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APPROXIMATION BY INTERPOLATING POLYNOMIALS IN WEIGHTED SYMMETRIC SMIRNOV SPACES

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Abstract

Let $\Gamma \subset \mathbb{C}$ be a closed BR curve without cusps. In this work approximation by complex interpolating polynomials in a Weighted Symmetric Smirnov Space is studied. It is proved that the convergence rate of complex interpolating polynomials and the convergence rate of best approximating algebraic polynomials are the same in the norm of Symmetric Smirnov Spaces.

Keywords: Curve of bounded rotation, Faber polynomials, Interpolating polynomial, Symmetric Smirnov space, Cauchy singular operator.

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1. Preliminaries and the main result

Let $\Gamma \subset \mathbb{C}$ be a closed rectifiable Jordan curve with Lebesgue length measure $|d\tau|$ and let $X(\Gamma)$ be a symmetric (rearrangement invariant) space over Γ generated by a rearrangement invariant function norm ρ, with associate space $X'(\Gamma)$. For each $f \in X(\Gamma)$ we define

$$\|f\|_{X(\Gamma)} := \rho(|f|), \; f \in X(\Gamma).$$

A symmetric space $X(\Gamma)$ equipped with norm $\|\cdot\|_{X(\Gamma)}$ is a Banach space [2, p. 3,5, Ths. 1.4 and 1.6].

For definitions and fundamental properties of general symmetric spaces we refer to [2].

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A function $\omega : \Gamma \to [0, \infty]$ is referred to as a *weight* if ω is measurable and the preimage $\omega^{-1}(\{0, \infty\})$ has measure zero. We set

$$X(\Gamma, \omega) := \{f \text{ measurable} : \ f\omega \in X(\Gamma)\},$$

which is equipped with the norm

$$\|f\|_{X(\Gamma,\omega)} := \|f\omega\|_{X(\Gamma)}.$$

The normed space $X(\Gamma, \omega)$ is called a *Weighted Symmetric Space on* Γ.

By $L^p(\Gamma)$, $1 \le p \le \infty$, we denote the Lebesgue space of measurable functions $f : \Gamma \to \mathbb{C}$. Let $E^p(G)$, $1 \le p \le \infty$ be the Smirnov space of functions analytic on G. It is well known that every function in $E^p(G)$, $1 \le p \le \infty$, has nontangential boundary values on $\Gamma := \partial G$. The boundary value of function f in $E^p(G)$, $1 \le p \le \infty$, will be denoted by f_*.

If $\omega \in X(\Gamma)$ and $1/\omega \in X'(\Gamma)$, then $X(\Gamma, \omega)$ is a Banach Function [2] Space and from Hölder's [2, p.9] inequality we have $X(\Gamma, \omega) \subset L^1(\Gamma)$.

By the Luxemburg representation theorem [2, Theorem 4.10, p. 62], there is a unique rearrangement invariant function norm $\bar{\rho}$ over the Lebesgue measure space $([0, |\Gamma|], m)$, where $|\Gamma|$ is the Lebesgue length of Γ, such that $\rho(f) = \bar{\rho}(f^*)$ for all non-negative and almost everywhere finite measurable functions f defined on Γ. Here f^* denotes the non-increasing rearrangement of f [2, p. 39]. The symmetric space over $([0, |\Gamma|], m)$ generated by $\bar{\rho}$ is called the *Luxemburg representation of* $X(\Gamma)$ and is denoted by $\bar{X}$.

Let g be a non-negative, almost everywhere finite and measurable function on $[0, |\Gamma|]$. For each $x > 0$ we set

$$(H_x g)(t) := \begin{cases} g(xt), & xt \in [0, |\Gamma|] \\ 0, & xt \notin [0, |\Gamma|] \end{cases}, \ t \in [0, |\Gamma|].$$

Then the operator $H_{1/x}$ is bounded on $\bar{X}$ [2, p. 165] with the operator norm

$$(h_X)(x) := \left\|H_{1/x}\right\|_{\mathcal{B}(\bar{X})},$$

where $\mathcal{B}\left(\bar{X}\right)$ is the Banach algebra of bounded linear operators on $\bar{X}$.

