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NORMAL DIFFERENTIAL OPERATORS OF THIRD-ORDER

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Abstract

In the Hilbert space of vector-functions $L^2(H,(a,b))$, where H is any separable Hilbert space, the general representation in terms of boundary values of all normal extensions of the formally normal minimal operator, generated by linear differential-operator expressions of third order in the form

$$l(u) = u'''(t) + A^3u(t),\ A : D(A) \subset H \to H,\ A = A^* \geq E,$$

is obtained in the first part of this study. Then, some spectral properties of these normal extensions are investigated. In particular, the case of $A^{-1} \in \mathfrak{S}_\infty(H)$, asymptotic estimates of normal extensions of eigenvalues has been established at infinity.

Keywords: Normal extension, Compact operator, Eigenvalue, Asymptotical behavior of eigenvalues.

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1. Introduction

It is known that operator theory plays an exceptionally important role in modern mathematics and physics, especially in boundary value problems, quantum mechanics and deformation theory. Also, spectral analysis of differential operators is one of the most important areas of modern mathematical physics [1]. In addition the investigation of different selfadjoint extensions of densely defined closed symmetric operators is among the fundamental mathematical problems arising in any physical model. By using the Calkin theory a survey of the selfadjoint (dissipative, accumulative) extensions and their spectral analysis has been analyzed in the case of symmetric differential-operator expressions first and second order in $L^2(H,(a,b))$, $a, b \in \mathbb{R}$ [6] (for a detail analysis of these problems see [9]). However, many physical problems oblige one to investigate normal extensions

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of formally normal differential-operators of lower order in the Hilbert space of vector-functions on a finite interval.

The basic results of this theory had been established and developed by E. A. Coddington [2]. Unfortunately, applications of this theory to the theory of differential operators in Hilbert space have not received the attention it deserves. In this sense, for first and second order differential operators, some results have been obtained in [7, 8].

In this study, the general representation of boundary values of all normal extensions of the formally normal minimal operator, generated by two-term linear differential-operator expression for third order with selfadjoint coefficient, in the Hilbert space of vector functions $L^2(H,(a,b))$ where H is any separable Hilbert space and $a,b \in \mathbb{R}$, is obtained. Furthermore, some spectral properties of these normal extensions are investigated. Finally, in the special case of operator coefficient asymptotically estimates of these normal extensions eigenvalues has been established at infinity.

2. The minimal and maximal operators

Let us start with a important definition.

2.1. Definition. A densely defined closed operator $\mathbb{N}$ in a Hilbert space is called *formally normal* if $D(\mathbb{N}) \subset D(\mathbb{N}^*)$ and $\|\mathbb{N}f\| = \|\mathbb{N}^*f\|$, for all $f \in D(\mathbb{N})$. If a formally normal operator has no formally normal non-trivial extension, then it is called a *maximal formally normal operator*. If a formally normal operator $\mathbb{N}$ satisfies the condition $D(\mathbb{N}) = D(\mathbb{N}^*)$, then it is called a *normal operator* [2].

In the space $L^2(H,(a,b))$ consider a linear differential-operator expression of third-order in the form

$$l(u) = u'''(t) + A^3(t)u(t), \tag{2.1}$$

where for each $t \in [a,b]$, $A(t)$ is a linear selfadjoint operator in H, $A(t) \geq E$ and $D(A(t)) = D$. It is clear that a formally adjoint expression in the space $L^2(H,(a,b))$ to (2.1), is in the form

$$l^+(v) = -v'''(t) + A^3(t)v(t). \tag{2.2}$$

Now let us define the operator L_0' on the dense manifold of the vector-functions D_0' in $L^2(H,(a,b))$

$$D_0' := \Big\{u(t) \in L^2 : \ u(t) = \sum_{k=1}^{n} \varphi_k(t) f_k, \ \varphi_k(t) \in C_0^\infty(a,b), \ f_k \in D(A), \\ k = 1,2,\ldots,n, n \in \mathbb{N}\Big\}$$

as $L_0' := l(u)$. Since for all $u \in D_0'$

$$\mathrm{Re}(L_0'u,u)_{L_2} = \mathrm{Re}(lu(t),u(t))_{L_2} = \mathrm{Re}(A^3(t)u(t),u(t))_{L_2} \geq \|u(t)\|_{L_2}^2 \geq 0,$$

then L_0' is an accretive operator, and so it has a closure in $L^2(H,(a,b))$ [3]. The closure of L_0' in $L^2(H,(a,b))$ is called the *minimal operator generated by the differential-operator expression* (2.1) and it is denoted by L_0. In a similar way, the minimal operator L_0^+, generated by the differential operator expression (2.2) in $L^2(H,(a,b))$ is constructed.

The adjoint operator of L_0^+ (L_0) in $L^2(H,(a,b))$ is called the *maximal operator generated by* (2.1) ((2.2)), and is denoted by L (L^+). It is clear that $L_0 \subset L$, $L_0^+ \subset L^+$.

