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ON SEMI-E-CONVEX AND QUASI-SEMI-E-CONVEX FUNCTIONS

Farzollah Mirzapour*

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Abstract

In this paper we give some necessary and sufficient conditions under which a lower semi-continuous function defined on a real normed space is a semi-E-convex or quasi-semi-E-convex function.

Keywords: Semi-E-convex, Quasi-semi-E-convex, Lower semi-continuous.

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1. Introduction

The concepts of E-convex set and semi-E-convex function were introduced in [2] and [5]. These concepts are generalizations of convex function and quasi-convex function. Let us recall some definitions and related results. Let X be a topological vector space. Then

(1) A set $U \subset X$ is said to be *E-convex* if and only if there is a map $E : X \longrightarrow X$ such that $\lambda E(x) + (1-\lambda)E(y) \in U$, for each $x, y \in U$ and $0 \leq \lambda \leq 1$.

(2) A function $f : X \longrightarrow \mathbb{R}$ is said to be *semi-E-convex* on a set $U \subseteq X$ if and only if there is a map $E : X \longrightarrow X$ such that U is an E-convex set and

$$f(\lambda E(x) + (1-\lambda)E(y)) \leq \lambda f(x) + (1-\lambda)f(y),$$

for each $x, y \in U$ and $0 \leq \lambda \leq 1$.

(3) The mapping $f : X \longrightarrow \mathbb{R}$ is said to be *quasi-semi-E-convex* on a set $U \subseteq X$ if

$$f(\lambda E(x) + (1-\lambda)E(y)) \leq \max\{f(x), f(y)\},$$

for each $x, y \in U$, $\lambda \in [0, 1]$ such that $\lambda E(x) + (1-\lambda)E(y) \in U$.

Let $E : X \longrightarrow X$ be a map and define $E \times I : X \times \mathbb{R} \longrightarrow X \times \mathbb{R}$ by $(E \times I)(x, t) = (E(x), t)$. It is easy to show that $U \subset X$ is E-convex if and only if $U \times \mathbb{R}$ is $E \times I$-convex. For a function $f : X \longrightarrow \mathbb{R}$ we denote by $\text{epi}(f)$ the epigraph of f; i.e.

$$\text{epi}(f) = \{(x, \alpha) : x \in U,\ \alpha \in \mathbb{R},\ f(x) \leq \alpha\}.$$

Also, let us recall from [1] the following two results that will be used in the sequel.

1.1. Proposition. *Let X be a topological vector space and $U \subseteq X$ convex. Then*

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*Department of Mathematics, Faculty of Sciences, University of Zanjan, 45195-313, Iran.
E-mail: f.mirza@znu.ac.ir

(a) *A function* $f : X \longrightarrow \mathbb{R}$ *is semi-E-convex function on* U *if and only if* $\mathrm{epi}(f)$ *is* $E \times I$*-convex on* $X \times \mathbb{R}$.
(b) *A function* f *is quasi-semi-E-convex on* U *if and only if the level set* $K_\alpha = \{x : x \in U, f(x) \leq \alpha\}$ *is E-convex for each* $\alpha \in \mathbb{R}$.

In this paper, for a lower semi continuous function defined on a real normed space we present some statements equivalent to semi-E-convexity and quasi-semi-E-convexity.

2. Results

First, we recall that if X is a normed space and $S \subset X$, a function $f : S \longrightarrow [-\infty, +\infty]$ is lower semi-continuous if and only if for every real number λ the set $\{x \in S : f(x) \leq \lambda\}$ is closed and this is true if and only if its epigraph $\mathrm{epi}(f) = \{(x, \lambda) \in S \times \mathbb{R} : f(x) \leq \lambda\}$ is closed (as a subset of $X \times \mathbb{R}$). See for example [3] and [4].

Let $(x, s), (y, t) \in X \times \mathbb{R}$, with $x, y \in X$ and $s, t \in \mathbb{R}$. The line segment $[(x, s), (y, t)]$ (endpoint (x, s) and (y, t)) is the segment $\{\gamma(x, s) + (1 - \gamma)(y, t) : 0 \leq \gamma \leq 1\}$. If $(x, s) \neq (y, t))$, the interior $((x, s), (y, t))$ of $[(x, s), (y, t)]$ is the segment $\{\gamma(x, s) + (1 - \gamma)(y, t) : 0 < \gamma < 1\}$. In a similar way, we can define $[(x, s), (y, t))$ and $((x, s), (y, t)]$.