The functions

$$\alpha_X := \lim_{x \to 0} \frac{\log h_X(x)}{\log x}, \ \beta_X := \lim_{x \to \infty} \frac{\log h_X(x)}{\log x}$$

are called the *lower and upper Boyd indices* [3] *of the symmetric space* $X(\Gamma)$. These indices satisfy $0 \le \alpha_X \le \beta_X \le 1$. The indices α_X and β_X are called *nontrivial* if $0 < \alpha_X$ and $\beta_X < 1$.

Let Γ be a closed rectifiable Jordan curve in the complex plane $\mathbb{C}$. The curve Γ separates the plane into two domains $G := \text{int}\Gamma$ and $G^- := \text{ext}\Gamma$. We set $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T} := \partial\mathbb{D}$ and $\mathbb{D}^- := \text{ext}\mathbb{T}$. Let $w = \phi(z)$ be the conformal map of G^- onto $\mathbb{D}^-$ normalized by the conditions

$$\phi(\infty) = \infty, \ \lim_{z \to \infty} \frac{\phi(z)}{z} > 0.$$

When $|z|$ is sufficiently large, ϕ has the Laurent expansion

$$\phi(z) = dz + d_0 + \frac{d_1}{z} + \cdots$$

and hence we have

$$[\phi(z)]^n = d^n z^n + \sum_{k=0}^{n-1} d_{n,k} z^k + \sum_{k<0} d_{n,k} z^k.$$

The polynomial

$$F_n(z) := d^n z^n + \sum_{k=0}^{n-1} d_{n,k} z^k$$

is called *nth Faber polynomial* with respect to $\overline{G}$.

Note that for every natural number n, F_n is a polynomial of degree n. For further information about the Faber polynomials, the monographs [4, Ch. I, §6], [11, Ch. II] and [12] can be consulted.

Let γ be an oriented rectifiable curve. For $z \in \gamma$, $\delta > 0$ we denote by $s_+(z,\delta)$ (respectively, $s_-(z,\delta)$) the subarc of γ in the positive (respectively negative) orientation of γ with starting point z, and arclength from z to each point less than δ.

If γ is a smooth curve and

$$\lim_{\delta\to 0}\left\{ \int\limits_{s_-(z,\delta)} |d_\varsigma \arg(\varsigma - z)| + \int\limits_{s_+(z,\delta)} |d_\varsigma \arg(\varsigma - z)| \right\} = 0$$

holds uniformly with respect to $z \in \gamma$, then it is said [13] that γ is of vanishing rotation (VR). As follows from this definition, the VR condition is stronger than smoothness. In [13] L. Zhong and L. Zhu proved that there exists a smooth curve which is not of VR. On the other hand, if the angle of inclination $\theta(s)$ of the tangent to γ as a function of the arclength s along γ satisfies the condition

$$\int_0^\delta \frac{\omega(t)}{t}\, dt < \infty,$$

where $\omega(t)$ is the modulus of continuity of $\theta(s)$, then [13] γ is VR.

1.1. Definition. Let γ be a rectifiable Jordan curve with length L and let $z = z(t)$ be its parametric representation with arclength $t \in [0, L]$. If $\beta(t) := \arg z'(t)$ can be defined on $[0, L]$ to become a function of bounded variation, then γ is called of *bounded rotation* ($\gamma \in$BR) and $\int_\Gamma |d\beta(t)|$ is called the *total rotation* of γ.

For example, a curve which is made up of finitely many convex arcs (corners are permitted), is of bounded rotation [4, p. 45]. If $\gamma \in$BR, then there are two half tangents at each point of γ. It is easily seen that every VR curve is a BR curve. Since a BR curve may have cusps or corners, there exists a BR curve which is not a VR curve (for example, a rectangle in the plane).