2.2. Lemma. *If $f \in L_2(a,b)$, $a,b \in \mathbb{R}$ and for all real-valued function $\varphi, \psi \in C_0^\infty(a,b)$ and*

$$\int_a^b f(t)\,(\varphi(t)\psi(t))'\,dt = 0,$$

then $f = constant$ almost everywhere in (a,b).

Proof. In this case,

$$\begin{aligned}
&\int_a^b f(t)(\varphi(t)\psi(t))'dt \\
&\qquad = \int_a^b f(t)\varphi'(t)\psi(t)\,dt + \int_a^b f(t)\varphi(t)\psi'(t)\,dt \\
&\qquad = \left(\int_a^t f(s)\varphi'(s)ds\right)\psi(t)\Bigg|_{t=a}^{t=b} - \int_a^b \left(\int_a^t f(s)\varphi'(s)\,ds\right)\psi'(t)\,dt \\
&\qquad\qquad + \int_a^b f(t)\varphi(t)\psi'(t)\,dt \\
&\qquad = \int_a^b \left(f(t)\varphi(t) - \int_a^t f(s)\varphi'(s)\,ds\right)\psi'(t)\,dt = 0.
\end{aligned}$$

Hence, there exists a real number $c \in \mathbb{R}$ for every $\varphi \in C_0^\infty(a,b)$, $f(t)\varphi(t) - \int_a^t f(s)\varphi'(s)\,ds$ $= c$ a.e. [4]. If $\varphi(t) \neq 0$, $t \in (a,b)$, then the function $f^*(t) := \dfrac{\int_a^t f(s)\varphi'(s)\,ds - c}{\phi(t)}$, $a < t < b$, is equal the function f a.e., and so

$$\int_a^t f^*(s)\varphi'(s)ds = f^*(t)\varphi(t) + c$$

holds. From this relation and continuity of the function f^* on (a,b), this function is differentiable and $f^{*\prime}(t) = 0$. Therefore the function f is constant a.e. in (a,b). □

2.3. Theorem. *Let $\sigma_x(t) := (A^3(t)x,x)_H$, $x \in D$. If the minimal operator L_0 is formally normal in $L^2(H,(a,b))$, then $\sigma_x = constant$ a.e. in (a,b).*

Proof. For every $u(t) \in D(L_0)$

$$\|L_0u(t)\|_{L^2}^2 - \|L^+u(t)\|_{L^2}^2 = 2\left[(u'''(t), A^3(t)u(t))_{L^2} + (A^3(t)u(t), u'''(t))_{L^2}\right] = 0.$$

By using this relation, for the special vector-function $u(t) = \varphi(t)x \in D_0'$, $x \in D$,

$$\begin{aligned}
&\left(u'''(t), A^3(t)u(t)\right)_{L^2} + \left(A^3(t)u(t), u'''(t)\right)_{L^2} \\
&\qquad = \int_a^b \sigma_x(t)\left(\varphi'''(t)\overline{\varphi(t)} + \varphi(t)\overline{\varphi'''(t)}\right)dt = 0.
\end{aligned}$$

If $\varphi \in C_0^\infty(a,b)$ is a real-valued function, then

$$\int_a^b \sigma_x(t)\varphi'''(t)\varphi(t)\,dt = 0$$

is true. For every real-valued function $\varphi \in C_0^\infty(a,b)$, $e^{it}\varphi(t)$ belong to $C_0^\infty(a,b)$. If these functions are used in the last equation

$$\int_a^b \sigma_x(t)\left(\left(e^{it}\varphi(t)\right)'''e^{-it}\varphi(t) + e^{it}\varphi(t)\overline{(e^{it}\varphi(t))}'''\right)dt$$

$$= \int_a^b \sigma_x(t)\left(2\varphi(t)\varphi'''(t) - 6\varphi(t)\varphi'(t)\right)dt$$

$$= -6\int_a^b \sigma_x(t)\varphi(t)\varphi'(t)\,dt = -3\int_a^b \sigma_x(t)\left((\varphi(t))^2\right)'dt = 0,$$

then for every real-valued functions $\varphi \in C_0^\infty(a,b)$

$$\int_a^b \sigma_x(t)\left((\varphi(t))^2\right)'dt = 0$$

is true. For every real-valued functions $\varphi,\ \psi \in C_0^\infty(a,b),\ \varphi+\psi \in C_0^\infty(a,b)$ and from the last equation

$$\int_a^b \sigma_x(t)\left((\varphi(t)+\psi(t))^2\right)'dt = 2\int_a^b \sigma_x(t)\left(\varphi(t)\psi(t)\right)'dt = 0$$

holds. Therefore, by Lemma 2.2, $\sigma_x(t) = \text{constant}$ a.e. for any $x \in D$. □

2.4. Corollary. *If $\overline{\alpha(t)} = \alpha(t) \in L_2(a,b)$ and $A(t) = \alpha(t)A$, $A^* = A \geq E$, then the minimal operator L_0 is formally normal in $L^2(H,(a,b))$ if and only if the function α is constant a.e. in (a,b).* □

2.5. Corollary. *If $dimH < +\infty$, then the minimal operator L_0 is formally normal in the Hilbert space $L^2(H,(a,b))$ if and only if $A(t) = costant$ a.e. in (a,b).* □

According to these results, throughout this work let us consider $A(t) = A$, $A \geq E$ in (a,b).

