$$

where

$$m_{n,k}^{q,\alpha}(x) = \begin{bmatrix} n+k \\ k \end{bmatrix}_q \frac{\prod\limits_{i=0}^{k-1}\left(x+\alpha [i]_q\right)\prod\limits_{s=0}^{n}\left(1-q^s x+\alpha [s]_q\right)}{\prod\limits_{i=0}^{n+k}\left(1+\alpha [i]_q\right)}. \tag{2.1}$$

As usual, we use the test functions $e_i(x) = x^i$ for $i = 0, 1, 2$.

2.1. Lemma. *For all $x \in [0,1)$ and $n \in \mathbb{N}$, we have*

$$M_n^{q,\alpha}(e_0;x) = 1, \tag{2.2}$$

$$M_n^{q,\alpha}(e_1;x) = q^n x, \tag{2.3}$$

$$\frac{q^{2n}}{1+\alpha} x(x+\alpha) \le M_n^{q,\alpha}(e_2;x) \le \frac{q^{2n+1}}{1+\alpha} x(x+\alpha) + \frac{q^{2n}}{[n]_q} x. \tag{2.4}$$

Proof. Item (2.2) easily follows by

$$\sum_{k=0}^{\infty} m_{n,k}^{q,\alpha}(x) = 1.$$

A direct computation yields

$$\begin{aligned} M_n^{q,\alpha}(e_1;x) &= q^n \sum_{k=1}^{\infty} \frac{[n+k-1]_q!}{[k-1]_q!\,[n]_q!} \frac{\prod\limits_{i=0}^{k-1}\left(x+\alpha [i]_q\right)\prod\limits_{s=0}^{n}\left(1-q^s x+\alpha [s]_q\right)}{\prod\limits_{i=0}^{n+k}\left(1+\alpha [i]_q\right)} \\ &= q^n \sum_{k=0}^{\infty} \frac{[n+k]_q!}{[k]_q!\,[n]_q!} \frac{x\prod\limits_{i=1}^{k}\left(x+\alpha [i]_q\right)\prod\limits_{s=0}^{n}\left(1-q^s x+\alpha [s]_q\right)}{\prod\limits_{i=1}^{n+k+1}\left(1+\alpha [i]_q\right)} \\ &= q^n \sum_{k=0}^{\infty} \frac{[n+k]_q!}{[k]_q!\,[n]_q!} \frac{x\prod\limits_{i=0}^{k-1}\left(x+\alpha [i+1]_q\right)\prod\limits_{s=0}^{n}\left(1-q^s x+\alpha [s]_q\right)}{\prod\limits_{i=0}^{n+k}\left(1+\alpha [i+1]_q\right)} \\ &= q^n x, \end{aligned}$$

which guarantees (2.3). Now we will prove (2.4). We immediately see

$$M_n^{q,\alpha}(e_2;x) = q^{2n} \sum_{k=1}^{\infty} \frac{[n+k-1]_q!}{[k-1]_q!\,[n]_q!} \frac{\prod\limits_{i=0}^{k-1}\left(x+\alpha [i]_q\right)\prod\limits_{s=0}^{n}\left(1-q^s x+\alpha [s]_q\right)}{\prod\limits_{i=0}^{n+k}\left(1+\alpha [i]_q\right)} \frac{[k]_q}{[n+k]_q}.$$

Since $[k]_q = q[k-1]_q + 1$ for $k \geq 1$, we get

$$M_n^{q,\alpha}(e_2;x) = q^{2n+1}\sum_{k=2}^{\infty}\frac{[n+k-2]_q!}{[k-2]_q!\,[n]_q!}\frac{\prod_{i=0}^{k-1}\left(x+\alpha[i]_q\right)\prod_{s=0}^{n}\left(1-q^sx+\alpha[s]_q\right)}{\prod_{i=0}^{n+k}\left(1+\alpha[i]_q\right)}$$

$$\frac{[n+k-1]_q!}{[n+k]_q} + q^{2n}\sum_{k=1}^{\infty}\frac{[n+k-1]_q!}{[k-1]_q!\,[n]_q!}\frac{\prod_{i=0}^{k-1}\left(x+\alpha[i]_q\right)\prod_{s=0}^{n}\left(1-q^sx+\alpha[s]_q\right)}{\prod_{i=0}^{n+k}\left(1+\alpha[i]_q\right)[n+k]_q}$$

