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CONVEXITY OF INTEGRAL OPERATORS OF p-VALENT FUNCTIONS

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Abstract

In this paper, we consider two general p-valent integral operators for certain analytic functions in the unit disc $\mathcal{U}$ and give some properties for these integral operators on some classes of univalent functions.

Keywords: Analytic functions, Integral operators, p-valently starlike functions, p-valently convex functions.

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1. Introduction and preliminaries

Let $\mathcal{A}(p,n)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \ (p,n \in \mathbb{N} = \{1,2,\dots\}), \tag{1.1}$$

which are analytic in the open disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also $\mathcal{A}(1,n) = \mathcal{A}(n)$, $\mathcal{A}(p,1) = \mathcal{A}(p)$ and $\mathcal{A}(1,1) = \mathcal{A}$.

A function $f \in \mathcal{A}(p,n)$ is said to be *p-valently starlike of order* α, $(0 \le \alpha < p)$, if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \ (z \in \mathcal{U}). \tag{1.2}$$

We denote by $S_p^*(\alpha)$ the class of all such functions. Also $S_1^*(\alpha) = S^*(\alpha)$. On the other hand, a function $f \in \mathcal{A}(p,n)$ is said to be *p-valently convex of order* α $(0 \le \alpha < p)$ if and only if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \ (z \in \mathcal{U}). \tag{1.3}$$

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Let $\mathcal{C}_p(\alpha)$ denote the class of all those functions which are p-valently convex of order α in $\mathcal{U}$. Also $\mathcal{C}_1(\alpha) = \mathcal{C}(\alpha)$. A function $f \in \mathcal{A}(p,n)$ is said to be class $R_p(\alpha)$, $(0 \le \alpha < p)$ if and only if

$$\Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha, \ (z \in \mathcal{U}). \tag{1.4}$$

Also $R_1(\alpha) = R(\alpha)$. For a function $f \in \mathcal{A}(p,n)$ we define the following operator

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= \frac{1}{p} z f'(z), \\ &\vdots \\ D^k f(z) &= D\left(D^{k-1} f(z)\right), \end{aligned} \tag{1.5}$$

where $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The differential operator D^k was studied by Shenan *et al.* (see [14]). When $p = 1$ we get the Sălăgean differential operator (see [12]).

We note that if $f \in \mathcal{A}(p,n)$, then

$$D^k f(z) = z^p + \sum_{j=n+p}^{\infty} \left(\frac{j}{p}\right)^k a_j z^j, \ (p,n \in \mathbb{N} = \{1,2,\ldots\}) \ (z \in \mathcal{U}).$$

Recently, A. Alb Lupaş (see [2]) define the family $\mathcal{B}S(p,m,\mu,\alpha)$, $\mu \ge 0$, $0 \le \alpha < 1$, $m \in \mathbb{N} \cup \{0\}$, $p,n \in \mathbb{N}$ so that it consists of functions $f \in \mathcal{A}(p,n)$ satisfying the condition

$$\left|\frac{D^{m+1} f(z)}{z^p}\left(\frac{z^p}{D^m f(z)}\right)^{\mu} - p\right| < p - \alpha, \ (z \in \mathcal{U}). \tag{1.6}$$

1.1. Remark. The family $\mathcal{B}S(p,m,\mu,\alpha)$ is a new comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones. For example, $\mathcal{B}S(1,0,1,\alpha) \equiv S^*(\alpha)$, $\mathcal{B}S(1,1,1,\alpha) \equiv \mathcal{C}(\alpha)$, $\mathcal{B}S(p,0,0,\alpha) = R_p(\alpha)$ and $\mathcal{B}S(1,0,0,\alpha) \equiv R(\alpha)$.

Another interesting subclass is the special case $\mathcal{B}S(1,0,2,\alpha) \equiv \mathcal{B}(\alpha)$ which has been introduced by Frasin and Darus (see [7]) and also the class $\mathcal{B}S(1,0,\mu,\alpha) \equiv \mathcal{B}(\mu,\alpha)$ which has been introduced by Frasin and Jahangiri (see [8]).