3. Domains of minimal and maximal operators

Let H be a separable Hilbert space, $A : D(A) \subset H \to H, A^* = A \geq E$ and define an inner product on the domain $D(A)$ by

$$(x,y)_{+1/2} := \left(A^{1/2}x, A^{1/2}y\right)_H .$$

With this inner product $D(A)$ is Hilbert space, which will be denoted by $H_{+1/2} = H_{+1/2}(A)$. Also let us introduce in H a new norm:

$$\|y\|_{H_{-1/2}} := \sup_{x\in H_{+1/2}} \frac{|(y,x)_H|}{\|x\|_{+1/2}}$$

The completion of H with respect to the norm $\|\cdot\|_{H-1/2}$ is denoted by $H_{-1/2} = H_{-1/2}(A)$. The elements of this space are called *generalized elements*. In this case

$A : H_{+1/2} \to H$ is a continuous operator and its adjoint operator is denoted by $\tilde{A} : H \to H_{-1/2}$. Also $\tilde{A}$ is an extension of A and $\tilde{A}^* = \tilde{A} \geq E$ [6].

From now on it will be considered that

$$\tilde{l}(u) = u'''(t) + \tilde{A}^3 u(t). \tag{3.1}$$

We state the following lemmas. Their proofs can be done similarly to those in [6].

3.1. Lemma. *The operators*

$$x \mapsto e^{-\tilde{A}(t-a)}x, \quad x \mapsto e^{\tilde{A}(t-b)}x$$

are continuous from $H_{-1/2}$ to $L^2(H,(a,b))$. □

3.2. Lemma. *The operators*

$$f(t) \mapsto \int_a^b e^{-\tilde{A}(t-a)} f(t)\,dt, \quad f(t) \mapsto \int_a^b e^{\tilde{A}(t-b)} f(t)\,dt$$

are continuous from $L^2(H,(a,b))$ into $H_{+1/2}$.

The following theorem can be proved by using results in [6, 10].

3.3. Theorem. *The domain $D(L)$ of the maximal operator L generated by expression (3.1) consists of the vector-function $f(t)$ that have a representation $Lu(t) = f(t)$, $f \in L_2(H,(a,b))$, where*

$$u(t) = e^{-(t-a)\tilde{A}}x_1 + e^{\frac{1-i\sqrt{3}}{2}(t-b)\tilde{A}}x_2 + e^{\frac{1+i\sqrt{3}}{2}(t-b)\tilde{A}}x_3 + \frac{1}{3}A^{-2}\int_a^t e^{-(t-s)A} f(s)\,ds$$
$$+ \frac{1-i\sqrt{3}}{6}A^{-2}\int_t^b e^{\frac{1-i\sqrt{3}}{2}(t-s)A} f(s)\,ds + \frac{1+i\sqrt{3}}{6}A^{-2}\int_t^b e^{\frac{1+i\sqrt{3}}{2}(t-s)A} f(s)\,ds,$$
$$x_i \in H_{-1/2},\ i = 1,2,3. \quad \square$$

Also the domain of the minimal operator L_0 consists of vector-functions $u(t) \in D(L)$ for which $u(a) = u(b) = u'(a) = u'(b) = u''(a) = u''(b) = 0$.

3.4. Corollary. *If L_n, $L_0 \subset L_n \subset L$ is a normal extension, then the domain $D(L_n)$ of L_n consists of the vector-functions $u(t)$ that have the representation*

$$u(t) = e^{-(t-a)\tilde{A}}x_1 + e^{\frac{1-i\sqrt{3}}{2}(t-b)\tilde{A}}x_2 + e^{\frac{1+i\sqrt{3}}{2}(t-b)\tilde{A}}x_3 + \frac{1}{3}A^{-2}\int_a^t e^{-(t-s)A} f(s)\,ds$$
$$+ \frac{1-i\sqrt{3}}{6}A^{-2}\int_t^b e^{\frac{1-i\sqrt{3}}{2}(t-s)A} f(s)\,ds + \frac{1+i\sqrt{3}}{6}A^{-2}\int_t^b e^{\frac{1+i\sqrt{3}}{2}(t-s)A} f(s)\,ds,$$
$$x_i \in H_{+1/2},\ i = 1,2,3,$$

where $L_n u(t) = f(t)$, $f \in L_2(H,(a,b))$.

