Hacettepe Journal of Mathematics and Statistics
Volume 41 (5) (2012), 627 – 633

ON OPERATORS OF STRONG TYPE B

Safak Alpay*

Received 17 : 06 : 2011 : Accepted 21 : 03 : 2012

Abstract

We discuss operators of strong type B between a Banach lattice and a Banach space and give necessary and sufficient conditions for this class of operators to coincide with weakly compact operators.

Keywords: Operators of strong type B, b-weakly compact operators, Banach lattice.

2000 AMS Classification: 46 A 40, 46 B 40, 46 B 42.

1. Introduction

A vector lattice E is an ordered vector space for which $\sup\{x, y\}$ exists for every pair of vectors x, y in E. Let E be a vector lattice. For $x, y \in E$ with $x \leq y$ in E, the set $[x, y] = \{t \in E : x \leq t \leq y\}$ is called an *order interval*. A subset of E is called *order bounded* if it is contained in some order interval. A Banach lattice E is a Banach space $(E, ||\cdot||)$ where E is also a vector lattice and its norm satisfies the following property: For each $x, y \in E$ with $|x| \leq |y|$, we have $||x|| \leq ||y||$. If E is a Banach lattice, its topological dual E' equipped with the dual norm and order is also a Banach lattice. A norm $||\cdot||$ on a Banach lattice E is called *order continuous* if for each net (x_α) with $x_\alpha \downarrow 0$ in E, (x_α) converges to zero for the norm $||\cdot||$, where $(x_\alpha) \downarrow 0$ means that (x_α) is decreasing, its infimum exists and is equal to zero.

A Banach lattice E is said to be a *KB-space* whenever every increasing norm bounded sequence in $E_+ = \{x \in E : 0 \leq x\}$ is norm convergent. Each KB-space has order continuous norm, but a Banach lattice with an order continuous norm is not necessarily a KB-space. Indeed, the Banach lattice c_0 has order continuous norm but it is not a KB-space. However, if E is a Banach lattice, the topological dual is a KB-space if and only if its norm is order continuous. A Banach lattice E is an abstract M-space (*AM-space* in short) if for each $x, y \in E$ with $\inf\{x, y\} = 0$, we have $||x + y|| = \max\{||x||, ||y||\}$. A Banach lattice E is an *AL-space* if its dual E' is an AM-space.

We will use the term operator to mean a bounded linear mapping. The space of bounded linear operators between Banach spaces E, F will be denoted by $L(E, F)$. All vector lattices considered in this note are assumed to have separating order duals. We refer the reader to [1] and [18] for further terminology and notation.

*Department of Mathematics, Middle East Technical University, Ankara, Turkey.
e-Mail: safak@metu.edu.tr

2. Operators of strong type B

A subset B of a vector lattice E is called *b-order bounded in E* if it is order bounded in the order bidual $E^{\sim\sim}$. Clearly every order bounded subset of E is b-order bounded. However, the converse does not hold in general. Indeed, the subset $B = \{e_n\}$, where (e_n) is the sequence of reals with all terms are zero except the n'th which is one, is a b-order bounded sequence in c_0 which is not order bounded in c_0. A vector lattice E is said to have the *b-property* if each b-order bounded subset of E is order bounded in E. The b-property and vector lattices with the b-property were defined and studied in [2,3,6,7], and [8].

Let E be a Banach lattice and X a Banach space. An operator $T : E \to X$ is called a *b-weakly compact operator* if T maps b-order bounded subsets of E into relatively weakly compact subsets of X. b-weakly compact operators are studied in the papers [4,5,7,8] and [10-15]. b-weakly compact operators also appear in the papers [16-17], and [18] under the name operators of type B. The space of b-weakly compact operators between a Banach lattice E and a Banach space X will be denoted by $W_b(E, X)$. Let us recall that an operator $T : E \to X$ is called *order weakly compact* if $T(B)$ is relatively weakly compact in X for each order bounded subset B of E. We refer the reader to [1] for an account of order weakly compact operators. The space of order weakly compact operators between a Banach lattice E and a Banach space X will be denoted by $W_o(E, X)$. If $W(E, X)$ is the space of weakly compact operators between E and X, then we have the following inclusions:

$$W(E, X) \subseteq W_b(E, X) \subseteq W_o(E, X)$$

between the classes of operators introduced above. Let us note that these inclusions may well be proper [5,17].