For $z \in \Gamma$ and $\epsilon > 0$ let $\Gamma(z,\epsilon)$ denote the portion of Γ which is inside the open disk of radius ϵ centered at z, i.e. $\Gamma(z,\epsilon) := \{t \in \Gamma : |t - z| < \epsilon\}$. Further, let $|\Gamma(z,\epsilon)|$ denote the length of $\Gamma(z,\epsilon)$. A rectifiable Jordan curve Γ is called a Carleson curve (Ahlfors-Regular curve, see e.g. [9, p.162, (12)]) if it satisfies

$$\sup_{\epsilon>0}\sup_{z\in\Gamma} \frac{1}{\epsilon} |\Gamma(z,\epsilon)| < \infty.$$

We consider the Cauchy-type integral

$$(\mathcal{H}f)(z) := \frac{1}{2\pi i} \int\limits_\Gamma \frac{f(\varsigma)}{\varsigma - z}\, d\varsigma,\ z \in G$$

and Cauchy's singular integral of $f \in L^1(\Gamma)$ defined as

$$S_\Gamma f(z) := \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int\limits_{\Gamma\setminus\Gamma(z,\epsilon)} \frac{f(\varsigma)}{\varsigma - z} d\varsigma,\ z \in \Gamma.$$

The linear operator $S_\Gamma : f \to S_\Gamma f$ is called the *Cauchy singular operator*.

For fixed $p \in (1,\infty)$ we define $q \in (1,\infty)$ by $(1/p) + (1/q) = 1$. The set of all weights $\omega : \Gamma \to [0,\infty]$ satisfying Muckenhoupt's A_p condition (see e.g. [5, p.254 (6.1)])

$$\sup_{z\in\Gamma}\sup_{\varepsilon>0}\left(\frac{1}{\varepsilon}\int\limits_{\Gamma(z,\varepsilon)}\omega(\tau)^p\,|d\tau|\right)^{1/p}\left(\frac{1}{\varepsilon}\int\limits_{\Gamma(z,\varepsilon)}[\omega(\tau)]^{-q}\,|d\tau|\right)^{1/q}<\infty$$

is denoted by $A_p(\Gamma)$.

1.2. Definition. Let ω be a weight on Γ and let

$$E_X(G,\omega) := \left\{ f \in E^1(G) : f_* \in X(\Gamma,\omega) \right\}.$$

The class of functions $E_X(G,\omega)$ will be called the *Weighted Symmetric Smirnov space* with respect to the domain G.

$E_X(G,\omega)$ is a natural generalization of Hardy, Hardy-Orlicz, Smirnov and Smirnov-Orlicz spaces. It is a Banach space with the norm $\|f\|_{E_X(G,\omega)} := \|f_*\|_{X(\Gamma,\omega)}$.

In this work we investigate the convergence property of interpolating polynomials based on the zeros of the Faber polynomials in Symmetric Smirnov Spaces under the assumption that Γ is a BR curve without cusps. Approximation by interpolating polynomials has been studied by several authors. In their work [10] under the assumption $\Gamma \in C(2,\alpha)$, $0 < \alpha < 1$, X. C. Shen and L. Zhong obtained a series of interpolation nodes in G and showed that interpolating polynomials and the best approximating polynomial have the same order of convergence in $E^p(G)$, $1 < p < \infty$. In [14], considering $\Gamma \in C(1,\alpha)$ and choosing the interpolation nodes as the zeros of the Faber polynomials, L. Y. Zhu obtained a similar result.

When Γ is a piecewise VR curve without cusps, L. Zhong and L. Zhu [13] showed that the interpolating polynomials based on the zeros of the Faber polynomials converge in the Smirnov class $E^p(G)$, $1 < p < \infty$.

In the case that all of the zeros of the nth Faber polynomial $F_n(z)$ are in G, we denote by $L_n(f,z)$ the $(n-1)$th interpolating polynomial to $f(z) \in E_X(G,\omega)$ based on the zeros of the Faber polynomials F_n.