Proof. Since L_n, $L_0 \subset L_n \subset L$ is a normal operator, then for every $u(t) \in D(L_n)$ we have $\tilde{A}u(t) \in L_2(H,(a,b))$. By using Theorem 3.3,

$$\begin{aligned}
&\tilde{A}u(t)\\
&= e^{(a-t)\tilde{A}}\tilde{A}x_1 + e^{\frac{1-i\sqrt{3}}{2}(t-b)\tilde{A}}\tilde{A}x_2 + e^{\frac{1+i\sqrt{3}}{2}(t-b)\tilde{A}}\tilde{A}x_3 + \frac{1}{3}A^{-1}\int_a^t e^{(s-t)A}f(s)\,ds\\
&+ \frac{1-i\sqrt{3}}{6}A^{-1}\int_t^b e^{\frac{1-i\sqrt{3}}{2}(t-s)A}f(s)\,ds + \frac{1+i\sqrt{3}}{6}A^{-1}\int_t^b e^{\frac{1+i\sqrt{3}}{2}(t-s)A}f(s)\,ds\\
&\in L_2(H,(a,b)).
\end{aligned}$$

Thus, $\tilde{A}x_i \in H_{+1/2},\ i=1,2,3$ and from Lemma 3.2, $x_i \in H_{+1/2},\ i=1,2,3$ are obtained. □

4. Description of normal extensions

The main purpose of this section is to describe all normal extensions of the minimal operator L_0 generated by (3.1). Since the index number of the minimal operator $\overline{Im(L_0)}$ is $(\dim H^3, \dim H^3)$, where $H^3 = H \oplus H \oplus H$, there exist at least one space of boundary values [6]. Let $(H^3, \gamma_1, \gamma_2)$ be a space of boundary values for the minimal operator $\overline{\mathrm{Im}(L_0)}$ and $\widehat{A} := \begin{pmatrix} \tilde{A} & 0 & 0 \\ 0 & \tilde{A} & 0 \\ 0 & 0 & \tilde{A} \end{pmatrix}$, $\widehat{A} : H^3 \to H^3$.

4.1. Theorem. *Let $A : D(A) \subset H \to H$, $A = A^* \geq E$ be a linear operator and $\tilde{A}W_2^3(H,(a,b)) \subset W_2^3(H,(a,b)$. Every normal extension L_n, $L_0 \subset L_n \subset L$, of the minimal operator L_0 in $L^2(H(a,b))$ is generated by the differential-operator expression (3.1) and the boundary condition*

$$(4.1) \qquad (W-E)\gamma_1(u) + i(W+E)\gamma_2(u) = 0,$$

where W and $\widehat{A}^{3/2}W\widehat{A}^{-3/2}$ are unitary operators in H^3. The unitary operator W is determined uniquely by the extension L_n, i.e. $L_n = L_W$.

On the contrary, the restriction of the maximal operator L to the manifold of vector-functions $u(t) \in W_2^3(H,(a,b))$ that satisfy (4.1) for any unitary operators W and $\widehat{A}^{3/2}W\widehat{A}^{-3/2}$ in H^3, is a normal extension of the minimal operator L_0 in $L_2(H,(a,b))$.

Proof. Let L_n be a normal extension of L_0. In this case

$$\begin{aligned}
&\mathrm{Re}(L_n)u = A^3u(t), \quad u \in D(L_n),\\
&\mathrm{Im}(L_n)u = -iu'''(t), \quad u \in D(L_n)
\end{aligned}$$

are selfadjoint operators in $L^2(H,(a,b))$. Firstly, for the minimal operator $\overline{\mathrm{Im}(L_0)}$ we prove that the triple $(H^3, \gamma_1, \gamma_2)$, where

$$\gamma_1(u) := \left\{-iu''(b), \frac{i}{2}\left(u'(b) - u'(a)\right), iu''(a)\right\},\ \gamma_2(u) := \left\{u(b), u'(b) + u'(a), u(a)\right\}$$

is a space of boundary values. For every $u, v \in W_2^3(H,(a,b))$

$$(\mathrm{Im}(L_0^*)u, v)_{L^2} - (u, \mathrm{Im}(L_0^*)v)_{L^2} = (\gamma_1(u), \gamma_2(v))_{H^3} - (\gamma_2(u), \gamma_1(v))_{H^3}$$

and if $\{x_1, x_2, x_3\}$, $\{y_1, y_2, y_3\}$ are arbitrary vectors from H^3, then the vector-function $u(t) := \alpha_1(t)y_3 + \alpha_2(t)(-2ix_2 + y_2) + i\alpha_3(t)y_3 + \beta_1(t)y_1 + \frac{1}{2}\beta_2(t)(y_2 + ix_2) + i\beta_3(t)x_3$, where the $\alpha_i, \beta_i \in W_2^3(a, b)$, $i = 1, 2, 3$ satisfy the conditions

$$\begin{aligned}
\alpha_1(a) = 1, &\quad \alpha_1'(a) = \alpha_1''(a) = \alpha_1(b) = \alpha_1'(b) = \alpha_1''(b) = 0,\\
\alpha_2''(a) = 1, &\quad \alpha_2(a) = \alpha_2''(a) = \alpha_2(b) = \alpha_2'(b) = \alpha_2''(b) = 0,\\
\alpha_3''(a) = 1, &\quad \alpha_3(a) = \alpha_3'(a) = \alpha_3(b) = \alpha_3'(b) = \alpha_3''(b) = 0,\\
\beta_1(b) = 1, &\quad \beta_1(a) = \beta_1'(a) = \beta_1''(a) = \beta_1'(b) = \beta_1''(b) = 0,\\
\beta_2'(b) = 1, &\quad \beta_2(a) = \beta_2'(a) = \beta_2''(a) = \beta_2(b) = \beta_1''(b) = 0,\\
\beta_3''(b) = 1, &\quad \beta_3(a) = \beta_3'(a) = \beta_3''(a) = \beta_1'(b) = \beta_1''(b) = 0
\end{aligned}$$

