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CONVERGENCE THEOREMS FOR A FINITE FAMILY OF NONEXPANSIVE AND TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract

In this paper, we define and study strong convergence of finite step iteration sequences with errors to a common fixed point for a pair consisting of a finite family of nonexpansive mappings and a finite family of total asymptotically nonexpansive mappings in a nonempty closed convex subset of uniformly convex Banach spaces. The results of this paper can be viewed as an improvement and extension of the corresponding results of F. Gu and Z. He (*Multi-step iterative process with errors for common fixed points of a finite family of nonexpansive mappings*, Math. Commun. **11** (1), 47–54, 2006), Z. Liu, R. P. Agarwal, C. Feng, and S. M. Kang (*Weak and strong convergence theorems of common fixed points for a pair of nonexpansive and asymptotically nonexpansive mappings*, Acta Univ. Palack. Olomuc. Fac. Rer Nat. Math. **44**, 83–96, 2005), Z. Liu, C. Feng, J. S. Ume and S. M. Kang (*Weak and strong convergence for common fixed points for a pair of nonexpansive and asymptotically nonexpansive mappings*, Taiwanese J. Math. **11**, 27–42, 2007), S. Saejung and K. Sitthikul (*Weak and strong convergence theorems for a finite family of nonexpansive and asymptotically nonexpansive mappings in Banach spaces*, Thai J. Math. Special Issue, 15–26, 2008) and others.

Keywords: Total asymptotically nonexpansive mappings, Common fixed point, Uniformly convex Banach space, Finite-step iteration scheme with errors with respect to a pair of mappings.

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1. Introduction

Let E be a real normed space and K be a nonempty subset of E. A mapping $T: K \to K$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. A mapping $T: K \to K$ is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.1}$$

for all $x, y \in K$ and $n \geq 1$. Goebel and Kirk [3] proved that if K is a nonempty closed and bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self-mapping has a fixed point.

A mapping T is said to be *asymptotically nonexpansive in the intermediate sense* (see, e.g., [2]) if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in K} (\|T^n x - T^n y\| - \|x - y\|) \leq 0. \tag{1.2}$$

If $F(T) := \{x \in K : Tx = x\} \neq \varnothing$ and (1.2) holds for all $x \in K$, $y \in F(T)$, then T is called *asymptotically quasi-nonexpansive in the intermediate sense*. Observe that if we define

$$a_n := \sup_{x,y\in K} (\|T^n x - T^n y\| - \|x - y\|), \text{ and } \sigma_n = \max\{0, a_n\}, \tag{1.3}$$

then $\sigma_n \to 0$ as $n \to \infty$ and (1.2) reduces to

$$\|T^n x - T^n y\| \leq \|x - y\| + \sigma_n, \text{ for all } x,\ y \in K,\ n \geq 1. \tag{1.4}$$

The class of mappings which are asymptotically nonexpansive in the intermediate sense was introduced by Bruck *et al.* [2]. It is known [5] that if K is a nonempty closed convex bounded subset of a uniformly convex Banach space E and T is a self-mapping of K which is asymptotically nonexpansive in the intermediate sense, then T has a fixed point. It is worth mentioning that the class of mappings which are asymptotically nonexpansive in the intermediate sense contains properly the class of asymptotically nonexpansive mappings.

Albert *et al.* [1] introduced a more general class of asymptotically nonexpansive mappings called total asymptotically nonexpansive mappings and studied methods of approximation of fixed points of mappings belonging to this class.

1.1. Definition. A mapping $T: K \to K$ is said to be *total asymptotically nonexpansive* if there exist nonnegative real sequences $\{\mu_n\}$ and $\{l_n\}$, $n \geq 1$ with $\mu_n,\ l_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x, y \in K$,

$$\|T^n x - T^n y\| \leq \|x - y\| + \mu_n \phi(\|x - y\|) + l_n,\ n \geq 1. \tag{1.5}$$

1.2. Remark. If $\phi(\lambda) = \lambda$, then (1.5) reduces to

$$\|T^n x - T^n y\| \leq (1 + \mu_n) \|x - y\| + l_n, n \geq 1. \tag{1.6}$$

In addition, if $l_n = 0$ for all $n \geq 1$, then total asymptotically nonexpansive mappings coincide with asymptotically nonexpansive mappings. If $\mu_n = 0$ and $l_n = 0$ for all $n \geq 1$, we obtain from (1.5) a class of mappings that includes the class of nonexpansive mappings. If $\mu_n = 0$ and $l_n = \sigma_n = \max\{0, a_n\}$, where $a_n := \sup_{x,y\in K} (\|T^n x - T^n y\| - \|x - y\|)$ for all $n \geq 1$, then (1.5) reduces to (1.4) which has been studied as mappings asymptotically nonexpansive in the intermediate sense.

1.3. Proposition. *Let K be a nonempty subset of E, $\{T_i\}_{i=1}^N : K \to K$ be N total asymptotically nonexpansive mappings. Then there exist nonnegative real sequences $\{\mu_n\}$ and $\{l_n\}$, $n \geq 1$ with $\mu_n,\ l_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x,\ y \in K$,*

$$\|T_i^n x - T_i^n y\| \leq \|x - y\| + \mu_n \phi\left(\|x - y\|\right) + l_n,\ n \geq 1, \tag{1.7}$$

for $i = 1, 2, \ldots, N$.

Proof. Since $T_i : K \to K$ is a total asymptotically nonexpansive mapping for $i = 1, 2, \ldots, N$, there exist nonnegative real sequences $\{\mu_{in}\}$, $\{l_{in}\}$, $n \geq 1$ with $\mu_{in},\ l_{in} \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi_i : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi_i\left(0\right) = 0$ such that for all $x,\ y \in K$,

$$\|T_i^n x - T_i^n y\| \leq \|x - y\| + \mu_{in} \phi_i\left(\|x - y\|\right) + l_{in},\ n \geq 1.$$

Setting

$$\begin{aligned} \mu_n &= \max\left\{\mu_{1n}, \mu_{2n}, \ldots, \mu_{Nn}\right\},\ l_n = \max\left\{l_{1n}, l_{2n}, \ldots, l_{Nn}\right\}, \\ \phi\left(a\right) &= \max\left\{\phi_1\left(a\right), \phi_2\left(a\right), \ldots, \phi_N\left(a\right)\right\} \text{ for } a \geq 0, \end{aligned}$$

then we get that there exist nonnegative real sequences $\{\mu_n\}$ and $\{l_n\}$, $n \geq 1$ with $\mu_n,\ l_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi\left(0\right) = 0$ such that

$$\begin{aligned} \|T_i^n x - T_i^n y\| &\leq \|x - y\| + \mu_{in} \phi_i\left(\|x - y\|\right) + l_{in} \\ &\leq \|x - y\| + \mu_n \phi\left(\|x - y\|\right) + l_n,\ n \geq 1, \end{aligned}$$

for all $x, y \in K$, and each $i = 1, 2, \ldots, N$. □

The main tool for approximation of fixed points of generalizations of nonexpansive mappings remains the iterative technique. Since Schu's results (see, [12, 13]), the modified Mann and Ishikawa iterative scheme have been studied extensively by several authors to approximate fixed points of generalizations of asymptotically nonexpansive mappings (see, e.g., [12, 13, 8, 15, 16]). Recently, Gu and He [4] studied a multi-step iterative sequence involving finite nonexpansive mappings in a uniformly convex Banach space. They obtained weak and strong convergence theorems for approximating common fixed points of nonexpansive mappings. Liu *et al.* in [6, 7] established new iterative methods, the modified two and the modified three-step iteration sequence with errors with respect to a pair of mappings. The results in [6] and [7] generalize, improve and unify many known results due to many authors.