1.2. Remark. Let $l = (l_1, l_2, \ldots, l_n) \in \mathbb{N}_0^n$, $\delta = (\delta_1, \delta_2, \ldots, \delta_n) \in \mathbb{R}_+^n$ for all $i = \{1,2,\ldots,n\}$, $n \in \mathbb{N}$. We define the following general integral operator

$$\begin{aligned} &\mathcal{I}_{n,p}^{l,\delta}(f_1, f_2, \ldots, f_n) : \mathcal{A}(p,n) \to \mathcal{A}(p,n), \\ &\mathcal{I}_{n,p}^{l,\delta}(f_1, f_2, \ldots, f_n) = F_{p,n,l,\delta}(z), \\ &F_{p,n,l,\delta}(z) = \int_0^z p t^{p-1} \prod_{i=1}^{n} \left(\frac{D^{l_i} f_i(t)}{t^p}\right)^{\delta_i} dt, \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} &\mathcal{J}_{n,p}^{l,\lambda}(g_1, g_2, \ldots, g_n) : \mathcal{A}(p,n) \to \mathcal{A}(p,n) \\ &\mathcal{J}_{n,p}^{l,\lambda}(g_1, g_2, \ldots, g_n) = \mathcal{G}_{p,n,l,\lambda}(z), \\ &\mathcal{G}_{p,n,l,\lambda}(z) = \int_0^z p t^{p-1} \prod_{i=1}^{n} \left(e^{D^{l_i} g_i(t) \backslash t^{p-1}}\right)^{\lambda} dt, \end{aligned} \tag{1.8}$$

where $f_i, g_i \in \mathcal{A}(p,n)$ for all $i = \{1,2,\ldots,n\}$ and D is defined by (1.5).

1.3. Remark. The integral operator (1.7) was studied and introduced by Saltık *et al.* (see [13]). We note that if $l_1 = l_2 = \ldots = l_n = 0$ for all $i = \{1, 2, \ldots, n\}$, then the integral operator $F_{p,n,l,\delta}(z)$ reduces to the operator $F_p(z)$ which was studied by Frasin (see [6]). Upon setting $p = 1$ in the operator (1.7), we can obtain the integral operator $D^k F(z)$ which was studied by Breaz *et al.* (see [4]). For $p = 1$ and $l_1 = l_2 = \ldots = l_n = 0$ in (1.7), the integral operator $F_{p,n,l,\delta}(z)$ reduces to the operator $F_n(z)$ which was studied by Breaz and Breaz (see [3]). Observe that for $p = n = 1$, $l_1 = 0$ and $\mu_1 = \mu$ we obtain the integral operator $I_\mu(f)(z)$ which was studied by Pescar and Owa (see [11]), for $\mu_1 = \mu \in [0, 1]$ the special case of the operator $I_\mu(f)(z)$ was studied by Miller *et al.* (see [10]). For $p = n = 1$, $l_1 = 0$ and $\mu_1 = 1$ in (1.7), we have the Alexander integral operator $I(f)(z)$ in [1]. For $l_1 = l_2 = \ldots = l_n = 0$ in (1.7), the integral operator was studied by E. Deniz (see [5]).

1.4. Remark. For $l_1 = l_2 = \ldots = l_n = 0$ in (1.8), the integral operator was studied by E. Deniz (see [5]).For $p = n = 1$ and $l_1 = l_2 = \ldots = l_n = 0$ in (1.8), the integral operator $\mathcal{G}_{p,n,l,\lambda}(z)$ was studied by Frasin in [9].

In this paper, we obtain the order of convexity of the operators $F_{p,n,l,\delta}(z)$ and $\mathcal{G}_{p,n,l,\lambda}(z)$ on the class $\mathcal{BS}(p, l_i, \mu, \alpha)$. As special cases, the order of convexity of the operators $\int_0^z \left(\frac{f(t)}{t}\right)^\delta dt$ and $\int_0^z \left(e^{g(t)}\right)^\lambda dt$ are given.