belongs to $D(\mathrm{Im}(L_0^*)$ and $\gamma_1(u) = \{x_1, x_2, x_3\}$, $\gamma_2(u) = \{y_1, y_2, y_3\}$. Thus we have proved that the triple $(H^3, \gamma_1, \gamma_2)$ is a space of boundary values for the minimal operator $\overline{Im(L_0)}$.

It is known that a selfadjoint extension $\mathrm{Im}(L_n)$ of the minimal operator $\mathrm{Im}(L_0)$ in $L^2(H, (a, b))$ is described by the following boundary condition

$$(W - E)\gamma_1(u) + i(W + E)\gamma_2(u) = 0,$$

with a uniquely unitary operator W in H^3 [6]. On the other hand since the extension L_n is a normal operator, then for every $u(t) \in D(L_n)$ the following equality holds

$$(\mathrm{Re}(L_n)u, \mathrm{Im}(L_n)u)_{L^2} = (\mathrm{Im}(L_n)u, \mathrm{Re}(L_n)u)_{L^2}.$$

In other words for every $u(t) \in D(L_n)$

$$\begin{aligned}
&(\mathrm{Re}(L_n)u(t), \mathrm{Im}(L_n)u(t))_{L^2} - (\mathrm{Im}(L_n)u(t), \mathrm{Re}(L_n)u(t))_{L^2}\\
&\qquad = \big(\gamma_1(\tilde{A}^{3/2}u), \gamma_2(\tilde{A}^{3/2}u)\big)_{H^3} - \big(\gamma_2(\tilde{A}^{3/2}u), \gamma_1(\tilde{A}^{3/2}u)\big)_{H^3} = 0.
\end{aligned}$$

From the above, the linear relation

$$\theta := \big\{\big\{\gamma_1(\tilde{A}^{3/2}u), \gamma_1(\tilde{A}^{3/2}u)\big\} : u \in D(L_n)\big\} \subset H^3 \oplus H^3$$

is selfadjoint, so there is a unitary operator $V : H^3 \to H^3$ such that

$$(V - E)\gamma_1(\tilde{A}^{3/2}u) + i(V + E)\gamma_2(\tilde{A}^{3/2}u) = 0.$$

Let us set $U := \tilde{A}^{-3/2}V\tilde{A}^{3/2}$. In this case the relation implies that

$$(U - E)\gamma_1(u) + i(U + E)\gamma_2(u) = 0,\ u \in D(L_n).$$

Since the unitary operator W is determined uniquely by the extension $\mathrm{Im}(L_n)$, then U is a unitary operator and $U = W$, i.e., $\widehat{A}^{3/2}W\widehat{A}^{-3/2}$ is a unitary operator in H^3. It is clear that the unitary operator W is determined uniquely by the extension L_n.

Now let L_W be an operator generated by the differential-operator expression $\tilde{l}(u)$ with the boundary condition (4.1) in $L^2(H, (a, b))$, that is,

$$\begin{aligned}
&L_W u = l(u),\\
&D(L_W) = \big\{u \in W_2^3(H, (a, b)) :\ (W - E)\gamma_1(u) + i(W + E)\gamma_2(u) = 0\big\},
\end{aligned}$$

where W and $\widehat{A}^{3/2}W\widehat{A}^{-3/2}$ are unitary operators in H^3. In this case the adjoint operator L_W^* is generated by the differential-operator expression $\tilde{l}^*(v) = -v'''(t) + \tilde{A}^3 v(t)$ with the boundary condition

$$(W^* - E)\gamma_1(v) - i(W^* + E)\gamma_2(v) = 0,\ v \in D(L_W^*).$$

It is easy to see that $D(L_W) = D(L_W^*)$ and the other conditions of normality extensions in L^2 can be easily verified. □

5. The spectrum of normal extensions and asymptotical behavior of their eigenvalues

In this section the spectrum of the normal extension L_W of minimal operator L_0 in L^2 generated by linear differential-operator expression (3.1) and boundary conditions (4.1) with unitary operators W and $\widehat{A}^{3/2}W\widehat{A}^{-3/2}$ in H^3 will be investigated.