Very recently, Saejung and Sitthikul [9] studied convergence theorems for a finite family of nonexpansive and asymptotically nonexpansive mappings in a uniformly convex Banach space. In 2010 Saluja [11] proved strong convergence to common fixed points of a pair of quasi-nonexpansive and asymptotically quasi-nonexpansive mappings.

Inspired and motivated by these facts, we define and study convergence theorems of finite step iterative sequences with errors involving a finite family of nonexpansive and a finite family of total asymptotically nonexpansive mappings in a nonempty closed convex subset of a uniformly convex Banach space. The results of this paper can be viewed as an improvement and extention of the corresponding results of [4, 6, 7, 10] and others. The scheme (1.8) is defined as follows.

Let K be a nonempty closed convex subset of a Banach space E. Let $S_1, S_2, \ldots, S_N : K \to K$ be N nonexpansive mappings, $T_1, T_2, \ldots, T_N : K \to K$ be N total asymptotically nonexpansive mappings. Then the sequence $\{x_n\}$ defined by

$$
\begin{aligned}
& x_1 \in K, \\
& x_{n+1} = x_n^{(N)} = a_n^{(N)} T_N^n x_n^{(N-1)} + b_n^{(N)} S_N x_n + c_n^{(N)} u_n^{(N)}, \\
& x_n^{(N-1)} = a_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + b_n^{(N-1)} S_{N-1} x_n + c_n^{(N-1)} u_n^{(N-1)}, \\
& \cdots\cdots\cdots\cdots\cdots\cdots \\
& x_n^{(3)} = a_n^{(3)} T_3^n x_n^{(2)} + b_n^{(3)} S_3 x_n + c_n^{(3)} u_n^{(3)}, \\
& x_n^{(2)} = a_n^{(2)} T_2^n x_n^{(1)} + b_n^{(2)} S_2 x_n + c_n^{(2)} u_n^{(2)} \\
& x_n^{(1)} = a_n^{(1)} T_1^n x_n + b_n^{(1)} S_1 x_n + c_n^{(1)} u_n^{(1)}, \ n \geq 1,
\end{aligned} \tag{1.8}
$$

is called an *N-step iterative sequence*, where $\left\{u_n^{(i)}\right\}$ are bounded sequences in K and $\left\{a_n^{(i)}\right\}_{n=1}^{\infty}$, $\left\{b_n^{(i)}\right\}_{n=1}^{\infty}$, $\left\{c_n^{(i)}\right\}_{n=1}^{\infty} \subset [0,1]$ are such that $a_n^{(i)} + b_n^{(i)} + c_n^{(i)} = 1$ for all $i = 1, 2, \ldots, N$.

In case $S_1 = S_2 = \ldots = S_N = I$, then (1.8) reduces to the multi-step iteration with errors for N total asymptotically nonexpansive mappings.

For $N = 3$, then (1.8) reduces to the modified three-step iteration:

$$
\begin{aligned}
& x_1 \in K, \\
& x_{n+1} = a_n^{(3)} T_3^n y_n + b_n^{(3)} S_3 x_n + c_n^{(3)} u_n^{(3)}, \\
& y_n = a_n^{(2)} T_2^n z_n + b_n^{(2)} S_2 x_n + c_n^{(2)} u_n^{(2)}, \\
& z_n = a_n^{(1)} T_1^n x_n + b_n^{(1)} S_1 x_n + c_n^{(1)} u_n^{(1)}, \ n \geq 1,
\end{aligned} \tag{1.9}
$$

where $\left\{u_n^{(i)}\right\}$ are bounded sequences in K and $\left\{a_n^{(i)}\right\}_{n=1}^{\infty}$, $\left\{b_n^{(i)}\right\}_{n=1}^{\infty}$, $\left\{c_n^{(i)}\right\}_{n=1}^{\infty} \subset [0,1]$ are such that $a_n^{(i)} + b_n^{(i)} + c_n^{(i)} = 1$ for all $i = 1, 2, 3$.

If $T_1 = T_2 = T_3 = T$ and $S_1 = S_2 = S_3 = S$ are self-mappings, then (1.9) reduces to the modified three-step iteration defined by Liu *et al.* [6]

$$
\begin{aligned}
& x_1 \in K, \\
& x_{n+1} = a_n^{(3)} T y_n + b_n^{(3)} S x_n + c_n^{(3)} u_n^{(3)}, \\
& y_n = a_n^{(2)} T z_n + b_n^{(2)} S x_n + c_n^{(2)} u_n^{(2)}, \\
& z_n = a_n^{(1)} T x_n + b_n^{(1)} S x_n + c_n^{(1)} u_n^{(1)}, \ n \geq 1,
\end{aligned} \tag{1.10}
$$

where $\left\{u_n^{(i)}\right\}$ are bounded sequences in K and $\left\{a_n^{(i)}\right\}_{n=1}^{\infty}$, $\left\{b_n^{(i)}\right\}_{n=1}^{\infty}$, $\left\{c_n^{(i)}\right\}_{n=1}^{\infty} \subset [0,1]$ are such that $a_n^{(i)} + b_n^{(i)} + c_n^{(i)} = 1$ for all $i = 1, 2, 3$.

In case $S = I$ and $a_n^{(1)} = c_n^{(1)} = 0$ for $n \geq 1$, the sequence $\{x_n\}_{n \geq 1}$ generated in (1.10) reduces to the usual modified Ishikawa iteration sequence with errors.

If $T_2 = T_3 = T$ and $S_2 = S_3 = S$ are self-mappings and $a_n^{(1)} = c_n^{(1)} = 0$ for $n \geq 1$, then (1.9) reduces to the modified Ishikawa iteration sequence with errors defined by Liu *et al.* [7]

$$
\begin{aligned}
& x_1 \in K, \\
& x_{n+1} = a_n^{(3)} T y_n + b_n^{(3)} S x_n + c_n^{(3)} u_n^{(3)}, \\
& y_n = a_n^{(2)} T x_n + b_n^{(2)} S x_n + c_n^{(2)} u_n^{(2)}, \ n \geq 1,
\end{aligned} \tag{1.11}
$$

where $\left\{u_n^{(2)}\right\}$ and $\left\{u_n^{(3)}\right\}$ are bounded sequences in K, $\left\{a_n^{(2)}\right\}_{n\geq 1}$, $\left\{b_n^{(2)}\right\}_{n\geq 1}$, $\left\{c_n^{(2)}\right\}_{n\geq 1}$, $\left\{a_n^{(3)}\right\}_{n\geq 1}$, $\left\{b_n^{(3)}\right\}_{n\geq 1}$ and $\left\{c_n^{(3)}\right\}_{n\geq 1}$ are sequences in $[0,1]$ such that $a_n^{(2)}+b_n^{(2)}+c_n^{(2)} = a_n^{(3)}+b_n^{(3)}+c_n^{(3)} = 1$.

In case $S = I$ and $a_n^{(2)} = c_n^{(2)} = 0$ for $n \geq 1$, the sequence $\{x_n\}_{n\geq 1}$ generated in (1.11) reduces to the usual modified Mann iteration sequence with errors.

The purpose of this paper is to study the strong convergence theorems of the finite-step iteration sequence $\{x_n\}$ with error terms defined by (1.8) to a common fixed point for a pair of a finite families of nonexpansive mappings and a finite family of total asymptotically nonexpansive mappings in a uniformly convex Banach space.

2. Preliminaries

Now, we recall some well-known concepts and results.