In the following theorems, we assume that X is a normed linear space and $E : X \longrightarrow X$ a map. Let $f : X \longrightarrow [-\infty, +\infty]$ be lower semi-continuous and $f(E(x)) \leq f(x)$, for all $x \in X$.

2.1. Theorem. *Let* $f : X \longrightarrow [-\infty, +\infty]$ *be lower semi-continuous and suppose that there exists* $\alpha \in (0, 1)$ *such that for all* $x, y \in X$ *and* $u, v \in \mathbb{R}$, $f(x) < u$, $f(y) < v$, *and*

$$f(\alpha E(x) + (1 - \alpha)E(y)) < \alpha u + (1 - \alpha)v.$$

Then $f : X \longrightarrow [-\infty, +\infty]$ *is semi-E-convex.*

Proof. It is sufficient to show that $\mathrm{epi}(f)$ is $E \times I$-convex on $X \times \mathbb{R}$. Suppose on the contrary that there exist $(x, \lambda_1), (y, \lambda_2) \in \mathrm{epi}(f)$ (with $x, y \in X$ and $\lambda_1, \lambda_2 \in \mathbb{R}$) and $\alpha_0 \in (0, 1)$ such that

$$(\alpha_0 E(x) + (1 - \alpha_0)E(y), \alpha_0\lambda_1 + (1 - \alpha_0)\lambda_2) \notin \mathrm{epi}(f).$$

Let us put $x_0 = \alpha_0 E(x) + (1 - \alpha_0)E(y)$ and $\lambda_0 = \alpha_0\lambda_1 + (1 - \alpha_0)\lambda_2$, then $(x_0, \lambda_0) \notin \mathrm{epi}(f)$, and

$$A = \mathrm{epi}(f) \cap [(x, \lambda_1), (x_0, \lambda_0)] \text{ and also } B = \mathrm{epi}(f) \cap [(x_0, \lambda_0), (y, \lambda_2)].$$

Since f is lower semi-continuous, $\mathrm{epi}(f)$ is a closed subset of $X \times \mathbb{R}$, Consequently A and B are bounded and closed subsets of $X \times \mathbb{R}$, $(x_0, \lambda_0) \notin A$, $(x_0, \lambda_0) \notin B$. Thus, there exist $(\widetilde{x}, s) \in A$ and $(\widetilde{y}, t) \in B$ with $\widetilde{x}, \widetilde{y} \in X$ and $s, t \in \mathbb{R}$ such that

$$\min_{a \in A} \|a - (x_0, \lambda_0)\| = \|(\widetilde{x}, s) - (x_0, \lambda_0)\|, \text{ and } \min_{b \in B} \|b - (x_0, \lambda_0)\| = \|(\widetilde{y}, t) - (x_0 \lambda_0)\|.$$

Hence we have

$$\mathrm{epi}(f) \cap ((\widetilde{x}, s), (\widetilde{y}, t)) = \emptyset. \tag{2.1}$$

Notice that $\widetilde{x} \neq \widetilde{y}$ and $s \neq t$ and $((\widetilde{x}, s), (\widetilde{y}, t)) \neq \emptyset$.

On the other hand, since $(\widetilde{x}, s), (\widetilde{y}, t) \in X \times \mathbb{R}$, we have $f(E(\widetilde{x})) < s + \epsilon$, $f(E(\widetilde{y})) < t + \epsilon$ for each $\epsilon > 0$. Since $\alpha(s + \epsilon) + (1 - \alpha)(t + \epsilon) = \alpha s + (1 - \alpha)t + \epsilon$, by hypothesis, we have

$$f(\alpha E(\widetilde{x}) + (1 - \alpha)E(\widetilde{y})) < \alpha s + (1 - \alpha)t + \epsilon.$$

Since ϵ is an arbitrary positive real number, it follows that

$$f(\alpha E(\widetilde{x}) + (1 - \alpha)E(\widetilde{y})) \leq \alpha s + (1 - \alpha)t.$$

Hence,

$$\alpha(\widetilde{x}, s) + (1-\alpha)(\widetilde{y}, t) \in \operatorname{epi}(f),$$

which contradicts (2.1). Thus we conclude that $\operatorname{epi}(f)$ is $E \times I$-convex. This completes the proof. $\square$

The next theorem gives a characterization of semi-E-convexity.