Now for any $\lambda \in \mathbb{R}$ we define two matrixes as

$$\Delta_1(\lambda) = \begin{pmatrix} \left(1-\lambda^{2/3}\frac{1+i\sqrt{3}}{2}\right)e^{\lambda^{1/3}\left(\frac{\sqrt{3}+i}{2}\right)(b-a)} & \left(1-\lambda^{2/3}\frac{1-i\sqrt{3}}{2}\right)e^{\lambda^{1/3}\left(\frac{-\sqrt{3}+i}{2}\right)(b-a)} & \left(1+\lambda^{2/3}\right)e^{-i\lambda^{1/3}(b-a)} \\ \lambda^{1/3}\left(1+3e^{\lambda^{1/3}\left(\frac{\sqrt{3}+i}{2}\right)(b-a)}\right)\frac{\sqrt{3}+i}{4} & \lambda^{1/3}\left(1+3e^{\lambda^{1/3}\left(\frac{-\sqrt{3}+i}{2}\right)(b-a)}\right)\frac{-\sqrt{3}+i}{4} & \frac{-i\lambda^{1/3}\left(1+3e^{-i\lambda^{1/3}(b-a)}\right)}{2} \\ 1+\lambda^{2/3}\frac{1+i\sqrt{3}}{2} & 1+\lambda^{2/3}\frac{1-i\sqrt{3}}{2} & 1-\lambda^{2/3} \end{pmatrix}$$

$$\Delta_2(\lambda) = \begin{pmatrix} \left(1+\lambda^{2/3}\frac{1+i\sqrt{3}}{2}\right)e^{\lambda^{1/3}\left(\frac{\sqrt{3}+i}{2}\right)(b-a)} & \left(1+\lambda^{2/3}\frac{1-i\sqrt{3}}{2}\right)e^{\lambda^{1/3}\left(\frac{-\sqrt{3}+i}{2}\right)(b-a)} & \left(1-\lambda^{2/3}\right)e^{-i\lambda^{1/3}(b-a)} \\ \lambda^{1/3}\left(3+e^{\lambda^{1/3}\left(\frac{\sqrt{3}+i}{2}\right)(b-a)}\right)\frac{\sqrt{3}+i}{4} & \lambda^{1/3}\left(3+e^{\lambda^{1/3}\left(\frac{-\sqrt{3}+i}{2}\right)(b-a)}\right)\frac{-\sqrt{3}+i}{4} & \frac{-i\lambda^{1/3}\left(3+e^{-i\lambda^{1/3}(b-a)}\right)}{2} \\ 1-\lambda^{2/3}\frac{1+i\sqrt{3}}{2} & 1-\lambda^{2/3}\frac{1-i\sqrt{3}}{2} & 1+\lambda^{2/3}. \end{pmatrix}$$

5.1. Theorem. *Let W and $\widehat{A}^{3/2}W\widehat{A}^{-3/2}$ be unitary operators in H^3 satisfying the condition $\tilde{A}W_2^3(H,(a,b)) \subset W_2^3(H,(a,b)$. The point spectrum of the normal extension L_W has the form $\lambda = \lambda_r + i\lambda_i \in \mathbb{C}$ if and only if $\lambda_r \in \sigma_p(\tilde{A}^3 \otimes E)$, $0 \in \sigma_p(W\Delta_1(\lambda_i) + \Delta_2(\lambda_i))$ and there exists a vector different from the zero vector in the intersection of the eigenspaces $H_{\lambda_r}(\tilde{A}^3 \otimes E)$ and $H_{\lambda_i}(ImL_W)$.*

Proof. Suppose that $\lambda = \lambda_r + i\lambda_i$ is an eigenvalue of the operator L_W. Since L_W is a normal operator, then

$$u_\lambda'''(t) + \tilde{A}^3 u(t) = \lambda u_\lambda(t)$$
$$-u_\lambda'''(t) + \tilde{A}^3 u(t) = \overline{\lambda} u_\lambda(t)$$

and from this

$$u_\lambda'''(t) = i\lambda_i u_\lambda(t)$$
$$\tilde{A}^3 u(t) = \lambda_r u_\lambda(t)$$

are obtained. It is clear that

$$u_\lambda(t) = e^{\alpha_1(t-a)}x_1 + e^{\alpha_2(t-a)}x_2 + e^{\alpha_3(t-a)}x_3 \neq 0,$$

where $\alpha_{k+1} = \lambda_i^{1/3}\left(\cos(\frac{\pi}{6} + \frac{2k\pi}{3}) + i\sin(\frac{\pi}{6} + \frac{2k\pi}{3})\right)$, $k = 0,1,2,\ x_j \in H,\ j = 1,2,3$ and $u_\lambda(t) \in H_{\lambda_r}(\tilde{A} \otimes E) \cap H_{\lambda_i}(\mathrm{Im}L_W)$. Also this eigenvalue provides the boundary conditions (4.1), so

$$(W\Delta_1(\lambda_i) + \Delta_2(\lambda_i))\{x_1, x_2, x_3\} = 0$$

is found. This implies that $0 \in \sigma_p(W\Delta_1(\lambda_i) + \Delta_2(\lambda_i))$.

The converse of the theorem can be easily seen. □

Now we will investigate the point spectrum spectrum of L_W in the special cases $W = \pm E$.