and hence

$$M_n^{q,\alpha}(e_2;x) = \tag{2.5}$$

$$q^{2n+1}\sum_{k=0}^{\infty}\frac{[n+k]_q!}{[k]_q!\,[n]_q!}\frac{\prod_{i=0}^{k+1}\left(x+\alpha[i]_q\right)\prod_{s=0}^{n}\left(1-q^sx+\alpha[s]_q\right)}{\prod_{i=0}^{n+k+2}\left(1+\alpha[i]_q\right)}\frac{[n+k+1]_q}{[n+k+2]_q}$$

$$+q^{2n}\sum_{k=0}^{\infty}\frac{[n+k]_q!}{[k]_q!\,[n]_q!}\frac{\prod_{i=0}^{k}\left(x+\alpha[i]_q\right)\prod_{s=0}^{n}\left(1-q^sx+\alpha[s]_q\right)}{\prod_{i=0}^{n+k+1}\left(1+\alpha[i]_q\right)[n+k+1]_q}$$

Since $q \in (0,1)$, $x \in [0,1)$ and by using $[n+k+1]_q = \frac{[n+k+2]_q - 1}{q}$ and $[n+k+1]_q < [n+k+2]_q$, we obtain

$$M_n^{q,\alpha}(e_2;x) = q^{2n}\sum_{k=0}^{\infty}\frac{[n+k]_q!}{[k]_q!\,[n]_q!}\frac{\prod\limits_{i=0}^{k+1}\left(x+\alpha[i]_q\right)\prod\limits_{s=0}^{n}\left(1-q^sx+\alpha[s]_q\right)}{\prod\limits_{i=0}^{n+k+2}\left(1+\alpha[i]_q\right)}$$

$$-q^{2n}\sum_{k=0}^{\infty}\frac{[n+k]_q!}{[k]_q!\,[n]_q!}\frac{\prod\limits_{i=0}^{k+1}\left(x+\alpha[i]_q\right)\prod\limits_{s=0}^{n}\left(1-q^sx+\alpha[s]_q\right)}{\prod\limits_{i=0}^{n+k+2}\left(1+\alpha[i]_q\right)[n+k+2]_q}$$

$$+q^{2n}\sum_{k=0}^{\infty}\frac{[n+k]_q!}{[k]_q!\,[n]_q!}\frac{\prod\limits_{i=0}^{k}\left(x+\alpha[i]_q\right)\prod\limits_{s=0}^{n}\left(1-q^sx+\alpha[s]_q\right)}{\prod\limits_{i=0}^{n+k+1}\left(1+\alpha[i]_q\right)[n+k+1]_q}$$

$$\geq \frac{q^{2n}}{1+\alpha}x(x+\alpha)$$

$$-\frac{q^{2n}}{1+\alpha}x(x+\alpha)\sum_{k=0}^{\infty}\frac{[n+k]_q!}{[k]_q!\,[n]_q!}\frac{\prod\limits_{i=2}^{k+1}\left(x+\alpha[i]_q\right)\prod\limits_{s=0}^{n}\left(1-q^sx+\alpha[s]_q\right)}{\prod\limits_{i=2}^{n+k+2}\left(1+\alpha[i]_q\right)[n+k+2]_q}$$

$$+\frac{q^{2n}}{1+\alpha}x(x+\alpha)\sum_{k=0}^{\infty}\frac{[n+k]_q!}{[k]_q!\,[n]_q!}\frac{\prod\limits_{i=1}^{k}\left(x+\alpha[i]_q\right)\prod\limits_{s=0}^{n}\left(1-q^sx+\alpha[s]_q\right)}{\prod\limits_{i=1}^{n+k+1}\left(1+\alpha[i]_q\right)[n+k+2]_q}$$

$$= \frac{q^{2n}}{1+\alpha}x(x+\alpha). \tag{2.6}$$

On the other hand, using the inequalities $[n+k+1]_q < [n+k+2]_q$, $[n+k+1]_q > [n]_q$ for all $k = 0, 1, 2, ..., n \in \mathbb{N}$ and from (2.5) it follows

$$M_n^{q,\alpha}(e_2;x) \leq \frac{q^{2n+1}}{1+\alpha} x(x+\alpha) \sum_{k=0}^{\infty} \frac{[n+k]_q!}{[k]_q!\,[n]_q!} \frac{\prod_{i=2}^{k+1}\left(x+\alpha[i]_q\right)\prod_{s=0}^{n}\left(1-q^s x+\alpha[s]_q\right)}{\prod_{i=2}^{n+k+2}\left(1+\alpha[i]_q\right)} +$$