2.2. Theorem. *Let $f : X \longrightarrow (-\infty, +\infty]$ be lower semi-continuous. Then f is semi-E-convex, if and only if for all $x, y \in X$, there exists $\alpha \in (0,1)$ (α depends x, y) such that*

$$f(\alpha E(x) + (1-\alpha)E(y)) \leq \alpha f(x) + (1-\alpha) f(y). \tag{2.2}$$

Proof. Let $f : X \longrightarrow (-\infty, +\infty]$ be semi-E-convex. It is easy to see that for all $\alpha \in (0,1)$ (2.2) holds. For the converse, it is sufficient to show that $\operatorname{epi}(f)$ is $E \times I$-convex set, as a subset of $X \times \mathbb{R}$. By contradiction suppose that there exist $(x, \lambda_1), (y, \lambda_2) \in \operatorname{epi}(f)$ (with $x, y \in X$ and $\lambda_1, \lambda_2 \in \mathbb{R}$) and $\alpha_0 \in (0,1)$ such that

$$(\alpha_0 E(x) + (1-\alpha_0)E(y), \alpha_0\lambda_1 + (1-\alpha_0)\lambda_2) \notin \operatorname{epi}(f).$$

Let $x_0 = \alpha_0 E(x) + (1-\alpha_0)E(y)$ and $\lambda_0 = \alpha_0\lambda_1 + (1-\alpha_0)\lambda_2$. Then $(x_0, \lambda_0) \notin \operatorname{epi}(f)$. By following the proof of Theorem 2.1, by defining $A, B, (\widetilde{x}, s)$, and $(\widetilde{y}, t)$, we find that

$$\operatorname{epi}(f) \cap ((\widetilde{x}, s), (\widetilde{y}, t)) = \emptyset. \tag{2.3}$$

Notice that $((\widetilde{x}, s), (\widetilde{y}, t)) \neq \emptyset$.

On the other hand, by the hypothesis of the theorem, for $\widetilde{x}, \widetilde{y} \in X$ there exists $\alpha \in (0,1)$ such that

$$f(\alpha E(\widetilde{x}) + (1-\alpha)E(\widetilde{y})) \leq \alpha f(\widetilde{x}) + (1-\alpha) f(\widetilde{y}). \tag{2.4}$$

Since $(\widetilde{x}, s), (\widetilde{y}, t) \in \operatorname{epi}(f)$, we have

$$f(\widetilde{x}) \leq s \text{ and } f(\widetilde{y}) \leq t. \tag{2.5}$$

Combining (2.4) and (2.5) we obtain

$$f(\alpha E(\widetilde{x}) + (1-\alpha)E(\widetilde{y})) \leq \alpha s + (1-\alpha)t.$$

So, $\alpha(\widetilde{x}, s) + (1-\alpha)(\widetilde{y}, t) \in \operatorname{epi}(f)$, which contradicts with (2.3). Thus, we conclude that $\operatorname{epi}(f)$ is $E \times I$-convex. Now the result follows. $\square$

2.3. Corollary. *Let $f : X \longrightarrow (-\infty, +\infty]$ be lower semi-continuous. Then f is semi-E-convex if and only if, for all $x, y \in X$,*

$$f\big(\tfrac{1}{2}E(x) + \tfrac{1}{2}E(y)\big) \leq \tfrac{1}{2} f(x) + \tfrac{1}{2} f(y).$$

2.4. Example. Let us define $E : [0,\infty) \longrightarrow [0,\infty)$ by

$$E(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n},\ (m,n) = 1, \\ 0 & \text{if } x \notin \mathbb{Q} \text{ or } x = 0, \end{cases}$$

and the function $f : [0,\infty) \longrightarrow [0,\infty)$ by

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x < 1, \\ x^2 & \text{if} x \geq 1. \end{cases}$$

It is clear that $[0,\infty)$ is E-convex set, and f is a lower semi-continuous function on $[0,\infty)$. Also for each $x, y \geq 0$, there exists $\lambda \in [0,1]$ such that

$$f(\lambda E(x) + (1-\lambda)E(y)) \leq \lambda f(x) + (1-\lambda) f(y),$$

but it is not semi-E-convex on $[0,\infty)$, because $f\big(\frac{9}{10}\big)=\frac{9}{5}$, $f\big(E\big(\frac{9}{10}\big)\big)=\frac{1}{5}$ or equivalently $f(E(x))\nleq f(x)$ and therefore $f\big(\frac{1}{2}E\big(\frac{1}{2}\big)+\frac{1}{2}E(1)\big)=\frac{3}{2}$ and $\frac{1}{2}f\big(\frac{1}{2}\big)+\frac{1}{2}f(1)=1$.