Let E be a Banach space with dimension $E \geq 2$. The modulus of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : \ \|x\| = \|y\| = 1,\ \varepsilon = \|x-y\|\right\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

A mapping $T : K \to K$ is called:

(1) *demicompact* if any bounded sequence $\{x_n\}$ in K such that $\{x_n - Tx_n\}$ converges has a convergent subsequence;

(2) *semicompact* (or *hemicompact*) if any bounded sequence $\{x_n\}$ in K such that $\{x_n - Tx_n\} \to 0$ as $n \to \infty$ has a convergent subsequence. Every demicompact mapping is semicompact but the converse is not true in general.

2.1. Lemma. [14] *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)\, a_n + b_n,\ \ n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, then

(i) $\lim\limits_{n\to\infty} a_n$ *exists;*

(ii) *In particular, if $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ converging to 0, then* $\lim\limits_{n\to\infty} a_n = 0$. □

2.2. Lemma. [12] *Let E be a uniformly convex Banach space, $\{t_n\}_{n\geq 1} \subseteq [b,c] \subset (0,1)$, $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ be sequences in E. If $\limsup_{n\to\infty}\|x_n\| \leq a$, $\limsup_{n\to\infty}\|y_n\| \leq a$ and $\lim_{n\to\infty}\|t_n x_n + (1-t_n)\, y_n\| = a$ for some constant $a \geq 0$, then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.* □

3. Main results

3.1. Lemma. *Let K be a nonempty convex subset of a real Banach space E. Let S_1, $S_2, \ldots, S_N : K \to K$ be a finite family of nonexpansive mappings and $T_1, T_2, \ldots, T_N : K \to K$ a finite family of total asymptotically nonexpansive mappings with sequences $\{\mu_n\}$ and $\{l_n\}$ defined by (1.7) such that $\sum\limits_{n=1}^{\infty} \mu_n < \infty$, $\sum\limits_{n=1}^{\infty} l_n < \infty$ and $\mathcal{F}(S,T) = \bigcap_{i=1}^{N} F(S_i) \cap F(T_i) \neq \emptyset$. Assume that there exists M, $M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$*

for all $\lambda \geq M$, $i \in \{1, 2, \ldots, N\}$. *For an arbitrary* $x_1 \in K$, *define the sequences* $\{x_n\}$ *by recursion (1.8). If*

$$\sum_{n=1}^{\infty} c_n^{(i)} < \infty \text{ for all } i = 1, 2, \ldots, N, \tag{3.1}$$

then $\lim_{n\to\infty} \|x_n - p\|$ *exists for any* $p \in \mathcal{F}(S, T)$.

Proof. Let $p \in \mathcal{F}(S,T) = \bigcap_{i=1}^{N} F(S_i) \cap F(T_i)$. Since $\left\{u_n^{(i)}\right\}$ for all $i = 1, 2, \ldots, N$ are bounded sequences in K, we have

$$K = \max\left\{\sup_{n\geq 1}\left\|u_n^{(1)} - p\right\|, \ldots, \sup_{n\geq 1}\left\|u_n^{(N)} - p\right\|\right\}.$$

Since $S_1, S_2, \ldots, S_N$ are nonexpansive mappings and $T_1, T_2, \ldots, T_N$ are total asymptotically nonexpansive mappings, it follows from (1.8) that

$$\begin{aligned}
\left\|x_n^{(1)} - p\right\| &= \left\|a_n^{(1)} T_1^n x_n + b_n^{(1)} S_1 x_n + c_n^{(1)} u_n^{(1)} - p\right\| \\
&\leq a_n^{(1)} \left\|T_1^n x_n - p\right\| + b_n^{(1)} \left\|S_1 x_n - p\right\| + c_n^{(1)} \left\|u_n^{(1)} - p\right\| \\
&\leq a_n^{(1)} \left[\|x_n - p\| + \mu_n \phi(\|x_n - p\|) + l_n\right] \\
&+ b_n^{(1)} \|x_n - p\| + c_n^{(1)} \left\|u_n^{(1)} - p\right\| \\
&\leq (a_n^{(1)} + b_n^{(1)}) \|x_n - p\| + a_n^{(1)} \mu_n \phi(\|x_n - p\|) \\
&\qquad + a_n^{(1)} l_n + c_n^{(1)} K \\
&\leq \left(1 - c_n^{(1)}\right) \|x_n - p\| + a_n^{(1)} \mu_n \phi(\|x_n - p\|) \\
&\qquad + a_n^{(1)} l_n + c_n^{(1)} K \\
&\leq \|x_n - p\| + a_n^{(1)} \mu_n \phi(\|x_n - p\|) + a_n^{(1)} l_n + \varphi_{(1)}^n, \qquad (3.2)
\end{aligned}$$

where $\varphi_{(1)}^n = c_n^{(1)} K$. Since $\sum_{n=1}^{\infty} c_n^{(1)} < \infty$, we can see that $\sum_{n=1}^{\infty} \varphi_{(1)}^n < \infty$. Note that ϕ is an increasing function, it follows that $\phi(\lambda) \leq \phi(M)$ whenever $\lambda \leq M$ and (by hypothesis) $\phi(\lambda) \leq M^* \lambda$ if $\lambda \geq M$. In either case, we have

$$\phi(\lambda) \leq \phi(M) + M^* \lambda \tag{3.3}$$

for some $M, M^* > 0$. Thus, from (3.2) and (3.3), we have

$$\begin{aligned}
\left\|x_n^{(1)} - p\right\| &\leq \|x_n - p\| + a_n^{(1)} \mu_n \left[\phi(M) + M^* \|x_n - p\|\right] + a_n^{(1)} l_n + \varphi_{(1)}^n \\
&\leq (1 + M_1 \mu_n) \|x_n - p\| + R_1 (\mu_n + l_n) + \varphi_{(1)}^n
\end{aligned} \tag{3.4}$$

for some constants M_1, $R_1 > 0$. It follows from (3.3) and (3.4) that

$$\begin{aligned}
\left\|x_n^{(2)} - p\right\| &\leq a_n^{(2)} \left\|T_2^n x_n^{(1)} - p\right\| + b_n^{(2)} \|S_2 x_n - p\| + c_n^{(2)} \left\|u_n^{(2)} - p\right\| \\
&\leq a_n^{(2)} \left[\left\|x_n^{(1)} - p\right\| + \mu_n \phi\left(\left\|x_n^{(1)} - p\right\|\right) + l_n\right] \\
&\qquad + b_n^{(2)} \|x_n - p\| + c_n^{(2)} \left\|u_n^{(2)} - p\right\| \\
&\leq a_n^{(2)} \left[(1 + M_1 \mu_n) \|x_n - p\| + R_1 (\mu_n + l_n) + \varphi_{(1)}^n\right] \\
&\qquad + a_n^{(2)} \mu_n \left[\phi(M) + M^* \left\|x_n^{(1)} - p\right\|\right] + a_n^{(2)} l_n \\
&\qquad\qquad + b_n^{(2)} \|x_n - p\| + c_n^{(2)} \left\|u_n^{(2)} - p\right\|
\end{aligned}$$

$$
\begin{aligned}
&\le \left(a_n^{(2)}+b_n^{(2)}\right)(1+M_1\mu_n)\|x_n-p\|+a_n^{(2)}R_1(\mu_n+l_n)\\
&\qquad +a_n^{(2)}\varphi_{(1)}^n+a_n^{(2)}\mu_n\phi(M)+a_n^{(2)}\mu_nM^*\left\|x_n^{(1)}-p\right\|\\
&\qquad\qquad +a_n^{(2)}l_n+c_n^{(2)}K\\
&\le \left(1-c_n^{(2)}\right)(1+M_1\mu_n)\|x_n-p\|+a_n^{(2)}R_1(\mu_n+l_n)\\
&\qquad +a_n^{(2)}\varphi_{(1)}^n+a_n^{(2)}\mu_n\phi(M)+a_n^{(2)}l_n+c_n^{(2)}K\\
&\qquad\qquad +a_n^{(2)}\mu_nM^*\left[(1+M_1\mu_n)\|x_n-p\|+R_1(\mu_n+l_n)+\varphi_{(1)}^n\right]\\
&\le \|x_n-p\|+\left(M_1+M^*a_n^{(2)}+a_n^{(2)}\mu_nM^*M_1\right)\mu_n\|x_n-p\|\\
&\qquad +a_n^{(2)}R_1(\mu_n+l_n)+a_n^{(2)}\mu_n\phi(M)+a_n^{(2)}l_n+a_n^{(2)}\mu_nM^*R_1(\mu_n+l_n)\\
&\qquad\qquad +a_n^{(2)}\mu_nM^*\varphi_{(1)}^n+a_n^{(2)}\varphi_{(1)}^n+c_n^{(2)}K
\end{aligned}
$$