In order to prove our main results, we recall the following lemma.

1.5. Lemma. (General Schwarz Lemma). *Let the function f be regular in the disk $\mathcal{U}_R$ with $|f(z)| < M$, M fixed. If f has at $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \quad z \in \mathcal{U}_R. \tag{1.9}$$

Equality (in the inequality (1.9) for $z \neq 0$) can hold only if $f(z) = e^{i\theta} \frac{M}{R^m} |z|^m$, where θ is constant. □

2. Main results

2.1. Theorem. *Let $l = (l_1, l_2, \ldots, l_n) \in \mathbb{N}_0^n$, $\delta = (\delta_1, \delta_2, \ldots, \delta_n) \in \mathbb{R}_+^n$, $0 \leq \alpha < p$, $\mu \geq 0$ and $f_i \in \mathcal{A}(p, n)$ be in the class $\mathcal{BS}(p, l_i, \mu, \alpha)$ for all $i = \{1, 2, \ldots, n\}$. If $\left|D^{l_i} f_i(z)\right| \leq M$, $(M \geq 1;\ z \in \mathcal{U})$, then the integral operator*

$$F_{p,n,l,\delta}(z) = \int_0^z pt^{p-1} \prod_{i=1}^n \left(\frac{D^{l_i} f_i(t)}{t^p}\right)^{\delta_i} dt$$

is in $\mathcal{C}_p(\beta)$, where

$$\beta = p \left[1 - \sum_{i=1}^n \delta_i((2p - \alpha) M^{\mu-1} + 1)\right], \tag{2.1}$$

and $\sum_{i=1}^n \delta_i((2p - \alpha) M^{\mu-1} + 1) \leq 1$ for all $i = \{1, 2, \ldots, n\}$.

Proof. Define the function $F_{p,n,l,\delta}(z)$ by

$$F_{p,n,l,\delta}(z) = \int_0^z pt^{p-1} \prod_{i=1}^n \left(\frac{D^{l_i} f_i(t)}{t^p}\right)^{\delta_i} dt,$$

for $f_i(z) \in \mathcal{B}S(p, l_i, \mu, \alpha)$. On the other hand it is easy to see that

$$(F_{p,n,l,\delta}(z))' = pz^{p-1} \prod_{i=1}^{n} \left(\frac{D^{l_i} f_i(z)}{z^p} \right)^{\delta_i}. \tag{2.2}$$

Now, we differentiate (2.2) logarithmically and multiply by z to obtain

$$1 + \frac{z\,(F_{p,n,l,\delta}(z))''}{(F_{p,n,l,\delta}(z))'} - p = \sum_{i=1}^{n} \delta_i \left(\frac{z\,(D^{l_i} f_i)'(z)}{(D^{l_i} f_i)(z)} - p \right). \tag{2.3}$$

It follows from (2.3) and $p\left(D^{l_i+1} f_i(z)\right) = z\left(D^{l_i} f_i(z)\right)'$ that

$$\begin{aligned} &\left| 1 + \frac{z\,(F_{p,n,l,\delta}(z))''}{(F_{p,n,l,\delta}(z))'} - p \right| \\ &\qquad \le p \sum_{i=1}^{n} \delta_i \left(\left| \frac{D^{l_i+1} f_i(z)}{D^{l_i} f_i(z)} \right| + 1 \right) \\ &\qquad \le p \sum_{i=1}^{n} \delta_i \left(\left| \frac{D^{l_i+1} f_i(z)}{z^p} \left(\frac{z^p}{D^{l_i} f_i(z)} \right)^{\mu} \right| \left| \frac{D^{l_i} f_i(z)}{z^p} \right|^{\mu-1} + 1 \right). \end{aligned} \tag{2.4}$$