The following class of operators was introduced in [19].

2.1. Definition. An operator : $E \to X$ from a Banach lattice E into a Banach space X is called an *operator of strong type B* if T'', the second adjoint of T, maps the band $B(E)$, generated by E in E'', into X.

The space of operators of strong type B will be denoted by $W_{sb}(E, X)$. Since E'' is Dedekind complete, every band in E'' is a projection band and in particular, there is a positive projection from E'' onto $B(E)$. Thus, operators of strong type B extend to E''. One of the open problems put forward in [19] was the existence of a b-weakly compact operator which is not of strong type B. This question was settled in [16] and it was shown that there does exist a b-weakly compact operator which is not of strong type B. Thus we have:

$$W(E, X) \subseteq W_{sb}(E, X) \subseteq W_b(E, X) \subseteq W_o(E, X),$$

where the inclusions may be proper. Let us remark that operators of strong type B are of substance only when they do not coincide with weakly compact or b-weakly compact operators. For example, when E has order continuous norm or is an AM-space then we have $W_{sb}(E, X) = W_b(E, X)$. On the other hand, when E is a KB-space, then $E = B(E)$ and each operator $T : E \to X$ is an operator of strong type B and we have:

$$W_{sb}(E, X) = W_b(E, X) = W_o(E, X) = L(E, X).$$

Similarly, when E' has order continuous norm (i.e. l^1 does not embed in E) and c_0 does not embed in X, then the Grothendieck-Ghoussoub-Johnson Theorem (cf. [1, Theorem 17.6]) asserts that each operator $T : E \to X$ is weakly compact and again we have $W(E, X) = W_{sb}(E, X) = W_b(E, X) = W_o(E, X) = L(E, X)$. The space of operators of strong type B is a norm closed subspace of $L(E,X)$ and is an order ideal whenever F is a Banach lattice.

Let E, F be Banach lattices and $T : E \to F$ a positive operator. Let $x \in B(E)$ be an arbitrary positive vector. Then there exists a net $\{x_\alpha\}$ of positive elements in the order ideal $I(E)$ generated by E in E'' such that $x_\alpha \uparrow x$ in E'', i.e. $x = \sup_\alpha x_\alpha$ (cf. [1, Theorem 3.4]). Since adjoint operators are order continuous (cf. [1, Theorem 5.8]), we have $T''x_\alpha \uparrow T''x$ in F''. As T is a positive operator, T'' maps the order ideal generated by E in E'' into the order ideal generated by F in F''. It follows that $T''x \in B(F)$. In particular, if F is a KB-space, it follows that each positive and therefore each regular (difference of two positive operators) operator is of strong type B. It also follows from this observation that if an operator $T : E \to X$ factors over a KB-space as $R \circ S$ where the first factor S is order bounded, then T is of strong type B. As a result of this observation and [1, Theorem 14.15], we see that if $T : E \to X$ is an operator from a Banach lattice E into a Banach space X which does not contain a copy of c_0, then T admits a factorization through a KB-space F as $T = S \circ Q$ where Q is a lattice homomorphism. Since a lattice homomorphism is a positive operator, it follows that T is of strong type B.

Utilizing [18, Theorem 3.5.8], we see that if $T : E \to X$ is an operator for which T'' is order weakly compact, then T factors over a KB-space F as $T = S \circ Q$, where $Q : E \to F$ is an interval preserving lattice homomorphism and $S \in L(F,X)$. Therefore if T'' is order weakly compact, then T is of strong type B.