$$+\frac{q^{2n}}{[n]_q} x \sum_{k=0}^{\infty} \frac{[n+k]_q!}{[k]_q!\,[n]_q!} \frac{\prod_{i=1}^{k}\left(x+\alpha[i]_q\right)\prod_{s=0}^{n}\left(1-q^s x+\alpha[s]_q\right)}{\prod_{i=1}^{n+k+1}\left(1+\alpha[i]_q\right)},$$

which guarantees

$$M_n^{q,\alpha}(e_2;x) \leq \frac{q^{2n+1}}{1+\alpha} x(x+\alpha) + \frac{q^{2n}}{[n]_q} x. \tag{2.7}$$

Then, by combining (2.6) and (2.7), the proof is completed. □

The well-known Korovkin theorem (see [3, 13]) was improved via the concept of statistical convergence by Gadjiev and Orhan in [10]. This theorem can be stated as the following:

2.2. Theorem. *If the sequence of positive linear operators $L_n : C[a,b] \to B[a,b]$ satisfies the conditions*

$$st - \lim_n \|L_n e_j - e_j\| = 0,$$

where $j \in \{0,1,2\}$, then, for any function $f \in C[a,b]$, we have

$$st - \lim_n \|L_n f - f\| = 0.$$

In other words, the sequence of functions $(L_n f)_{n\geq 1}$ is statistically uniform convergent to f on $C[a,b]$. Here $B[a,b]$ stands for the space of all real valued bounded functions defined on $[a,b]$, endowed with the sup-norm.

Theorem 2.2 is true for $A-$statistical convergence, where A is non-negative regular summability matrix [7].

Now, we replace q and α in the definition of $M_n^{q,\alpha}$, by sequences (q_n), $0 < q_n \leq 1$, and (α_n), $\alpha_n \geq 0$, respectively, so that

$$st_A - \lim_n q_n^n = 1,\ st_A - \lim_n \frac{1}{[n]_{q_n}} = 0 \text{ and } st_A - \lim_n \alpha_n = 0. \tag{2.8}$$

For example, take $A = C_1$, the Cesáro matrix of order one, and define (q_n) and (α_n) sequences by

$$q_n = \begin{cases} \frac{1}{2}, & \text{if } n = m^2 \quad (m = 1,2,3...) \\ 1 - \frac{e^{-n}}{n}, & \text{if } n \neq m^2. \end{cases}$$

$$\alpha_n = \begin{cases} e^n, & \text{if } n = m^2 \quad (m = 1,2,3...) \\ 0, & \text{if } n \neq m^2. \end{cases}.$$

Since the C_1-density (or natural density) of the set of all squares is zero, $st_A - \lim_n q_n^n = 1$, $st_A - \lim_n \frac{1}{[n]_{q_n}} = 0$ and $st_A - \lim_n \alpha_n = 0$. It is observed that (q_n) and (α_n) satisfy the equalities in (2.8) but they do not converge in the ordinary case.

2.3. Theorem. *Let $A = (a_{nk})$ be a non-negative regular summability matrix and let (q_n) and (α_n) be two sequences satisfying (2.8). Then, for $f \in C[0,1]$, the sequence $\{M_n^{q_n,\alpha_n}(f;.)\}$ is $A-$statistically uniform convergent to f on the interval $[0,1]$.*

Proof. By (2.2) and the definition of the operators $M_n^{q_n,\alpha_n}$ in the case of $x = 1$, it is clear that

$$st_A - \lim_n \|M_n^{q_n,\alpha_n} e_0 - e_0\| = 0.$$

Taking into account the case $x = 1$ and from (2.3),

$$\|M_n^{q_n,\alpha_n} e_1 - e_1\| \leq 1 - q_n^n. \tag{2.9}$$

For a given $\varepsilon > 0$, we define the following sets;

$$U := \{n : \|M_n^{q_n,\alpha_n} e_1 - e_1\| \geq \varepsilon\}, \; U' := \{n : 1 - q_n^n \geq \varepsilon\}.$$