$$
\le (1+M_2\mu_n)\|x_n-p\|+R_2(\mu_n+l_n)+\varphi_{(2)}^n, \tag{3.5}
$$

where $\varphi_{(2)}^n=a_n^{(2)}\mu_nM^*\varphi_{(1)}^n+a_n^{(2)}\varphi_{(1)}^n+c_n^{(2)}K$ and for some constants $M_2,R_2>0$. Since $\sum_{n=1}^{\infty}\varphi_{(1)}^n<\infty$, $\sum_{n=1}^{\infty}\mu_n<\infty$ and $\sum_{n=1}^{\infty}c_n^{(2)}<\infty$, we can see that $\sum_{n=1}^{\infty}\varphi_{(2)}^n<\infty$. By induction, it follows (1.8), (3.4) and (3.5) that we have

$$
\left\|x_n^{(j)}-p\right\|\le(1+M_j\mu_n)\|x_n-p\|+R_j(\mu_n+l_n)+\varphi_{(j)}^n \tag{3.6}
$$

for $j=1,2,\dots,N-1$. Therefore, it follows from (1.8) and (3.6) that

$$
\begin{aligned}
\|x_{n+1}-p\| &= \left\|x_n^{(N)}-p\right\|\\
&\le a_n^{(N)}\left\|T_N^nx_n^{(N-1)}-p\right\|+b_n^{(N)}\|S_Nx_n-p\|+c_n^{(N)}\left\|u_n^{(N)}-p\right\|\\
&\le a_n^{(N)}\left[\left\|x_n^{(N-1)}-p\right\|+\mu_n\phi\left(\left\|x_n^{(N-1)}-p\right\|\right)+l_n\right]\\
&\qquad +b_n^{(N)}\|x_n-p\|+c_n^{(N)}\left\|u_n^{(N)}-p\right\|\\
&\le a_n^{(N)}\left[\left(1+M_{(N-1)}\mu_n\right)\|x_n-p\|+R_{(N-1)}(\mu_n+l_n)+\varphi_{(N-1)}^n\right]\\
&\qquad +a_n^{(N)}\mu_n\left[\phi(M)+M^*\left\|x_n^{(N-1)}-p\right\|\right]+a_n^{(N)}l_n\\
&\qquad\qquad +b_n^{(N)}\|x_n-p\|+c_n^{(N)}\left\|u_n^{(N)}-p\right\|\\
&\le \left(a_n^{(N)}+b_n^{(N)}\right)\left(1+M_{(N-1)}\mu_n\right)\|x_n-p\|+a_n^{(N)}R_{(N-1)}(\mu_n+l_n)\\
&\qquad +a_n^{(N)}\varphi_{(N-1)}^n+a_n^{(N)}\mu_n\phi(M)+a_n^{(N)}\mu_nM^*\left\|x_n^{(N-1)}-p\right\|\\
&\qquad\qquad +a_n^{(N)}l_n+c_n^{(N)}K\\
&\le \left(1-c_n^{(N)}\right)\left(1+M_{(N-1)}\mu_n\right)\|x_n-p\|+a_n^{(N)}R_{(N-1)}(\mu_n+l_n)\\
&\qquad +a_n^{(N)}\varphi_{(N-1)}^n+a_n^{(N)}\mu_n\phi(M)+a_n^{(N)}l_n+c_n^{(N)}K\\
&\qquad\qquad +a_n^{(N)}\mu_nM^*\left[\left(1+M_{(N-1)}\mu_n\right)\|x_n-p\|\right.\\
&\qquad\qquad\qquad \left.+R_{(N-1)}(\mu_n+l_n)+\varphi_{(N-1)}^n\right]
\end{aligned}
$$

$$\begin{aligned}&\le \|x_n - p\| + \left(M_{(N-1)} + a_n^{(N)} M^* + a_n^{(N)} \mu_n M^* M_{(N-1)}\right) \mu_n \|x_n - p\| \\ &\quad + a_n^{(N)} R_{(N-1)} (\mu_n + l_n) + a_n^{(N)} \mu_n \phi(M) + a_n^{(N)} l_n \\ &\quad + a_n^{(N)} \mu_n M^* R_{(N-1)} (\mu_n + l_n) + a_n^{(N)} \mu_n M^* \varphi^n_{(N-1)} \\ &\quad + a_n^{(N)} \varphi^n_{(N-1)} + c_n^{(N)} K \\ &\le (1 + M_N \mu_n) \|x_n - p\| + R_N (\mu_n + l_n) + \varphi^n_{(N)}, \qquad (3.7)\end{aligned}$$

where $\varphi^n_{(N)} = a_n^{(N)} \mu_n M^* \varphi^n_{(N-1)} + a_n^{(N)} \varphi^n_{(N-1)} + c_n^{(N)} K$ and for some constants M_N, $R_N > 0$. Since $\sum_{n=1}^{\infty} \varphi^n_{(N-1)} < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} c_n^{(N)} < \infty$, we can see that $\sum_{n=1}^{\infty} \varphi^n_{(N)} < \infty$. Also, since $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} l_n < \infty$ and $\sum_{n=1}^{\infty} \varphi^n_{(N)} < \infty$, by Lemma 2.1, we get that $\lim_{n\to\infty} \|x_n - p\|$ exists. This completes the proof. □

3.2. Lemma. *Let K be a nonempty convex subset of a uniformly convex Banach space E. Let $S_1, S_2, \ldots, S_N : K \to K$ be a finite family of nonexpansive mappings and $T_1, T_2, \ldots, T_N : K \to K$ a finite family of total asymptotically nonexpansive mappings with sequences $\{\mu_n\}$ and $\{l_n\}$ defined by (1.7) such that*

$$\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} l_n < \infty \qquad (3.8)$$

and $\mathcal{F}(S,T) = \bigcap_{i=1}^{N} F(S_i) \cap F(T_i) \neq \emptyset$. Assume that there exists M, $M^* > 0$ such that $\phi(\lambda) \le M^* \lambda$ for all $\lambda \ge M$, $i \in \{1, 2, \ldots, N\}$. Suppose that

$$\|x - T_i y\| \le \|S_i x - T_i y\| \qquad (3.9)$$

for all $x, y \in K$ and $i = 1, 2, \ldots, N$. For an arbitrary $x_1 \in K$, define the sequences $\{x_n\}$ by recursion (1.8) and for some $\eta_1, \eta_2 \in (0,1)$ with the following restrictions:

(i)

$$0 < \eta_1 \le a_n^{(i)} \le \eta_2 < 1, \ \forall n \ge n_0 \text{ for some } n_0 \in \mathbb{N}, \qquad (3.10)$$

(ii)

$$\sum_{n=1}^{\infty} c_n^{(i)} < \infty \text{ for all } i = 1, 2, \ldots, N,$$

then

$$\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i^n x_n\| = 0$$

for all $i = 1, 2, \ldots, N$.