Finally, let E, F be Banach lattices and X a Banach space. Let $T : E \to F, S : F \to X$ be two operators. If F has order continuous norm and S is b-weakly compact, then S factors over a KB-space [8]. Consequently, the operator $S \circ T$ is of strong type B.

We close this section with an intrinsic characterization of operators of strong type B. Let E be a Banach lattice and $\overline{I(E)}$ the closed order ideal in E'' generated by E. According to the main Theorem in [17], T is of strong type B if and only if the restriction of T'' to $\overline{I(E)}$ does not preserve c_0. This enables us to give the following:

2.2. Proposition. *The operator $T : E \to X$ is of strong type B if and only if $(T''x_n)$ is convergent for each increasing sequence (x_n) in $\overline{I(E)}$ with $||x_n|| \le 1$ for all n.*

Proof. $\Longrightarrow$ If T is of strong type B, then $T''|_{\overline{I(E)}}$ does not preserve a copy of c_0. Hence, by [18, Theorem 3.4.11], (Tx_n) is convergent for each increasing sequence in $B_{\overline{I(E)}}$, where $B_{\overline{I(E)}}$ is the closed unit ball of $\overline{I(E)}$.

$\Longleftarrow$ If $T''(x_n)$ is convergent for each increasing sequence in $B_{\overline{I(E)}}$, then $T''|_{\overline{I(E)}}$ does not preserve a subspace isomorphic to c_0 and consequently T is of strong type B. □

3. When is $W_{sb}(E,X) = W(E,X)$?

The following result was claimed for b-weakly compact operators in [11]. In one part of the proof the authors were misguided by an erroneous part of [5, Proposition 2]. However, their claim is still true for operators of strong type B. We now state and prove this result. The proof follows the same lines as in [11]. In order to identify when we have $W(E,X) = W_{sb}(E,X)$, we need a lemma which is a combination of theorems 116.1 and 116.3 in [20].

3.1. Lemma. *Suppose E' does not have order continuous norm. Then there exist a disjoint sequence (u_n) of positive elements in E with $||u_n|| \le 1$ for all n, $0 \le \phi \in E'$ and $\epsilon > 0$ satisfying $\phi(u_n) > \epsilon$ for all n. Moreover, the components ϕ_n of ϕ in the carriers C_{u_n} form an order bounded disjoint sequence in E'_+ such that $\phi_n(u_n) = \phi(u_n)$ for all n and $\phi_n(u_m) = 0$ if $n \ne m$.* □

3.2. Proposition. *Let E be a Banach lattice and X a Banach space. The following are equivalent:*

1) $W(E,X) = W_{sb}(E,X)$.
2) *One of the following hold:*
   a) *E' has order continuous norm,*
   b) *X is reflexive.*

Proof. 2a $\Longrightarrow$ 1 Let T be an operator of strong type B and let $B(E)$ be the band generated by E in E''. Then $T''B(E) \subseteq X$. Since E' has order continuous norm $E'' = B(E)$ and $T''(E'') \subseteq X$. Thus T is weakly compact.

2b $\Longrightarrow$ 1 In this case each operator $T : E \to X$ is weakly compact.

1 $\Longrightarrow$ 2 Suppose E' is not a KB-space and X is not reflexive. Then we construct an operator of strong type B which is not weakly compact. Since E' is not a KB-space, it follows from the preceding lemma that there exist a disjoint sequence (u_n) in E_+ with $||u_n|| \leq 1$ for all n and some $\phi \in E'_+$, $\epsilon > 0$ such that $\phi(u_n) > \epsilon$ for all n. The components ϕ_n of ϕ in the carriers C_{u_n} of u_n form an order bounded disjoint sequence in E'_+ such that $\phi(u_n) = \phi_n(u_n)$ and $\phi_m(u_n) = 0$ if $n \neq m$. Note that we have $0 \leq \phi_n \leq \phi$ for all n. Let us define $T_1 : E \to l^1$ by

$$T_1(x) = \left(\frac{\phi_n(x)}{\phi(u_n)}\right)$$

for each $x \in E$. Since

$$\sum_{n=1}^{\infty}\left|\frac{\phi_n(x)}{\phi(u_n)}\right| \leq \frac{1}{\epsilon}\sum_{n=1}^{\infty}\phi_n(|x|) \leq \frac{1}{\epsilon}\phi(|x|)$$

the operator T_1 is well-defined, positive and as l^1 is a KB-space, it is of strong type B.