Since $\left|D^{l_i} f_i(z)\right| \le M$, $(M \ge 1,\ z \in \mathcal{U})$ for all $i = \{1, 2, \dots, n\}$, applying the General Schwarz Lemma, we have

$$\left|D^{l_i} f_i(z)\right| \le M\,|z|^p.$$

Therefore, from (2.4), we obtain

$$\left| 1 + \frac{z\,(F_{p,n,l,\delta}(z))''}{(F_{p,n,l,\delta}(z))'} - p \right| \le p \sum_{i=1}^{n} \delta_i \left(\left| \frac{D^{l_i+1} f_i(z)}{z^p} \left(\frac{z^p}{D^{l_i} f_i(z)} \right)^{\mu} \right| M^{\mu-1} + 1 \right). \tag{2.5}$$

From (2.5) and (1.6), we see that

$$\begin{aligned} &\left| 1 + \frac{z\,(F_{p,n,l,\delta}(z))''}{(F_{p,n,l,\delta}(z))'} - p \right| \\ &\qquad \le p \sum_{i=1}^{n} \delta_i \left(\left(\left| \frac{D^{l_i+1} f_i(z)}{z^p} \left(\frac{z^p}{D^{l_i} f_i(z)} \right)^{\mu} - p \right| + p \right) M^{\mu-1} + 1 \right) \\ &\qquad \le p \sum_{i=1}^{n} \delta_i ((2p - \alpha)\, M^{\mu-1} + 1), \\ &\qquad = p - \beta. \end{aligned} \tag{2.6}$$

This completes the proof. □

2.2. Corollary. *Let* $l = (l_1, l_2, \dots, l_n) \in \mathbb{N}_0^n$, $\delta = (\delta_1, \delta_2, \dots, \delta_n) \in \mathbb{R}_+^n$, $0 \le \alpha < p$, $\mu \ge 0$ *and* $f_i \in \mathcal{A}(p, n)$ *is in the class* $\mathcal{B}S(p, l_i, \mu, \alpha)$ *for all* $i = \{1, 2, \dots, n\}$. *If* $\left|D^{l_i} f_i(z)\right| \le M$, $(M \ge 1;\ z \in \mathcal{U})$, *then the integral operator* $F_{p,n,l,\delta}(z)$ *is convex in* $\mathcal{U}$ *and*

$$\sum_{i=1}^{n} \delta_i = \frac{1}{(2p - \alpha)\, M^{\mu-1} + 1}.$$ □

Letting $p = 1$, $l_i = 0$ in Theorem 2.1 for all $i = \{1, 2, \dots, n\}$, we have

2.3. Corollary. *Let* $\delta = (\delta_1, \delta_2, \dots, \delta_n) \in \mathbb{R}_+^n$, $\mu \ge 0$, $0 \le \alpha < 1$ *and* $f_i \in \mathcal{A}(n)$ *is in the class* $\mathcal{B}(\mu, \alpha)$ *for all* $i = \{1, 2, \dots, n\}$. *If* $|f_i(z)| \le M$, $(M \ge 1;\ z \in \mathcal{U})$, *then the integral* *operator* $F_{1,n,0,\delta}(z) \in \mathcal{C}(\beta)$ *is in* $\mathcal{U}$ *and*

$$\beta = 1 - \sum_{i=1}^{n} \delta_i \left[(2-\alpha) M^{\mu-1} + 1\right],$$

where $\sum_{i=1}^{n} \delta_i \left[(2-\alpha) M^{\mu-1} + 1\right] \leq 1$ *for all* $i = \{1, 2, \ldots, n\}$. □

Letting $n = 1$ in Corollary 2.2, we have

2.4. Corollary. *Let* $\delta \in \mathbb{R}^+$, $\mu \geq 0$, $0 \leq \alpha < 1$ *and let* $f \in \mathcal{A}$ *be in the class* $\mathcal{B}(\mu, \alpha)$. *If* $|f(z)| \leq M$, $(M \geq 1;\ z \in \mathcal{U})$, *then the integral operator* $F_{1,1,0,\delta}(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\delta} dt \in \mathcal{C}(\beta)$ *is in* $\mathcal{U}$, *and*

$$\beta = 1 - \delta \left[(2-\alpha) M^{\mu-1} + 1\right],$$

where $\delta \left[(2-\alpha) M^{\mu-1} + 1\right] \leq 1$. □

Letting $p = 1$, $l_i = 0$, $\mu = 1$ in Theorem 2.1 for all $i = \{1, 2, \ldots, n\}$, we have