Our last result give a necessary and sufficient condition for a real-valued lower semi-continuous to be quasi-semi-E-convex.

2.5. Theorem. *Let $f : X \longrightarrow \mathbb{R}$ be lower semi-continuous. Then f is quasi-semi-E-convex if and only if, for all $x, y \in X$, there exists an $\alpha \in (0,1)$ (α depends on x, y) such that*

$$f(\alpha E(x)+(1-\alpha)E(y)) \leq \max\{f(x), f(y)\}.$$

Proof. By part (b) of Proposition 1.1, it can be easily checked that $f : X \longrightarrow \mathbb{R}$ is quasi-semi-E-convex if and only if for every real number λ, the level set $\{x \in X : \ f(x) \leq \lambda\}$ is E-convex. Suppose on the contrary that there exists a real number λ^* such that the set $F_{\lambda^*}=\{x \in X : \ f(x) \leq \lambda^*\}$ is not a E-convex set. Thus there exist $x, y \in F_{\lambda^*}$, and $\alpha_0 \in (0,1)$ such that $\alpha_0 E(x)+(1-\alpha_0)E(y) \notin F_{\lambda^*}$. Let $x_0=\alpha_0 E(x)+(1-\alpha_0)E(y)$, then $x_0 \notin F_{\lambda^*}$. Let

$$A=F_{\lambda^*}\cap[x,x_0] \text{ and } B=F_{\lambda^*}\cap[x_0,y],$$

where $[x,x_0]=\{\gamma x+(1-\gamma)x_0 : 0\leq\gamma\leq 1\}$ and $[x_0,y]=\{\gamma x_0+(1-\gamma)y : 0\leq\gamma\leq 1\}$. Notice that F_{λ^*} is a closed set [3]. Consequently A and B are bounded and closed subsets of X, and $x_0 \notin A,\ x_0 \notin B$. Thus there exist $\widetilde{x}\in A$ and $\widetilde{y}\in B$ such that

$$\min_{a\in A}\|a-x_0\|=\|\widetilde{x}-x_0\|$$

and

$$\min_{b\in B}\|b-x_0\|=\|\widetilde{y}-x_0\|,$$

where $\|\cdot\|$ is the norm on X. Hence we have

$$F_{\lambda^*}\cap[\widetilde{x},x_0]=\emptyset \text{ and } F_{\lambda^*}\cap[x_0,\widetilde{y}]=\emptyset.$$

Therefore,

$$F_{\lambda^*}\cap(\widetilde{x},\widetilde{y})=\emptyset. \tag{2.6}$$

Notice that $\widetilde{x}\neq\widetilde{y}$ and so $(\widetilde{x},\widetilde{y})\neq\emptyset$.

On the other hand, by the hypothesis of the theorem, for $\widetilde{x},\widetilde{y}\in X$, there exists an $\alpha\in(0,1)$ such that

$$f(\alpha E(\widetilde{x})+(1-\alpha)E(\widetilde{y}))\leq\max\{f(\widetilde{x}),f(\widetilde{y})\}. \tag{2.7}$$

Since $\widetilde{x},\widetilde{y}\in F_{\lambda^*}$ we have

$$f(\widetilde{x})\leq\lambda^* \text{ and } f(\widetilde{y})\leq\lambda^*. \tag{2.8}$$

Combining (2.7) and (2.8), we obtain

$$f(\alpha E(\widetilde{x})+(1-\alpha)E(\widetilde{y}))\leq\lambda^*.$$

So $\alpha E(\widetilde{x})+(1-\alpha)E(\widetilde{y})\in F_{\lambda^*}$, which contradicts (6). Thus we conclude that F_{λ^*} is convex. This completes the proof. □

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References

[1] Chen, X. *Some Properties of semi-E-convex Functions* J. Math. Anal. Appl. **275**, 251–262, 2002.

[2] Syau, Y. R. and Lee, E. S. *Some properties of E-convex functions*, Elsevier Applied Mathematics Letters **18**, 1074–1080, 2005.

[3] Syau, Y. R. *A Note On Convex Functions*, Internal. J. Math. Scl. **22** (3), 525–534, 1999.

[4] Yang, X. M. *Technical Note On E-convex Sets, E-convex Functions, and E-convex Programing*, Journal of Optimization Theory and Applications **109** (3), 699–704, 2001.

[5] Youness, E. A. *E-convex sets, E-convex Functions, and E-convex Programming*, Journal of Optimization Theory and Applications **102** (2), 439–450, 1999.