On the other hand, since X is not reflexive, the closed unit ball B_X of X is not weakly compact. Hence we can find a sequence (y_n) in B_X without any weakly convergent subsequences. Let us define the operator $T_2 : l^1 \to X$ by $T_2(\alpha_n) = \Sigma\alpha_n y_n$. Since $\Sigma||\alpha_n y_n|| \leq \Sigma|\alpha_n|$, T_2 is well-defined. We consider the operator $T = T_2 \circ T_1 : E \to l^1 \to X$ defined by

$$T(x) = \sum_n \frac{\phi_n(x)}{\phi(u_n)} y_n$$

for each $x \in E$. As T factors over the KB-space l^1, and the first factor T_1 is positive, T is of strong type B. However as $T(u_n) = y_n$ and since (y_n) is chosen not to have any weakly convergent subsequences, T is not weakly compact. □

The preceding theorem cannot be extended to b-weakly compact operators. There is a Banach lattice E with E' a KB-space and a non-weakly compact operator $T : E \to c_0$ which is b-weakly compact [4].

We now give some consequences of the preceding result.

3.3. Corollary. *Let E, F be Banach lattices. The following are equivalent:*

1) *Each strong type B operator $T : E \to F$ is weakly compact.*
2) *Each positive strong type B operator $T : E \to F$ is weakly compact.*
3) *One of the following holds:*
   a) *E' is a KB-space.*
   b) *F is reflexive.* □

The next result yields a characterization of the order continuity of the dual norm.

3.4. Corollary. *The following are equivalent:*

1) *Each strong type B operator T on E is weakly compact.*
2) *Each positive strong type B operator on E is weakly compact.*
3) *E' is a KB-space.* □

3.5. Corollary. *Let E be an infinite dimensional AL-space. Then the following are equivalent:*

1) *Each strong type B operator $T : E \to F$ is weakly compact.*
2) *F is reflexive.*

The proposition above enables us to recapture a recent result proved in [15].

3.6. Corollary. *Let X be a non-reflexive Banach lattice. Then the following are equivalent:*

1) $W_o(E,X) = W(E,X)$.
2) *One of the following holds:*
   a) *E has the positive Grothendieck property.*
   b) *E' is a KB-space and c_0 is not embeddable in X.*

Proof. 2a $\Longrightarrow$ 1 This is [18, Theorem 5.3.13].

2b $\Longrightarrow$ 1 This follows from [1, Theorem 5.27].

1 $\Longrightarrow$ 2 We first show that E' has order continuous norm. If this is not the case then there exists an operator $T : E \to X$ which is of strong type B but is not weakly compact. As T is order weakly compact, this is a contradiction to (1).

Assume now that c_0 is embeddable in X and let $S : c_0 \to X$ be this embedding. We have to show that for each (x'_n) in E'_+ which is $\sigma(E', E)$ convergent to 0 in E', (x'_n) is $\sigma(E', E'')$ convergent to zero. Let (x'_n) be such a sequence and define $T : E \to c_0$ by $T(x) = (x'_n(x))$ for each $x \in E$. As c_0 has order continuous norm and T is positive, T is order weakly compact. Hence the operator $S \circ T : E \to c_0 \to X$ is also order weakly compact and consequently, weakly compact. As S is an embedding, it follows that $T : E \to c_0$ is also weakly compact. Therefore its adjoint $T' : l^1 \to E'$ is weakly compact. As $T'(\alpha_n) = \Sigma_n \alpha_n x'_n$ for each $(\alpha_n) \in l^1$, the subset (x'_n) in $T'(B_{l^1})$ is relatively weakly compact, where B_{l^1} is the closed unit ball of l^1. Thus $x'_n \to 0$ in $\sigma(E', E'')$. □

The proof of the following is similar.