2.5. Corollary. *Let* $\delta = (\delta_1, \delta_2, \ldots, \delta_n) \in \mathbb{R}_+^n$, $0 \leq \alpha < 1$ *and let* $f_i \in \mathcal{A}(n)$ *be in the class* $S^*(\alpha)$ *for all* $i = \{1, 2, \ldots, n\}$. *Then the integral operator* $F_{1,n,0,\delta}(z) \in \mathcal{C}(\beta)$ *is in* $\mathcal{U}$, *where*

$$\beta = 1 - \sum_{i=1}^{n} \delta_i (3-\alpha),$$

where $\sum_{i=1}^{n} \delta_i (3-\alpha) \leq 1$ *for all* $i = \{1, 2, \ldots, n\}$. □

Letting $n = 1, \delta = \frac{1}{3}$ and $\alpha = 0$ in Corollary 2.5, we have

2.6. Corollary. *Let* $f \in \mathcal{A}$ *be starlike in* $\mathcal{U}$. *If* $|f(z)| \leq M$, $(M \geq 1;\ z \in \mathcal{U})$, *then the integral operator* $F_{1,1,0,\frac{1}{3}}(z)$ *is convex in* $\mathcal{U}$. □

2.7. Remark. Letting δ_i by $\frac{1}{\beta_i}$, $p = 1$, $l_i = 0$ in Theorem 2.1 for all $i = \{1, 2, \ldots, n\}$ we obtain Theorem 2.1 (see [9]).

2.8. Theorem. *Let* $l = (l_1, l_2, \ldots, l_n) \in \mathbb{N}_0^n$, $\lambda \in \mathbb{R}_+^n$, $0 \leq \alpha < p$, $\mu \geq 0$ *and* $g_i \in \mathcal{A}(p, n)$ *be in the class* $\mathcal{B}S(p, l_i, \mu, \alpha)$ *for all* $i = \{1, 2, \ldots, n\}$. *If* $\left|D^{l_i} g_i(z)\right| \leq M$, $(M \geq 1;\ z \in \mathcal{U})$, *then the integral operator*

$$\mathcal{G}_{p,n,l,\lambda}(z) = \int_0^z p t^{p-1} \prod_{i=1}^{n} \left(e^{D^{l_i} g_i(t) \backslash t^{p-1}}\right)^{\lambda} dt, \tag{2.7}$$

is in $\mathcal{C}_p(\beta)$, *where*

$$\beta = p - \left[\lambda n \left\{\left(p^2 + (1-\alpha)p\right) M^{\mu} + (p-1) M\right\}\right],$$

and $\lambda \leq \frac{p}{n\left\{\left(p^2 + (1-\alpha)p\right) M^{\mu} + (p-1)M\right\}}$.

Proof. Define the function $\mathcal{G}_{p,n,l,\lambda}(z)$ by

$$\mathcal{G}_{p,n,l,\lambda}(z) = \int_0^z p t^{p-1} \prod_{i=1}^{n} \left(e^{D^{l_i} g_i(t) \backslash t^{p-1}}\right)^{\lambda} dt,$$

for $g_i(z) \in \mathcal{B}S(p, l_i, \mu, \alpha)$. It follows that

$$1 + \frac{z\left(\mathcal{G}_{p,n,l,\lambda}(z)\right)''}{\left(\mathcal{G}_{p,n,l,\lambda}(z)\right)'} - p = \lambda \sum_{i=1}^{n} \left[\frac{\left(D^{l_i} g_i(z)\right)'}{z^{p-1}} - (p-1) \frac{D^{l_i} g_i(z)}{z^p}\right] z. \tag{2.8}$$