Then by (2.9), we can see $U \subseteq U'$. So, for all $n \in \mathbb{N}$,

$$0 \leq \sum_{k \in U} a_{nk} \leq \sum_{k \in U'} a_{nk}.$$

Letting $n \to \infty$ and using (2.8), we conclude that $\lim_n \sum_{k \in U} a_{nk} = 0$, which gives

$$st_A - \lim_n \|M_n^{q_n,\alpha_n} e_1 - e_1\| = 0.$$

Finally, by (2.4) and the definition of the operators $M_n^{q_n,\alpha_n}$ in the case of $x = 1$, we get

$$\begin{aligned} |M_n^{q_n,\alpha}(e_2;x) - e_2| &\leq \left| \frac{q^{2n+1}}{1+\alpha} x(x+\alpha) + \frac{q^{2n}}{[n]_q} x - x^2 \right| \\ &\leq \left(1 - \frac{q^{2n+1}}{1+\alpha}\right) x^2 + \left(\frac{\alpha q^{2n+1}}{1+\alpha} + \frac{q^{2n}}{[n]_q}\right) x \\ &\leq 1 + \frac{q^{2n}}{[n]_q} - \frac{1-\alpha}{1+\alpha} q^{2n+1} \end{aligned}$$

So, we can write

$$\|M_n^{q_n,\alpha_n} e_2 - e_2\| \leq 1 - \frac{1-\alpha_n}{1+\alpha_n} q_n^{2n+1} + \frac{q_n^{2n}}{[n]_{q_n}}. \tag{2.10}$$

Since, for all $n \in \mathbb{N}$, $0 < q_n^n \leq q_n \leq 1$, one can get $st_A - \lim_n q_n = 1$. So, by (2.8) observe that

$$st_A - \lim_n \left(1 - \frac{1-\alpha_n}{1+\alpha_n} q_n^{2n+1}\right) = st_A - \lim_n \frac{q_n^{2n}}{[n]_{q_n}} = 0.$$

We define the following sets;

$$D := \{n : \|M_n^{q_n,\alpha_n} e_2 - e_2\| \geq \varepsilon\},$$

$$D_1 := \left\{n : 1 - \frac{1-\alpha_n}{1+\alpha_n} q_n^{2n+1} \geq \varepsilon\right\} \text{ and } D_2 := \left\{n : \frac{q_n^{2n}}{[n]_{q_n}} \geq \varepsilon\right\}.$$

Then, by (2.10), we get $D \subseteq D_1 \cup D_2$. Hence, for all $n \in \mathbb{N}$

$$\sum_{k \in D} a_{nk} \leq \sum_{k \in D_1} a_{nk} + \sum_{k \in D_2} a_{nk}.$$

Taking limit as $j \to \infty$, we get

$$st_A - \lim_n \|M_n^{q_n,\alpha_n} e_2 - e_2\| = 0.$$

So, from Theorem 2.2, the proof is completed. □

Now we will give a theorem on a degree of approximation of continuous function f by the sequence of $M_n^{q,\alpha}(f;x)$. For this aim, we will use the modulus of continuity of function $f \in C[0,1]$ defined by

$$\omega(f;\delta) = \sup\{|f(t) - f(x)| : t, x \in [0,1], \ |t-x| \leq \delta\}$$

for any positive number δ.

2.4. Theorem. *Let $x \in [0,1]$ and f be a continuous function defined on $[0,1]$. Then, for $n \in \mathbb{N}$, the following inequalities holds:*

$$|M_n^{q,\alpha}(f;x) - f(x)| \leq 2\omega\left(f, \delta_n^{q,\alpha}(x)\right)$$

where

$$\delta_n^{q,\alpha}(x) = \left\{\left(1 - 2q^n + \frac{q^{2n+1}}{1+\alpha}\right)x^2 + \left(\frac{\alpha q^{2n+1}}{1+\alpha} + \frac{q^{2n}}{[n]_q}\right)x\right\}^{\frac{1}{2}}. \tag{2.11}$$

Proof. Since the case of $x = 1$ is obvious, assume that $x \in [0,1)$ and $f \in C[0,1]$. By the known properties of modulus of continuity, we can write for any $\delta > 0$, that

$$|f(t) - f(x)| \leq \omega(f,\delta)\left(1 + \frac{|t-x|}{\delta}\right).$$

Therefore, by the linearity and monotonicity of the operators $M_n^{q,\alpha}$, we obtain for any $\delta > 0$,

$$\begin{aligned} |M_n^{q,\alpha}(f;x) - f(x)| &\leq M_n^{q,\alpha}(|f(t) - f(x)|;x) \\ &\leq \omega(f,\delta)\left\{1 + \frac{1}{\delta}M_n^{q,\alpha}(|t-x|;x)\right\}. \end{aligned}$$