3.7. Corollary. *Let F be a non-reflexive Banach lattice. Then the following are equivalent:*

1) $W_o(E,F) = W(E,F)$.
2) *One of the following holds:*
   a) *E has the positive Grothendieck property.*
   b) *E' and F are KB-spaces.* □

3.8. Corollary. *Consider the scheme of operators*

$$E \xrightarrow{S_1} F \xrightarrow{S_2} G \xrightarrow{S_3} H$$

between Banach lattices where E' has order continuous norm. If S_1 is dominated by an operator of strong type B, S_2 is compact, and S_3 is dominated by an order weakly compact operator, then $S_3 \circ S_2 \circ S_1$ is a compact operator.

Proof. By the proposition S_1 is weakly compact, therefore $S_2 \circ S_1$ factors over a reflexive Banach lattice X as $T_1 \circ T_2$, where $T_2 : X \to G$ is positive and is dominated by a positive compact operator. By [1, Theorem 19.14], $S_3 \circ T_2$ is a Dunford-Pettis operator. Then $S_3 \circ S_2 \circ S_1 = (S_3 \circ T_2) \circ T_1$ is a compact operator. □

The square of an operator of strong type B need not be weakly compact. Consider for example the identity operator on $L^1[0,1]$.

To this end we have the following:

3.9. Corollary. *The following are equivalent:*

1) *Let* $0 \leq S, T : E \to E, 0 \leq S \leq T, T \in W_{sb}(E,E)$*, then* S *is weakly compact.*
2) *Let* $0 \leq T : E \to E$ *be of strong type B, then* T *is weakly compact.*
3) *If* $0 \leq T : E \to E$ *is of strong type B, then* T^2 *is weakly compact.*
4) E' *is a KB-space.*

Proof. $3 \implies 4$ Suppose E' is not a KB-space. Then we construct a positive operator of strong type B, say T, such that T^2 is not weakly compact. There exist a disjoint sequence (u_n) in E_+ with $||u_n|| \leq 1$ and $\phi \in E'_+$ with $\epsilon < \phi(u_n)$ for some ϵ and all n. The components ϕ_n of ϕ in the carriers C_{u_n} of u_n form a disjoint sequence in E'_+ such that $\phi_n(u_n) = \phi(u_n)$ for all n and $\phi_n(u_n) = 0$ if $n \neq m$. Clearly, (u_n) does not have any weakly convergent subsequences as $\phi(u_n) > \epsilon$ for all n. Let us define the operator T on E by $T(x) = \sum_n \frac{\phi_n(x)}{\phi(u_n)} u_n$ for all $x \in E$. Then the operator T admits a factorization through l^1 and is therefore of strong type B. However since $T(u_n) = u_n$, T^2 is not weakly compact. □

3.10. Definition. Let E, F be Banach lattices. An operator $T : E \to F$ is called *semi-compact* if for each $\epsilon > 0$, there exists $u \in F_+$ such that $T(B_E) \subseteq [-u,u] + \epsilon B_F$.

3.11. Corollary. *Suppose* E' *is a KB-space and that* E *has the Dunford-Pettis property. Then each positive operator of strong type B is semi-compact.*

Proof. Let $T : E \to F$ be of strong type B. Then T is weakly compact. As E has the Dunford-Pettis property, T is a Dunford-Pettis operator. Semi-compactness of T follows from the order continuity of the norm in E' by [14, Theorem 2.2]. □

The following was stated as Theorem 2.3 in [13]. We offer a generalization. It is a generalization since operators of strong type B are b-weakly compact.

3.12. Proposition. *Let* E, F *be Banach lattices. If each positive strong type B operator is compact, then one of the following is true:*

1) E' *is a KB-space.*
2) F *is finite dimensional.*

Proof. Suppose that the statements are not true. We construct a positive strong type B operator which is not compact.