Proof. Let $p \in \mathcal{F}(S,T) = \bigcap_{i=1}^{N} F(S_i) \cap F(T_i)$. By Lemma 3.1, we have that $\lim_{n\to\infty} \|x_n - p\|$ exists. Let $\lim_{n\to\infty} \|x_n - p\| = r$ for some $r \ge 0$. We have that

$$\left\|x_n^{(N-1)} - p\right\| \le \left(1 + M_{(N-1)} \mu_n\right) \|x_n - p\| + R_{(N-1)} (\mu_n + l_n) + \varphi^n_{(N-1)}, \qquad (3.11)$$

where $\sum_{n=1}^{\infty} \varphi^n_{(N-1)} < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} l_n < \infty$. It follows that

$$\begin{aligned}\limsup_{n\to\infty} \left\|x_n^{(N-1)} - p\right\| &\le \limsup_{n\to\infty} \left[\left(1 + M_{(N-1)} \mu_n\right) \|x_n - p\| + R_{(N-1)} (\mu_n + l_n) + \varphi^n_{(N-1)}\right] \\ &\le r \end{aligned} \qquad (3.12)$$

and by (3.8) and (3.12)

$$\limsup_{n\to\infty}\left\|T_N^n x_n^{(N-1)}-p\right\| \leq \limsup_{n\to\infty}\left[\left\|x_n^{(N-1)}-p\right\| + \mu_n\phi\left(\left\|x_n^{(N-1)}-p\right\|\right)+l_n\right] \leq r. \tag{3.13}$$

Since S_N is nonexpansive, we get

$$\limsup_{n\to\infty}\|S_N x_n-p\| \leq \limsup_{n\to\infty}\|x_n-p\| \leq r. \tag{3.14}$$

Next, consider

$$\left\|T_N^n x_n^{(N-1)}-p+c_n^{(N)}(u_n^{(N)}-x_n)\right\| \leq \left\|T_N^n x_n^{(N-1)}-p\right\|+c_n^{(N)}\left\|u_n^{(N)}-x_n\right\|.$$

Therefore, we have

$$\limsup_{n\to\infty}\left\|T_N^n x_n^{(N-1)}-p+c_n^{(N)}(u_n^{(N)}-x_n)\right\| \leq r. \tag{3.15}$$

Also,

$$\left\|S_N x_n-p+c_n^{(N)}(u_n^{(N)}-x_n)\right\| \leq \|S_N x_n-p\|+c_n^{(N)}\left\|u_n^{(N)}-x_n\right\|,$$

which implies that

$$\limsup_{n\to\infty}\left\|S_N x_n-p+c_n^{(N)}(u_n^{(N)}-x_n)\right\| \leq r, \tag{3.16}$$

and we have that

$$x_n^{(N)}-p = a_n^{(N)}\left(T_N^n x_n^{(N-1)}-p+c_n^{(N)}(u_n^{(N)}-x_n)\right) + \left(1-a_n^{(N)}\right)\left(S_N x_n-p+c_n^{(N)}(u_n^{(N)}-x_n)\right).$$

Hence,

$$\begin{aligned} r &= \lim_{n\to\infty}\left\|x_n^{(N)}-p\right\| \\ &= \lim_{n\to\infty}\left\|a_n^{(N)}\left(T_N^n x_n^{(N-1)}-p+c_n^{(N)}(u_n^{(N)}-x_n)\right) + \left(1-a_n^{(N)}\right)\left(S_N x_n-p+c_n^{(N)}(u_n^{(N)}-x_n)\right)\right\| \end{aligned} \tag{3.17}$$

Using (3.15), (3.16), (3.17) and Lemma 2.2, we find

$$\lim_{n\to\infty}\left\|T_N^n x_n^{(N-1)}-S_N x_n\right\| = 0. \tag{3.18}$$

It follows from (3.9) that

$$\lim_{n\to\infty}\left\|T_N^n x_n^{(N-1)}-x_n\right\| = 0. \tag{3.19}$$

Now, we shall show that $\lim_{n\to\infty}\left\|T_{N-1}^n x_n^{(N-2)}-S_{N-1}x_n\right\|=0$. For each $n\geq 1$,

$$\begin{aligned} \|x_n-p\| &\leq \left\|T_N^n x_n^{(N-1)}-x_n\right\|+\left\|T_N^n x_n^{(N-1)}-p\right\| \\ &\leq \left\|T_N^n x_n^{(N-1)}-x_n\right\| \\ &\quad + \left[\left\|x_n^{(N-1)}-p\right\|+\mu_n\phi\left(\left\|x_n^{(N-1)}-p\right\|\right)+l_n\right]. \end{aligned}$$

By using (3.8) and (3.19) , we obtain

$$r=\lim_{n\to\infty}\|x_n-p\|\le\lim\inf_{n\to\infty}\left\|x_n^{(N-1)}-p\right\|.$$

It follows that

$$r\le\lim\inf_{n\to\infty}\left\|x_n^{(N-1)}-p\right\|\le\lim\sup_{n\to\infty}\left\|x_n^{(N-1)}-p\right\|\le r.$$

This implies that

$$\lim_{n\to\infty}\left\|x_n^{(N-1)}-p\right\|=r. \tag{3.20}$$

On the other hand, we get

$$\left\|x_n^{(N-2)}-p\right\|\le\left(1+M_{(N-2)}\mu_n\right)\|x_n-p\|+R_{(N-2)}\left(\mu_n+l_n\right)+\varphi^n_{(N-2)},$$

where $\sum\limits_{n=1}^{\infty}\varphi^n_{(N-2)}<\infty$, $\sum\limits_{n=1}^{\infty}\mu_n<\infty$ and $\sum\limits_{n=1}^{\infty}l_n<\infty$. Hence

$$\begin{aligned}\lim\sup_{n\to\infty}\left\|x_n^{(N-2)}-p\right\|\le&\ \lim\sup\nolimits_{n\to\infty}\begin{array}{l}\left[\left(1+M_{(N-2)}\mu_n\right)\|x_n-p\|\right.\\ \qquad\left.+R_{(N-2)}\left(\mu_n+l_n\right)+\varphi^n_{(N-2)}\right]\end{array}\\ \le&\ r,\end{aligned} \tag{3.21}$$

and by (3.8)

$$\begin{aligned}\lim\sup_{n\to\infty}\left\|T^n_{N-1}x_n^{(N-2)}-p\right\|\le&\ \lim\sup\nolimits_{n\to\infty}\begin{array}{l}\left[\left\|x_n^{(N-2)}-p\right\|\right.\\ \qquad\left.+\mu_n\phi\left(\left\|x_n^{(N-2)}-p\right\|\right)+l_n\right]\end{array}\\ \le&\ r.\end{aligned} \tag{3.22}$$

Since S_{N-1} is nonexpansive, we get

$$\begin{aligned}\lim\sup_{n\to\infty}\|S_{N-1}x_n-p\|&\le\lim\sup_{n\to\infty}\|x_n-p\|\\ &\le r.\end{aligned} \tag{3.23}$$

Next, consider

$$\begin{aligned}&\left\|T^n_{N-1}x_n^{(N-2)}-p+c_n^{(N-1)}(u_n^{(N-1)}-x_n)\right\|\\ &\qquad\le\left\|T^n_{N-1}x_n^{(N-2)}-p\right\|+c_n^{(N-1)}\left\|u_n^{(N-1)}-x_n\right\|.\end{aligned}$$

Therefore, we have

$$\lim\sup_{n\to\infty}\left\|T^n_{N-1}x_n^{(N-2)}-p+c_n^{(N-1)}(u_n^{(N-1)}-x_n)\right\|\le r. \tag{3.24}$$