Therefore from (2.8) and $p\left(D^{l_i+1}f_i\right)(z) = z\left(D^{l_i}f_i(z)\right)'$, we obtain

$$\begin{aligned}&\left|1+\frac{z\left(\mathcal{G}_{p,n,l,\lambda}(z)\right)''}{\left(\mathcal{G}_{p,n,l,\lambda}(z)\right)'}-p\right| \\ &\quad\le \lambda\left(\sum_{i=1}^{n}\left[p\left|\frac{D^{l_i+1}g_i(z)}{z^p}\right|+(p-1)\left|\frac{D^{l_i}g_i(z)}{z^p}\right|\right]\right) \\ &\quad\le \lambda\left(\sum_{i=1}^{n}\left[p\left|\frac{D^{l_i+1}g_i(z)}{z^p}\left(\frac{z^p}{D^{l_i}g_i(z)}\right)^{\mu}\right|\left|\frac{D^{l_i}g_i(z)}{z^p}\right|^{\mu}+(p-1)\left|\frac{D^{l_i}g_i(z)}{z^p}\right|\right]\right).\end{aligned}$$

Applying the General Schwarz Lemma once again, we have

$$\left|\frac{D^{l_i}g_i(z)}{z^p}\right| \le M,\ \ (z\in\mathcal{U}),$$

and hence

$$\begin{aligned}&\left|1+\frac{z\left(\mathcal{G}_{p,n,l,\lambda}(z)\right)''}{\left(\mathcal{G}_{p,n,l,\lambda}(z)\right)'}-p\right| \\ &\quad\le \lambda\left(\sum_{i=1}^{n}\left[p\left|\frac{D^{l_i+1}g_i(z)}{z^p}\left(\frac{z^p}{D^{l_i}g_i(z)}\right)^{\mu}\right|M^{\mu}+(p-1)M\right]\right).\end{aligned}\tag{2.9}$$

Therefore from (2.9), we obtain

$$\begin{aligned}&\left|1+\frac{z\left(\mathcal{G}_{p,n,l,\lambda}(z)\right)''}{\left(\mathcal{G}_{p,n,l,\lambda}(z)\right)'}-p\right| \\ &\quad\le \lambda\left(\sum_{i=1}^{n}\left[\left(p\left|\frac{D^{l_i+1}g_i(z)}{z^p}\left(\frac{z^p}{D^{l_i}g_i(z)}\right)^{\mu}-p\right|+p\right)M^{\mu}+(p-1)M\right]\right) \\ &\quad\le \lambda n\left\{\left(p^2+(1-\alpha)p\right)M^{\mu}+(p-1)M\right\} \\ &\quad= p-\beta.\end{aligned}$$

This completes the proof. □

Letting $l_i = 0$, $\mu = 0$ in Theorem 2.8 for all $i = \{1, 2, \ldots, n\}$, we have

2.9. Corollary. *Let $g_i \in \mathcal{A}(n)$ be in the class $R_p(\alpha)$, $\lambda \in \mathbb{R}_+^n$, $0 \le \alpha < p$. If $|g_i(z)| \le M$, $(M \ge 1;\ z \in \mathcal{U})$, then the integral operator $\mathcal{G}_{p,n,0,\lambda}(z)$ is in $\mathcal{C}_p(\beta)$ in $\mathcal{U}$, where*

$$\beta = p - \left\{\lambda n\left[\left(p^2+(1-\alpha)p\right)+(p-1)M\right]\right\},$$

and $\lambda n\left[\left(p^2+(1-\alpha)p\right)+(p-1)M\right] \le p.$ □

Letting $n = 1$, $p = 1$, $l = 0$ in Theorem 2.8, we have

2.10. Corollary. *Let $\lambda \in \mathbb{R}^+$, $0 \le \alpha < 1$, $\mu \ge 0$ and let $g \in \mathcal{A}$ be in the class $\mathcal{B}(\mu, \alpha)$. If $|g(z)| \le M$, $(M \ge 1;\ z \in \mathcal{U})$, then the integral operator $\mathcal{G}_{1,1,0,\lambda}(z) = \int_0^z \left(e^{g(t)}\right)^{\lambda} dt$ is in $\mathcal{C}(\beta)$ in $\mathcal{U}$, where*

$$\beta = 1 - \lambda(2-\alpha)M^{\mu},$$

and $\lambda(2-\alpha)M^{\mu} \le 1.$ □

Letting $p = 1$, $l_i = 0$, $\mu = 1$ in Theorem 2.8 for all $i = \{1, 2, \ldots, n\}$, we have