As the norm of E' is not order continuous, E contains a sublattice which is isomorphic to l^1 and there exists a positive projection $P : E \to l^1$ by [1, Theorem 14.21].

On the other hand, since F is infinite dimensional, there exists a disjoint norm bounded sequence (y_n) in F_+ which does not converge to zero in norm. Let us consider the operator $S : l^1 \to F$ defined by $S(\lambda_n) = \sum_{i=1}^{\infty} \lambda_n y_n$ for each (λ_n) in l^1. Since $S(e_n) = y_n$ for each n, S is not compact. Consider the operator $T = S \circ P : E \to l^1 \to F$. Since T factors over a KB-space where the first factor is positive, it is of strong type B. However, T is not compact. Because if it were, then $S = T \circ i$ would also be compact, where i is the inclusion operator $i : l^1 \to E$, which is obviously not true. □

References

[1] Aliprantis, C. D. and Burkinshaw, O. *Positive Operators* (Academic Press, Orlando, San Diego, New York, 1985).

[2] Alpay, S., Altin, B. and Tonyali, C. *A note on Riesz spaces with b-property*, Positivity **7**, 135–139, 2003).

[3] Alpay, S. and Altin, B. *On property (b) of vector lattices*, Czech Math. J. **56** (131), 765–772, 2006.

[4] Alpay, S. and Altin, B. *On Riesz spaces with b-property and b-weakly compact operators*, Vladikavkaz Mat. Zh. **11** (2), 19–26, 2009.

[5] Alpay, S. and Altin, B. *A note on b-weakly compact operators*, Positivity **11**,575–582, 2007.

[6] Alpay, S. and Altin, B. *On Riesz spaces with b-property and strongly order bounded operators*, Rend. Circ. Mat. Palermo **60** (2), 1–12, 2011.

[7] Alpay, S. and Ercan, Z. *Characterization of Riesz spaces with b-property*, Positivity **13**, 21–30, 2009.

[8] Altin, B. *On b-weakly compact operators*, Taiwanese J. Math. **11**, 143–150, 2007.

[9] Altin, B. *On Banach lattices with Levi norms*, Proc. Amer. Math. Soc. **135**, 1059–1063, 2007.

[10] Aqzzouz, B., Elbour, A., Moussa, M. and Hmichane, J. *Some characterization of b-weakly compact operators*, Math. Rep. (Bucur) **12** (62), 315–324, 2010.

[11] Aqzzouz, B. and Elbour, A. *On the weak compactness of b-weakly compact operators*, Positivity **14**, 75–81, 2010.

[12] Aqzzouz, B., Elbour, A. and Hmichane, J. *The duality problem of b-weakly comopact operators*, Positivity **13**, 683–692, 2009.

[13] Aqzzouz, B., Nouira, R. and Zraoula, I. *Semi-compactness of positive Dunford-Pettis operators on Banach lattices*, Proc. Amer. Math. Soc. **136**, 1957–2006, 2006.

[14] Aqzzouz, B., Elbour, A. and Aboutafail, O. *Compactness of b-weakly compact operators*, Preprint.

[15] Aqzzouz, B. and Elbour, A. *On the weak compactness of order weakly compact operators*, Preprint.

[16] Ghoussoub, N. and Johnson, W. B. *Counterexamples to several problems on the factorization of bounded linear operators*, Proc. Amer. Math. Soc. **92**, 233–238, 1984.

[17] Ghoussoub, N. and Johnson, W. B. *Operators which factor through Banach lattices not containing* c_0 (Lecture Notes in Math. **1470**, Springer, Berlin, 1991).

[18] Meyer-Nieberg, P. *Banach Lattices* (Springer, Berlin, Heidelberg, New York, 1991).

[19] Niculescu, C. *Order* σ*-continuous operators on Banach lattices* (Lecture Notes in Math. **991**, Springer, Berlin, Heidelberg, New York, 1983), 188–201.

[20] Zaanen, A. C. *Riesz spaces II* (North-Holland, Amsterdam, New York, 1983).