By the Cauchy-Schwarz inequality we have

$$|M_n^{q,\alpha}(f;x) - f(x)| \leq \omega(f,\delta)\left\{1 + \frac{1}{\delta}\sqrt{M_n^{q,\alpha}\left((t-x)^2;x\right)}\right\}. \tag{2.12}$$

Then, we can write from Lemma 2.1

$$\begin{aligned} M_n^{q,\alpha}\left((t-x)^2;x\right) &\leq \frac{q^{2n+1}}{1+\alpha}x(x+\alpha) + (1-2q^n)x^2 + \frac{q^{2n}}{[n]_q}x \\ &= \left(1 - 2q^n + \frac{q^{2n+1}}{1+\alpha}\right)x^2 + \left(\frac{\alpha q^{2n+1}}{1+\alpha} + \frac{q^{2n}}{[n]_q}\right)x. \end{aligned}$$

In the inequality (2.12), taking

$$\delta = \delta_n^{q,\alpha}(x) = \left\{\left(1 - 2q^n + \frac{q^{2n+1}}{1+\alpha}\right)x^2 + \left(\frac{\alpha q^{2n+1}}{1+\alpha} + \frac{q^{2n}}{[n]_q}\right)x\right\}^{\frac{1}{2}},$$

the proof is completed. □

2.5. Remark. Let (q_n) and (α_n) be two sequences satisfying (2.8). If we take $q = q_n$ and $\alpha = \alpha_n$ in Theorem 2.4, then, we obtain the rate of $A-$statistical convergence of our operators to the function f being approximated.

3. An rth order generalization of $M_n^{q,\alpha}$

Kirov and Popova [14] proposed a generalization of the rth order of positive linear operators such that it keeps the linearity property but loose the positivity. Also, this generalization is sensitive to the degree of smoothness of the function f as approximations to f. In [19], using the similar method introduced by Kirov and Popova, Özarslan and Duman consider a generalization of the MKZ-type operators on the $q-$integers. Recently, Agratini [2] introduced a generalization of the rth order of Stancu type operators. For every integer $n \geq 1$, $L_n : C[a,b] \to C[a,b]$ be the operators defined by

$$(L_n f)(x) = \sum_{k\in J_n} p_{n,k}(x) f(x_{n,k}), \quad x \in [a,b]$$

where $J_n \subset \mathbb{N}_0 := \{0\} \cup \mathbb{N}$ and a net on the compact $[a,b]$ namely $(x_{n,k})_{k\in J_n}$ and $\sum_{k\in J_n} p_{n,k}(x) = 1$. For $r \in \mathbb{N}_0$ by $f \in C^r[a,b]$ we mean the space of all functions f for which their rth derivative are continuous on the interval $[a,b]$ and $(T_r f)(x_{n,k};.)$ be Taylor's polynomial of r degree associated to the function f on the point $x_{n,k}$, $k \in J_n$. Agratini [2] considered the linear operators $L_{n,r} : C^r[a,b] \to C[a,b]$,

$$\begin{aligned}(L_{n,r} f)(x) &= \sum_{k\in J_n} (T_r f)(x_{n,k};.)\, p_{n,k}(x) \\ &= \sum_{k\in J_n}\sum_{i=0}^{r} \frac{f^{(i)}(x_{n,k})}{i!}(x-x_{n,k})^i p_{n,k}(x), \ x \in [a,b].\end{aligned}$$

Now, we introduce a generalization of the positive linear operators $M_n^{q,\alpha}$, by using the Agratini's method.

$$\begin{aligned}M_{n,r}^{q,\alpha}(f;x) &= \sum_{k=0}^{\infty} (T_r f)(x_{n,k};.)\, m_{n,k}^{q,\alpha}(x) \\ &= \sum_{k=0}^{\infty}\sum_{i=0}^{r} \frac{f^{(i)}(x_{n,k})}{i!}(x-x_{n,k})^i m_{n,k}^{q,\alpha}(x),\end{aligned} \tag{3.1}$$

where $m_{n,k}^{q,\alpha}(x)$ is given by (2.1), $x_{n,k} = \frac{q^n [k]_q}{[n+k]_q}$, $f \in C^r[0,1]$ and $x \in [0,1)$. If $x = 1$ we define that $M_{n,r}^{q,\alpha}(f;1) = f(1)$ as stated before. Clearly, $M_{n,0}^{q,\alpha} = M_n^{q,\alpha}, n \in \mathbb{N}$ for every $f \in C[0,1]$, $x \in [0,1]$, $q \in (0,1]$ and $n \in \mathbb{N}$.