2.11. Corollary. *Let $g_i \in \mathcal{A}(n)$ be in the class $S^*(\alpha)$, $\lambda \in \mathbb{R}_+^n$, $0 \leq \alpha < 1$, for all $i = \{1, 2, \ldots, n\}$. If $|g_i(z)| \leq M$, $(M \geq 1;\ z \in \mathcal{U})$, then the integral operator $\mathcal{G}_{1,n,0,\lambda}(z)$ is in $\mathcal{C}(\beta)$ in $\mathcal{U}$, where*

$$\beta = 1 - \lambda n(2 - \alpha)M,$$

and $\lambda \leq \frac{1}{n(2-\alpha)M}$. □

Letting $\alpha = 0$, $M = n = 1$ and $\lambda = \frac{1}{2}$ in Corollary 2.12, we have

2.12. Corollary. *Let $g \in \mathcal{A}$ be starlike in $\mathcal{U}$ for all $i = \{1, 2, \ldots, n\}$. If $|g(z)| \leq 1$, $(z \in \mathcal{U})$, then the integral operator $\mathcal{G}_{1,1,0,\frac{1}{2}}(z)$ is convex in $\mathcal{U}$.* □

References

[1] Alexander, J. W.*Functions which map the interior of the unit circle upon simple regions*, Annals of Mathematics **17** (1), 12–22, 1915.

[2] Alb Lupaş, A. *A subclass of analytic functions defined by differential Sălăgean operator*, Acta Universitatis Apulensis **20**, 259–263, 2009.

[3] Breaz D. and Breaz, N. *Two integral operators*, Studia Universitatis Babes-Bolyai Mathematica **47** (3), 13–19, 2002.

[4] Breaz, D., Güney, H. Ö. and Sălăgean, G. S. *A new general integral operator*, Tamsui Oxford Journal of Mathematical Sciences **25** (4), 407–414, 2009.

[5] Deniz, E., Çağlar, M. and Orhan, H. *The order of convexity of two p-valent integral operators*, American Institute of Physics, International Symposium of Mathematical Science, Bolu, Turkey, (1309), 234–240, 2010.

[6] Frasin, B. A. *Convexity of integral operators of p-valent functions*, Math. Comput. Model. **51**, 601–605, 2010.

[7] Frasin, B. A. and Darus, M. *On certain analytic univalent functions*, Internat. J. Math. and Math. Sci. **25** (5), 305–310, 2001.

[8] Frasin, B. A. and Jahangiri, J. *A new and comprehensive class of analytic functions*, Anal. Univ. Ordea Fasc. Math. **XV**, 59–62, 2008.

[9] Frasin, B. A. and Ahmad, A. *The order of convexity of two integral operators*, Studia Univ. "Babeş-Bolyai", Mathematica **LV** (2)(6), 113-117, 2010.

[10] Miller, S. S., Mocanu, P. T. and Reade, M. O. *Starlike integral operators*, Pacific Journal of Mathematics **79** (1), 157–168, 1978.

[11] Pescar V. and Owa, S. *Sufficient conditions for univalence of certain integral operators*, Indian Journal of Mathematics **42** (3), 347–351, 2000.

[12] Sălăgean, G. St. *Subclases of univalent functions* (Lecture Notes in Math., Springer Verlag, Berlin, 1983), 362–372.

[13] Saltık, G., Deniz E., and Kadıoğlu, E. *Two new general p-valent integral operators*, Math. Comput. Model. **52**, 1605–1609, 2010.

[14] Shenan, G. M., Salim, T. O. and Marouf, M. S. *A certain class of multivalent prestarlike functions involving the Srivastava-Saigo-Owa fractional integral operator*, Kyungpook Math. J. **44**, 353–362, 2004.