Also,

$$\left\|S_{N-1}x_n-p+c_n^{(N-1)}(u_n^{(N-1)}-x_n)\right\|\le\|S_{N-1}x_n-p\|+c_n^{(N-1)}\left\|u_n^{(N-1)}-x_n\right\|,$$

which implies that

$$\lim\sup_{n\to\infty}\left\|S_{N-1}x_n-p+c_n^{(N-1)}(u_n^{(N-1)}-x_n)\right\|\le r, \tag{3.25}$$

and we have that

$$\begin{aligned}x_n^{(N-1)}-p=&\ a_n^{(N-1)}\left(T^n_{N-1}x_n^{(N-2)}-p+c_n^{(N-1)}(u_n^{(N-1)}-x_n)\right)\\ &+\left(1-a_n^{(N-1)}\right)\left(S_{N-1}x_n-p+c_n^{(N-1)}(u_n^{(N-1)}-x_n)\right).\end{aligned}$$

Hence,

(3.26)
$$\begin{aligned} r &= \lim_{n\to\infty} \left\| x_n^{(N-1)} - p \right\| \\ &= \lim_{n\to\infty} \begin{Vmatrix} a_n^{(N-1)} \left(T_{N-1}^n x_n^{(N-2)} - p + c_n^{(N-1)}(u_n^{(N-1)} - x_n) \right) \\ \qquad + \left(1 - a_n^{(N-1)}\right) \left(S_{N-1} x_n - p + c_n^{(N-1)}(u_n^{(N-1)} - x_n) \right) \end{Vmatrix} \end{aligned}$$

Using (3.24), (3.25), (3.26) and Lemma 2.2, we find

(3.27) $$\lim_{n\to\infty} \left\| T_{N-1}^n x_n^{(N-2)} - S_{N-1} x_n \right\| = 0.$$

It follows from (3.9) that

(3.28) $$\lim_{n\to\infty} \left\| T_{N-1}^n x_n^{(N-2)} - x_n \right\| = 0.$$

Continuing a similar process, we have

(3.29) $$\lim_{n\to\infty} \left\| T_{N-i}^n x_n^{(N-i-1)} - x_n \right\| = 0, \ 0 \le i \le (N-2).$$

Now,

$$\left\| T_1^n x_n - p + c_n^{(1)}(u_n^{(1)} - x_n) \right\| \le \| T_1^n x_n - p \| + c_n^{(1)} \left\| u_n^{(1)} - x_n \right\|.$$

Hence,

(3.30) $$\limsup_{n\to\infty} \left\| T_1^n x_n - p + c_n^{(1)}(u_n^{(1)} - x_n) \right\| \le r.$$

Now, since S_1 is nonexpansive, we get

$$\begin{aligned} \limsup_{n\to\infty} \| S_1 x_n - p \| &\le \limsup_{n\to\infty} \| x_n - p \| \\ &\le r. \end{aligned}$$

Also,

$$\left\| S_1 x_n - p + c_n^{(1)}(u_n^{(1)} - x_n) \right\| \le \| S_1 x_n - p \| + c_n^{(1)} \left\| u_n^{(1)} - x_n \right\|,$$

which implies that

(3.31) $$\limsup_{n\to\infty} \left\| S_1 x_n - p + c_n^{(1)}(u_n^{(1)} - x_n) \right\| \le r,$$

and therefore

(3.32)
$$\begin{aligned} r &= \lim_{n\to\infty} \left\| x_n^{(1)} - p \right\| \\ &= \lim_{n\to\infty} \begin{Vmatrix} a_n^{(1)} \left(T_1^n x_n - p + c_n^{(1)}(u_n^{(1)} - x_n) \right) \\ \qquad + \left(1 - a_n^{(1)}\right) \left(S_1 x_n - p + c_n^{(1)}(u_n^{(1)} - x_n) \right) \end{Vmatrix}. \end{aligned}$$

Using (3.30), (3.31), (3.32) and Lemma 2.2, we find

(3.33) $$\lim_{n\to\infty} \| T_1^n x_n - S_1 x_n \| = 0.$$

It follows from (3.9) that

(3.34) $$\lim_{n\to\infty} \| T_1^n x_n - x_n \| = 0.$$

Similarly, by using the same argument as in the proof above, we have

(3.35) $$\lim_{n\to\infty} \| T_2^n x_n - x_n \| = 0.$$

Continuing a similar process, we have

$$\lim_{n\to\infty} \|T_i^n x_n - x_n\| = 0. \tag{3.36}$$

for all $i = 1, 2, \ldots, N$.

From (3.18), (3.19), (3.27), (3.28), (3.29), (3.33) and (3.34), we have

$$\lim_{n\to\infty} \|x_n - S_i x_n\| = 0 \tag{3.37}$$

for all $i = 1, 2, \ldots, N$. This completes the proof. □

3.3. Theorem. *Let K be a nonempty convex subset of a real Banach space E. Let $S_1, S_2, \ldots, S_N : K \to K$ be a finite family of continuous nonexpansive mappings and $T_1, T_2, \ldots, T_N : K \to K$ a finite family of continuous total asymptotically nonexpansive mappings with sequences $\{\mu_n\}$ and $\{l_n\}$ defined by (1.7) such that $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} l_n < \infty$ and $\mathcal{F}(S,T) = \bigcap_{i=1}^{N} F(S_i) \cap F(T_i) \neq \emptyset$. Assume that there exists $M, M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$, $i \in \{1, 2, \ldots, N\}$. Suppose that the family $\{S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N\}$ satisfies (3.1), (3.9) and (3.10). For an arbitrary $x_1 \in K$, define the sequences $\{x_n\}$ by recursion (1.8). Then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N\}$ if and only if $\liminf_{n\to\infty} d(x_n, \mathcal{F}(S,T)) = 0$, where $d(x_n, \mathcal{F}(S,T)) = \inf_{q\in\mathcal{F}(S,T)} \|x_n - q\|$, $n \geq 1$.*

Proof. Necessity is obvious. Indeed, if $x_n \to x^* \in \mathcal{F}(S,T)$ $(n \to \infty)$, then

$$d(x_n, \mathcal{F}(S,T)) = \inf_{x^*\in\mathcal{F}(S,T)} d(x_n, x^*) \leq \|x_n - x^*\| \to 0 \ \ (n \to \infty).$$

Now we prove sufficiency. It follows from (3.7) that for $x^* \in \mathcal{F}(S,T)$, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \left\|x_n^{(N)} - p\right\| \\ &\leq (1 + M_N\mu_n)\|x_n - p\| + R_N(\mu_n + l_n) + \varphi_{(N)}^n \\ &= \|x_n - p\| + \delta_n, \end{aligned} \tag{3.38}$$

where $\delta_n = M_N\mu_n\|x_n - p\| + R_N(\mu_n + l_n) + \varphi_{(N)}^n$. Since $\{x_n - p\}$ is bounded and $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} l_n < \infty$ and $\sum_{n=1}^{\infty} \varphi_{(N)}^n < \infty$, we obtain $\sum_{n=1}^{\infty} \delta_n < \infty$. Hence, (3.38) implies

$$\inf_{p\in\mathcal{F}(S,T)} \|x_{n+1} - p\| \leq \inf_{p\in\mathcal{F}(S,T)} \|x_n - p\| + \delta_n,$$

that is

$$d(x_{n+1}, \mathcal{F}(S,T)) \leq d(x_n, \mathcal{F}(S,T)) + \delta_n, \tag{3.39}$$

by Lemma 2.1 (i), it follows from (3.39) that we have $\lim_{n\to\infty} d(x_n, \mathcal{F}(S,T))$ exists. Noticing $\liminf_{n\to\infty} d(x_n, \mathcal{F}(S,T)) = 0$, it follows from (3.39) and Lemma 2.1 (ii) that we have $\lim_{n\to\infty} d(x_n, \mathcal{F}(S,T)) = 0$.