We have the following approximation theorem for the operators $M_{n,r}^{q,\alpha}$.

3.1. Theorem. *Let $A = (a_{nk})$ be a non-negative regular summability matrix and let (q_n) and (α_n) be two sequences satisfying (2.8). Let $r \in \mathbb{N}$ and $x \in [0,1]$. Then, for all $f \in C^r[0,1]$ such that $f^{(r)} \in Lip_M(\alpha)$ and for any $n \in \mathbb{N}$, we have*

$$st_A - \left\| M_{n,r}^{q_n,\alpha_n} f - f \right\| = 0.$$

Proof. For $x \in [0,1)$, we obtain from (3.1) that

$$f(x) - M_{n,r}^{q,\alpha}(f;x) = \sum_{k=0}^{\infty} m_{n,k}^{q,\alpha}(x)\{f(x) - (T_r f)(x_{n,k};x)\}. \tag{3.2}$$

Using the Taylor's formula, we can write

$$f(x) - (T_r f)(x_{n,k};x) = \frac{(x-x_{n,k})^r}{(r-1)!}\int_0^1 (1-t)^{r-1} \left[f^{(r)}(x_{n,k} + t(x-x_{n,k})) - f^{(r)}(x_{n,k})\right] dt.$$

Since $f^{(r)} \in Lip_M(\alpha)$, we obtain

$$|f(x) - (T_r f)(x_{n,k}; x)| \le M \frac{|x - x_{n,k}|^{r+\alpha}}{(r-1)!} B(r, \alpha+1) = M \frac{B(r,\alpha+1)}{(r-1)!} \varphi_x^{r+\alpha}(x_{n,k}), \tag{3.3}$$

where $\varphi_x(t) = |x - t|$ and $B(r, \alpha+1) = \prod_{j=1}^{r} (\alpha + j)^{-1}$, B being Beta function.

In view of relations (3.2) and (3.3), we get

$$\left| f(x) - M_{n,r}^{q,\alpha}(f; x) \right| \le M \frac{B(r,\alpha+1)}{(r-1)!} \sum_{k=0}^{\infty} m_{n,k}^{q,\alpha}(x) \varphi_x^{r+\alpha}(x_{n,k}).$$

Since the case of $x = 1$ is clear, we deduce

$$\left\| M_{n,r}^{q,\alpha} f - f \right\| \le M \frac{B(r,\alpha+1)}{(r-1)!} \left\| M_n^{q,\alpha} \varphi_x^{r+\alpha} \right\|. \tag{3.4}$$

Firstly, we replace q and α by two sequences (q_n) and (α_n) respectively, such that (2.8) holds. Then, for $\varepsilon > 0$, we define the following sets,

$$U := \left\{ n \in \mathbb{N} : \left\| M_{n,r}^{q_n,\alpha_n} f - f \right\| \ge \varepsilon \right\} \text{ and } U' := \left\{ n \in \mathbb{N} : \left\| M_n^{q_n,\alpha_n} \varphi_x^{r+\alpha} \right\| \ge \varepsilon \right\}.$$

From (3.4), we have $U \subseteq U'$. So, for all $n \in \mathbb{N}$, that

$$\sum_{k \in U} a_{nk} \le \sum_{k \in U'} a_{nk}.$$

Because ε is arbitrary and $\varphi_x^{r+\alpha} \in C[0,1]$, $\lim_{n\to\infty} \sum_{k \in U'} a_{nk} = 0$ from Theorem 2.3. So, the proof is completed. □

Since $\varphi_x^{r+\alpha}(t) = |x - t|^{r+\alpha} \in C[0,1]$ and $\varphi_x^{r+\alpha}(x) = 0$, we can give the following result.

3.2. Corollary. *Let $x \in [0,1]$ and $r \in \mathbb{N}$. Then, for all $f \in C^r[0,1]$ such that $f^{(r)} \in Lip_M(\alpha)$ and for any $n \in \mathbb{N}$, we have*

$$\left| M_{n,r}^{q,\alpha}(f; x) - f(x) \right| \le 2 \frac{MB(r,\alpha+1)}{(r-1)!} \omega\left(\varphi_x^{r+\alpha}, \delta_n^{q,\alpha}(x) \right)$$

where $\delta_n^{q,\alpha}(x)$ is given by (2.11).

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