For $f \in E_X(G,\omega)$, we denote by

$$E_n(f,G,\omega)_X := \inf\left\{ \|f - p_n\|_{E_X(G,\omega)} : p_n \text{ is a polynomial of degree} \le n \right\}$$

the minimal error of approximation of f by polynomials of degree at most n.

The main result of this work is the following.

1.3. Theorem. *Let Γ be a* BR *curve without cusps, ω a weight on Γ and let $X(\Gamma,\omega)$ be a weighted symmetric space on Γ having nontrivial Boyd indices α_X and β_X. If $\omega \in A_{1/\alpha_X} \cap A_{1/\beta_X}$, then for a sufficiently large natural number n, the roots of the Faber polynomials are in G and for every $f \in E_X(G,\omega)$*

$$\|f(\cdot) - L_n(f,\cdot)\|_{E_X(G,\omega)} \le cE_{n-1}(f,G,\omega)_X,$$

with a positive constant c depending only on Γ and X.

When Γ is a piecewise VR curve without cups and $\omega \equiv 1$, this theorem was proved in [13]. For a BR curve without cups and $\omega \equiv 1$, this theorem was proved in [7].

We use $c, c_1, c_2, \ldots$ to denote constants (which may, in general, differ in different contexts) depending only on numbers that are not important for the question of our interest.

2. Auxiliary results

Let Γ be a BR curve without cusps. Then (see, for example, Pommerenke [8])

$$F_n(z) = \frac{1}{\pi} \int_\Gamma [\phi(\varsigma)]^n \, d_\varsigma \arg(\varsigma - z), \; z \in \Gamma,$$

where the jump of $\arg(\varsigma - z)$ at $\varsigma = z$ is equal to the exterior angle $\alpha_z \pi$. Hence we have

$$0 \le \max_{z \in \Gamma} |\alpha_z - 1| < 1. \tag{2.1}$$

2.1. Lemma. [1] *Let Γ be a* BR *curve without cusps. Then for arbitrary $\epsilon > 0$, there exists a positive integer n_0 such that*

$$|F_n(z) - [\phi(z)]^n| < |\alpha_z - 1| + \epsilon, \; z \in \Gamma$$

holds for $n > n_0$. □

2.2. Lemma. [6] *Let Γ be a rectifiable Jordan curve, ω a weight on Γ and let $X(\Gamma,\omega)$ be a weighted symmetric space on Γ having nontrivial Boyd indices α_X and β_X. If $\omega \in A_{1/\alpha_X} \cap A_{1/\beta_X}$, then the singular operator S_Γ is bounded on $X(\Gamma,\omega)$, i.e.*

$$\|S_\Gamma f\|_{X(\Gamma,\omega)} \le c \|f\|_{X(\Gamma,\omega)}, \; f \in X(\Gamma,\omega)$$

for some constant $c > 0$. □

2.3. Lemma. *Let Γ be a* BR *curve without cusps. Then for a sufficiently large natural number n, the roots of the Faber polynomials F_n are in G.*

Proof. Let $\kappa := \max_{z \in \Gamma} |\alpha_z - 1|$, $z \in \Gamma$. Then by (2.1) we have $0 \le \kappa < 1$. Setting $\epsilon := \frac{1-\kappa}{2}$ in Lemma 2.1, for sufficiently large n we get

$$|F_n(z) - [\phi(z)]^n| < \frac{1+\kappa}{2}, \; z \in \Gamma. \tag{2.2}$$

Since $F_n(z) - [\phi(z)]^n$ is analytic on $C\overline{G} := \overline{\mathbb{C}} \backslash \overline{G}$, by the maximum principle we have

$$|F_n(z) - [\phi(z)]^n| < \frac{1+\kappa}{2}, \; z \in CG,$$

and therefore

$$|F_n(z)| \ge |\phi(z)|^n - \frac{1+\kappa}{2} \ge \frac{1-\kappa}{2} > 0, \; z \in \mathrm{CG}.$$

This gives to us that for sufficiently large n, all the zeros of the Faber polynomials F_n are in G. □