5.2. Lemma. *The function* $f(\lambda) := -\sin(2\lambda) - \sin\lambda\cosh\left(\sqrt{3}\lambda\right) + \sqrt{3}\cos\lambda\sinh\left(\sqrt{3}\lambda\right)$ *has only one root in the interval* $\left(\frac{2n-1}{2}\pi,\ \frac{2n+1}{2}\pi\right)$ *for all* $n \in \mathbb{Z}$.

Proof. In this case for all $n \in \mathbb{Z}$, $f\left(\frac{2n-1}{2}\pi\right) f\left(\frac{2n+1}{2}\pi\right) < 0$ and so there exits a root in the interval $\left(\frac{2n-1}{2}\pi,\ \frac{2n+1}{2}\pi\right)$. But

$$f''(\lambda) = 4\sin 2\lambda - 8\sin\lambda\cosh\left(\sqrt{3}\lambda\right) = 8\sin\lambda\left(\cos\lambda - \cosh\left(\sqrt{3}\lambda\right)\right).$$

This shows that for any interval $\left(\frac{2n-1}{2}\pi,\ \frac{2n+1}{2}\pi\right)$, $n \in \mathbb{Z}$ the function f is convex in a half interval which is $\left(\frac{2n-1}{2}\pi,\ n\pi\right)$ or $\left(n\pi,\ \frac{2n+1}{2}\pi\right)$, and concave in the other half interval. This result implies that the root is unique for every interval $\left(\frac{2n-1}{2}\pi,\ \frac{2n+1}{2}\pi\right)$, $n \in \mathbb{Z}$. □

5.3. Corollary. *The point spectrum of the normal operator* L_E *has the form*

$$\sigma_p(L_E) = \Big\{\lambda_r + i\lambda_i : \lambda_r \in \sigma_p\big(\tilde{A}^3 \otimes E\big),\ \lambda_i \text{ is the unique solution of the equation } f(\lambda) = 0 \text{ in every interval } \left(\left(\frac{2n-1}{b-a}\pi\right)^3, \left(\frac{2n+1}{b-a}\pi\right)^3\right),\ n \in \mathbb{Z}\Big\}.$$

Proof. If $\lambda = \lambda_r + i\lambda_i$ is an arbitrary element in $\sigma_p(L_E)$, then according to the proof of theorem 5.2 there exits an eigenvector $u_\lambda(t) = e^{\alpha_1(t-a)}x_1 + e^{\alpha_2(t-a)}x_2 + e^{\alpha_3(t-a)}x_3 \neq 0$, where $\alpha_{k+1} = \lambda_i^{1/3}\left(\cos(\frac{\pi}{6} + \frac{2k\pi}{3}) + i\sin(\frac{\pi}{6} + \frac{2k\pi}{3})\right)$, $k = 0, 1, 2$, $x_j \in H$, $j = 1, 2, 3$. This vector satisfies the boundary condition (4.1) with $W = E$, and so

$$\begin{pmatrix} e^{\alpha_1(b-a)} & e^{\alpha_2(b-a)} & e^{\alpha_3(b-a)} \\ \alpha_1 e^{\alpha_1(b-a)} + \alpha_1 & \alpha_2 e^{\alpha_2(b-a)} + \alpha_2 & \alpha_3 e^{\alpha_1(b-a)} + \alpha_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

This is possible when the matrix determinant is equal to zero. Consequently, compute this matrix determinant:

$$-2\sqrt{3}i\sin\left(\lambda_i^{1/3}(b-a)\right) + 6i\cos\left(\frac{1}{2}\lambda_i^{1/3}(b-a)\right)\sinh\left(\frac{\sqrt{3}}{2}\lambda_i^{1/3}(b-a)\right) - 2\sqrt{3}i\sin\left(\frac{1}{2}\lambda_i^{1/3}(b-a)\right)\cosh\left(\frac{\sqrt{3}}{2}\lambda_i^{1/3}(b-a)\right) = 0.$$

Therefore, by using Theorem 5.1 and Lemma 5.2 the proof is completed. □

5.4. Lemma. *The function* $g(\lambda) := \cos(2\lambda) - \cos\lambda\cosh\left(\sqrt{3}\lambda\right) + \sqrt{3}\sin\lambda\sinh\left(\sqrt{3}\lambda\right)$ *has only one root in the interval* $\left(\frac{2n-1}{2}\pi,\ \frac{2n+1}{2}\pi\right)$ *for all* $n \in \mathbb{Z}$.

Proof. In this case for all $n \in \mathbb{Z}$, $g\left(\frac{2n-1}{2}\pi\right) g\left(\frac{2n+1}{2}\pi\right) < 0$ and so there exits a root in the interval $\left(\frac{2n-1}{2}\pi,\ \frac{2n+1}{2}\pi\right)$. Also,

$$g'(\lambda) = 4\sin\lambda\left(\cosh\left(\sqrt{3}\lambda\right) - \cos\lambda\right).$$

From this, for any interval $\left(\frac{2n-1}{2}\pi,\ \frac{2n+1}{2}\pi\right)$, $n \in \mathbb{Z}$, $\lambda = n\pi$ is the only local extremum point of the function g. This means that the root is unique for every interval $\left(\frac{2n-1}{2}\pi,\ \frac{2n+1}{2}\pi\right)$, $n \in \mathbb{Z}$. □