Now, since $\lim_{n\to\infty} d(x_n, \mathcal{F}(S,T)) = 0$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, given $\epsilon > 0$, there exists a positive integer N_1 such that $d(x_n, \mathcal{F}(S,T)) \leq \frac{\epsilon}{4}$ and $\sum_{j=n}^{\infty} \delta_j \leq \frac{\epsilon}{4}$ for all $n \geq N_1$. So, we have $d(x_{N_1}, \mathcal{F}(S,T)) \leq \frac{\epsilon}{4}$ and $\sum_{j=N_1}^{\infty} \delta_j \leq \frac{\epsilon}{4}$. This means that there exists a $q_1 \in \mathcal{F}(S,T)$

such that $\|x_{N_1} - q_1\| \leq \frac{\epsilon}{4}$. It follows from (3.38) that when $n \geq N_1$, $m \geq 1$,

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - q_1\| + \|x_n - q_1\| \\ &\leq \|x_{N_1} - q_1\| + \sum_{j=N_1}^{n+m-1} \delta_j + \|x_{N_1} - q_1\| + \sum_{j=N_1}^{n-1} \delta_j \\ &\leq \|x_{N_1} - q_1\| + \sum_{j=N_1}^{\infty} \delta_j + \|x_{N_1} - q_1\| + \sum_{j=N_1}^{\infty} \delta_j \\ &\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} \\ &= \epsilon.\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in E; and since E is complete there exists $p \in E$ such that $x_n \to p$ as $n \to \infty$. We show that p is a common fixed point of $\{S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N\}$, that is we have that $p \in \mathcal{F}(S,T)$.

Assume for contradiction that $p \in \mathcal{F}^c(S,T)$ (where $\mathcal{F}^c(S,T)$ denotes the complement of $\mathcal{F}(S,T)$). Since $\mathcal{F}(S,T)$ is a closed subset of E (recall each $\{S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N\}$ is continuous), we have that $d(p, \mathcal{F}(S,T)) > 0$. But for all $x^* \in \mathcal{F}(S,T)$, we have

$$\|p - x^*\| \leq \|p - x_n\| + \|x_n - x^*\|,$$

which implies

$$d(p, \mathcal{F}(S,T)) \leq \|x_n - p\| + d(x_n, \mathcal{F}(S,T)),$$

so that as $n \to \infty$ we have $d(p, \mathcal{F}(S,T)) = 0$, which contradicts $d(p, \mathcal{F}(S,T)) > 0$. Thus, p is a common fixed point of $\{S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N\}$. This completes the proof. □

3.4. Theorem. *Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E. Let $S_1, S_2, \ldots, S_N : K \to K$ be a finite family of uniformly continuous nonexpansive mappings and $T_1, T_2, \ldots, T_N : K \to K$ a finite family of uniformly continuous total asymptotically nonexpansive mappings with sequences $\{\mu_n\}$ and $\{l_n\}$ defined by (1.7) such that $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} l_n < \infty$ and $\mathcal{F}(S,T) = \bigcap_{i=1}^{N} F(S_i) \cap F(T_i) \neq \emptyset$.*

Assume that there exists $M, M^* > 0$ such that $\phi(\lambda) \leq M^* \lambda$ for all $\lambda \geq M$, $i \in \{1, 2, \ldots, N\}$. Suppose that the family $\{S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N\}$ satisfies (3.1), (3.9) and (3.10). If one of the mappings in $\{T_1, T_2, \ldots, T_N\}$ is compact, then the sequence $\{x_n\}$ as defined in (1.8) converges strongly to a common fixed of the mappings $\{S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N\}$.

Proof. We obtain from Lemma 3.2 that

$$\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i^n x_n\| = 0, \text{ for all } i = 1, 2, \ldots, N. \tag{3.40}$$

Since T_N is a total asymptotically nonexpansive mapping and S_N is a nonexpansive mapping, it follows from (1.8), (3.18), (3.37) and condition $\sum_{n=1}^{\infty} c_n^{(N)} < \infty$ hat we have

$$\begin{aligned}\|x_{n+1}-x_n\| &= \left\|a_n^{(N)}T_N^n x_n^{(N-1)} + b_n^{(N)}S_N x_n + c_n^{(N)}u_n^{(N)} - x_n\right\| \\ &= \left\|a_n^{(N)}\left(T_N^n x_n^{(N-1)} - S_N x_n\right) + b_n^{(N)}\left(S_N x_n - x_n\right)\right. \\ &\qquad \left. + c_n^{(N)}\left(u_n^{(N)} - x_n\right)\right\| \\ &\le a_n^{(N)}\left\|T_N^n x_n^{(N-1)} - S_N x_n\right\| + c_n^{(N)}\left\|u_n^{(N)} - x_n\right\| \\ &\qquad + \left(1 - a_n^{(N)} - c_n^{(N)}\right)\|S_N x_n - x_n\| \\ &\to 0, \text{ as } n\to\infty. \end{aligned} \tag{3.41}$$

Let T_1 be compact. Since T_1 is continuous and compact, it is completely continuous. Hence, there exists a subsequence $\{T_1^{n_j}x_{n_j}\}$ of $\{T_1^n x_n\}$ such that $T_1^{n_j}x_{n_j}\to p$ as $j\to\infty$ for some $p\in E$. Hence $T_1^{n_j+1}x_{n_j}\to T_1 p$ as $j\to\infty$, and from (3.40) we have that $\lim_{j\to\infty} x_{n_j} = p$. Also from (3.40), $T_2^{n_j}x_{n_j}\to p$, $T_3^{n_j}x_{n_j}\to p,\ldots,T_N^{n_j}x_{n_j}\to p$ as $j\to\infty$. Hence, $T_2^{n_j+1}x_{n_j}\to T_2 p$, $T_3^{n_j+1}x_{n_j}\to T_3 p,\ldots,T_N^{n_j+1}x_{n_j}\to T_N p$ as $j\to\infty$. Using (3.41), it follows that $x_{n_j+1}\to p$ as $j\to\infty$.

Next, we show that $p\in\mathcal{F}(S,T)$. Observe that

$$\begin{aligned}\|p-T_i p\| &\le \left\|p - x_{n_j+1}\right\| + \left\|x_{n_j+1} - T_i^{n_j+1}x_{n_j+1}\right\| \\ &\quad + \left\|T_i^{n_j+1}x_{n_j+1} - T_i^{n_j+1}x_{n_j}\right\| + \left\|T_i^{n_j+1}x_{n_j} - T_i p\right\|\end{aligned}$$

for all $i=1,2,\ldots,N$. Taking the limit as $j\to\infty$ and using the fact that T_i is uniformly continuous we have that $p=T_i p$ and so $p\in F(T_i)$ for all $i=1,2,\ldots,N$.

Also by the continuity of all the mappings S_i and Lemma 3.2, we conclude that

$$\|S_i p - p\| = \lim_{j\to\infty}\left\|S_i x_{n_j} - x_{n_j}\right\| = 0,$$

for all $i=1,2,\ldots,N$. That is, $p\in\mathcal{F}(S,T)=\bigcap_{i=1}^N F(T_i)\cap F(S_i)$. It follows from Lemma 3.1 that $\lim_{n\to\infty}\|x_n-p\|$ exists, $p\in\mathcal{F}(S,T)$. Hence, $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{S_1,S_2,\ldots,S_N,T_1,T_2,\ldots,T_N\}$. This completes the proof. □

3.5. Remark. If $T_1, T_2,\ldots,T_N$ are asymptotically nonexpansive mappings, then $l_n=0$ and $\phi(\lambda)=\lambda$ so that the assumption that there exist $M, M^*>0$ such that $\phi(\lambda)\le M^*\lambda$ for all $\lambda\ge M$, $i\in\{1,2,\ldots,N\}$ in the above theorems is no longer needed. Hence, the results in the above theorems also hold for asymptotically nonexpansive mappings. Thus, the results in this paper improve and extend the corresponding results of [4, 6, 7] and [10] from asymptotically nonexpansive (or nonexpansive) mappings to total asymptotically nonexpansive mappings under general conditions.