2.4. Lemma. *Let Γ be a* BR *curve without cusps, ω a weight on Γ and let $X(\Gamma,\omega)$ be a weighted symmetric space on Γ having nontrivial Boyd indices α_X and β_X. If $\omega \in A_{1/\alpha_X} \cap A_{1/\beta_X}$, then for a sufficiently large natural number n, $L_n(f,\cdot)$ is uniformly bounded in $E_X(G,\omega)$.*

Proof. Choosing the interpolation nodes as the zeros of the Faber polynomials we have for $z' \in G$,

$$f(z') - L_n(f,z') = \frac{F_n(z')}{2\pi i} \int_\Gamma \frac{f(\varsigma)}{F_n(\varsigma)(\varsigma - z')} d\varsigma = F_n(z') \left(\mathcal{H}\left[\frac{f}{F_n}\right] \right)(z').$$

Taking the limit $z' \to z \in \Gamma$ along all nontangential paths inside of Γ we get

$$\begin{aligned}\|f(z) - L_n(f,z)\|_{E_X(G,\omega)} &= \left\| F_n(z) \cdot \left(S_\Gamma \left[\frac{f}{F_n} \right] \right)(z) \right\|_{X(\Gamma,\omega)} \\ &\le \left\{ \max_{z\in\Gamma} |F_n(z)| \right\} \cdot \left\| S_\Gamma \left[\frac{f}{F_n} \right] \right\|_{X(\Gamma,\omega)},\end{aligned}$$

and later, by Lemma 2.2,

$$\begin{aligned}\|f(z) - L_n(f,z)\|_{E_X(G,\omega)} &\le c \left\{ \max_{z\in\Gamma} |F_n(z)| \right\} \cdot \left\| \frac{f}{F_n} \right\|_{X(\Gamma,\omega)} \\ &\le c \left\{ \max_{z,\varsigma\in\Gamma} \left| \frac{F_n(z)}{F_n(\varsigma)} \right| \right\} \|f\|_{X(\Gamma,\omega)}.\end{aligned}$$

From (2.2)

$$\frac{1-\kappa}{2} < |F_n(z)| < \frac{3+\kappa}{2}, \ z \in \Gamma,$$

and hence

$$\|f(z) - L_n(f,z)\|_{E_X(G,\omega)} \le c\frac{3+\kappa}{1-\kappa} \cdot \|f(z)\|_{X(\Gamma,\omega)}, \ z \in \Gamma.$$

Since

$$\begin{aligned}\|L_n(f,\cdot)\|_{E_X(G,\omega)} &\le \|f\|_{E_X(G,\omega)} + \|f(\cdot) - L_n(f,\cdot)\|_{E_X(G,\omega)} \\ &\le \left(1 + c\frac{3+\kappa}{1-\kappa}\right) \|f\|_{X(\Gamma,\omega)},\end{aligned}$$

by choosing $c_2 := 1 + c\frac{3+\kappa}{1-\kappa}$ we obtain that $\|L_n\| \le c_2$ and the assertion holds. □

3. Proof of the theorem

The first part of the main theorem was proved in Lemma 2.3. Let P_{n-1} be the $(n-1)$th best approximating polynomial to f in $E_X(G,\omega)$. Since $L_n(f,\cdot)$ is a linear operator we get

$$\begin{aligned}\|f(\cdot) - L_n(f,\cdot)\|_{E_X(G,\omega)} &= \|f(\cdot) - P_{n-1}(\cdot) - L_n(f - P_{n-1},\cdot)\|_{E_X(G,\omega)} \\ &\le (1 + \|L_n\|) \|f(\cdot) - P_{n-1}(\cdot)\|_{E_X(G,\omega)}.\end{aligned}$$

Hence we conclude by Lemma 2.4 that

$$\begin{aligned}\|f(\cdot) - L_n(f,\cdot)\|_{E_X(G,\omega)} &\le (1 + c_2) \|f(\cdot) - P_{n-1}(\cdot)\|_{E_M(G)} \\ &= cE_{n-1}(f,G,\omega)_X,\end{aligned}$$

and the proof of the main theorem is completed. □

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