3.6. Corollary. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E. Let S_1, $S_2,\ldots,S_N: K\to K$ be a finite family of continuous nonexpansive mappings and $T_1,T_2,\ldots,T_N: K\to K$ a finite family of continuous asymptotically nonexpansive mappings with sequences $\{\mu_{in}\}\subset[0,\infty]$ such that $\sum_{n=1}^{\infty}\mu_{in}<\infty$ and $\mathcal{F}(S,T)=\bigcap_{i=1}^N F(S_i)\cap F(T_i)\ne\emptyset$. For an arbitrary $x_1\in K$, define the sequences $\{x_n\}$ by recursion (1.8). Suppose that $\sum_{n=1}^{\infty}c_n^{(i)}<\infty$ for all $i=1,2,\ldots,N$,*

(i) $\|x-T_i y\|\le\|S_i x - T_i y\|$ *for all $x,y\in K$ and $i=1,2,\ldots,N$,*

(ii) *there are* $\eta_1, \eta_2 \in (0,1)$ *such that* $0 < \eta_1 \leq a_n^{(i)} \leq \eta_2 < 1$, $\forall n \geq n_0$ *for some* $n_0 \in \mathbb{N}$.

Then:

(a) *The sequence* $\{x_n\}$ *converges strongly to a common fixed point of* $\{S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N\}$ *if and only if* $\liminf_{n\to\infty} d(x_n, \mathcal{F}(S,T)) = 0$, *where* $d(x_n, \mathcal{F}(S,T)) = \inf_{q\in\mathcal{F}(S,T)} \|x_n - q\|$, $n \geq 1$.

(b) *If one of the mappings in* $\{T_1, T_2, \ldots, T_N\}$ *is compact, then the sequence* $\{x_n\}$ *as defined in (1.8) converges strongly to a common fixed of the mappings* $\{S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N\}$. □

3.7. Corollary. *Let* K *be a nonempty closed convex subset of a uniformly convex Banach space* E. *Let* $S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N : K \to K$ *be a finite family of continuous nonexpansive mappings and suppose that* $\mathcal{F}(S,T) = \bigcap_{i=1}^N F(S_i) \cap F(T_i) \neq \emptyset$. *Let* $\{x_n\}$ *be the sequence defined by*

$$
\begin{aligned}
x_1 &\in K,\\
x_{n+1} = x_n^{(N)} &= a_n^{(N)} T_N x_n^{(N-1)} + b_n^{(N)} S_N x_n + c_n^{(N)} u_n^{(N)},\\
x_n^{(N-1)} &= a_n^{(N-1)} T_{N-1} x_n^{(N-2)} + b_n^{(N-1)} S_{N-1} x_n + c_n^{(N-1)} u_n^{(N-1)},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
x_n^{(3)} &= a_n^{(3)} T_3 x_n^{(2)} + b_n^{(3)} S_3 x_n + c_n^{(3)} u_n^{(3)},\\
x_n^{(2)} &= a_n^{(2)} T_2 x_n^{(1)} + b_n^{(2)} S_2 x_n + c_n^{(2)} u_n^{(2)},\\
x_n^{(1)} &= a_n^{(1)} T_1 x_n + b_n^{(1)} S_1 x_n + c_n^{(1)} u_n^{(1)},\ n \geq 1,
\end{aligned}
\tag{3.42}
$$

(called an N-step iterative sequence*), where* $\left\{u_n^{(i)}\right\}$ *are bounded sequences in* K *and* $\left\{a_n^{(i)}\right\}_{n=1}^\infty$, $\left\{b_n^{(i)}\right\}_{n=1}^\infty$, $\left\{c_n^{(i)}\right\}_{n=1}^\infty \subset [0,1]$ *are such that* $a_n^{(i)} + b_n^{(i)} + c_n^{(i)} = 1$, *for all* $i = 1, 2, \ldots, N$.

Suppose that $\sum_{n=1}^{\infty} c_n^{(i)} < \infty$ *for all* $i = 1, 2, \ldots, N$,

(i) $\|x - T_i y\| \leq \|S_i x - T_i y\|$ *for all* $x, y \in K$ *and* $i = 1, 2, \ldots, N$,

(ii) *there are* $\eta_1, \eta_2 \in (0,1)$ *such that* $0 < \eta_1 \leq a_n^{(i)} \leq \eta_2 < 1$, $\forall n \geq n_0$ *for some* $n_0 \in \mathbb{N}$.

Then

(a) *The sequence* $\{x_n\}$ *converges strongly to a common fixed point of* $\{S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N\}$ *if and only if* $\liminf_{n\to\infty} d(x_n, \mathcal{F}(S,T)) = 0$, *where* $d(x_n, \mathcal{F}(S,T)) = \inf_{q\in\mathcal{F}(S,T)} \|x_n - q\|$, $n \geq 1$.

(b) *If one of the mappings in* $\{T_1, T_2, \ldots, T_N\}$ *is compact, then the sequence* $\{x_n\}$ *as defined in (3.42) converges strongly to a common fixed of the mappings* $\{S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N\}$. □

3.8. Remark. Let K be a nonempty closed convex subset of a Banach space E. Let S_1, $S_2, \ldots, S_N : K \to E$ be N nonself nonexpansive mappings, let $T_1, T_2, \ldots, T_N : K \to E$ be N nonself total asymptotically nonexpansive mappings; assuming the existence of common fixed points of these operators, our theorems and method of proof easily carry over to this class of mappings using the iterative sequence $\{x_n\}$ defined by,

$$\begin{aligned}
x_1 &\in K,\\
x_{n+1} = x_n^{(N)} &= P\left[a_n^{(N)} T_N \left(PT_N\right)^{n-1} x_n^{(N-1)} + b_n^{(N)} S_N x_n + c_n^{(N)} u_n^{(N)}\right],\\
x_n^{(N-1)} &= P\left[a_n^{(N-1)} T_{N-1} \left(PT_{N-1}\right)^{n-1} x_n^{(N-2)} + b_n^{(N-1)} S_{N-1} x_n + c_n^{(N-1)} u_n^{(N-1)}\right],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
x_n^{(3)} &= P\left[a_n^{(3)} T_3 \left(PT_3\right)^{n-1} x_n^{(2)} + b_n^{(3)} S_3 x_n + c_n^{(3)} u_n^{(3)}\right],\\
x_n^{(2)} &= P\left[a_n^{(2)} T_2 \left(PT_2\right)^{n-1} x_n^{(1)} + b_n^{(2)} S_2 x_n + c_n^{(2)} u_n^{(2)}\right],\\
x_n^{(1)} &= P\left[a_n^{(1)} T_1 \left(PT_1\right)^{n-1} x_n + b_n^{(1)} S_1 x_n + c_n^{(1)} u_n^{(1)}\right],\ n \geq 1,
\end{aligned}$$

where $\left\{u_n^{(i)}\right\}$ are bounded sequences in K and $\left\{a_n^{(i)}\right\}_{n=1}^{\infty}$, $\left\{b_n^{(i)}\right\}_{n=1}^{\infty}$, $\left\{c_n^{(i)}\right\}_{n=1}^{\infty} \subset [0,1]$ such that $a_n^{(i)} + b_n^{(i)} + c_n^{(i)} = 1$, for all $i = 1, 2, \ldots, N$.

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