5.5. Corollary. *The point spectrum of the normal operator* L_{-E} *has the form*

$$\sigma_p(L_{-E}) = \Big\{\lambda_r + i\lambda_i : \lambda_r \in \sigma_p\Big(\tilde{A}^3 \otimes E\Big),\ \lambda_i \text{ is the unique solution of the equation } g(\lambda) = 0 \text{ in every interval } \left(\left(\frac{2n-1}{b-a}\pi\right)^3, \left(\frac{2n+1}{b-a}\pi\right)^3\right), n \in \mathbb{Z}\Big\}.$$

Proof. The proof of corollary is similar to Corollary 5.3 by using to Lemma 5.4. □

5.6. Corollary. *If* $\dim H = m < +\infty$, *then each normal extension* L_W *has a pure point spectrum and their eigenvalue numbers have the same asymptotics*

$$\lambda_n(\mathrm{Im} L_W) \sim \left(\frac{2n+1}{m(b-a)}\pi\right)^3, \text{ as } n \to \infty.$$

Let now the linear operators $\hat{A}_1$ and $\hat{A}_2$ be defined in H^3 as follows

$$\hat{A}_1 := \begin{pmatrix} (-i\tilde{A}^2+iE)e^{(a-b)\tilde{A}} & -\frac{\sqrt{3}-i}{2}\tilde{A}^2+iE & \frac{\sqrt{3}+i}{2}\tilde{A}^2+iE \\ -\frac{i}{2}\tilde{A}(3e^{(a-b)\tilde{A}}+E) & \frac{\sqrt{3}+i}{4}\tilde{A}\left(e^{\frac{1-i\sqrt{3}}{2}(a-b)\tilde{A}}+3E\right) & \frac{-\sqrt{3}+i}{4}\tilde{A}\left(e^{\frac{1+i\sqrt{3}}{2}(a-b)\tilde{A}}+3E\right) \\ i\tilde{A}+iE & \left(\frac{\sqrt{3}-i}{2}\tilde{A}^2+iE\right)e^{\frac{1-i\sqrt{3}}{2}(a-b)\tilde{A}} & \left(-\frac{\sqrt{3}+i}{2}\tilde{A}^2+iE\right)e^{\frac{1+i\sqrt{3}}{2}(a-b)\tilde{A}} \end{pmatrix},$$

$$\hat{A}_2 := \begin{pmatrix} \left(i\tilde{A}^2+iE\right)e^{(a-b)\tilde{A}} & \frac{\sqrt{3}-i}{2}\tilde{A}^2+iE & -\frac{\sqrt{3}+i}{2}\tilde{A}^2+iE \\ -\frac{i}{2}\tilde{A}\left(e^{(a-b)\tilde{A}}+3E\right) & \frac{\sqrt{3}-i}{4}\tilde{A}\left(e^{\frac{1-i\sqrt{3}}{2}(a-b)\tilde{A}}-E\right) & \frac{\sqrt{3}-i}{4}\tilde{A}\left(e^{\frac{1+i\sqrt{3}}{2}(a-b)\tilde{A}}+E\right) \\ -i\tilde{A}+iE & \left(-\frac{\sqrt{3}+i}{2}\tilde{A}^2+iE\right)e^{\frac{1-i\sqrt{3}}{2}(a-b)\tilde{A}} & \left(\frac{-\sqrt{3}+i}{2}\tilde{A}^2+iE\right)e^{\frac{1+i\sqrt{3}}{2}(a-b)\tilde{A}} \end{pmatrix}.$$

5.7. Theorem. *If* $A^{-1} \in \mathfrak{S}_\infty(H)$, $0 \in \rho(W\hat{A}_1 + \hat{A}_2)$ *and the operator* L_W *is any normal extension of minimal operator* L_0, *then* $L_W^{-1} \in \mathfrak{S}_\infty\left(L^2\right)$.

Proof. In this case, because the equation

$$\begin{aligned} L_W &= \tilde{A} \otimes E_{L_2(a,b)} + iE_H \otimes \left(-i\frac{d_W^3}{dt^3}\right) \\ &= \tilde{A} \otimes E_{L_2(a,b)}\left(E + (i\tilde{A}^{-1} \otimes E_{L_2(a,b)})E_H \otimes \left(-i\frac{d_W^3}{dt^3}\right)\right) \end{aligned}$$

is valid, L_W has the inverse

$$L_W^{-1} = \tilde{A}^{-1} \otimes E_{L_2(a,b)}\left(E + (i\tilde{A}^{-1} \otimes E_{L_2(a,b)})E_H \otimes \left(-i\frac{d_W^3}{dt^3}\right)\right)^{-1}$$

and this operator is bounded. According to Theorem 3.3 and the domain of L_W,

$$\begin{aligned} &L_W^{-1}f(t) \\ &\quad = e^{(a-t)\tilde{A}}x_1 + e^{\frac{1-i\sqrt{3}}{2}(t-b)\tilde{A}}x_2 + e^{\frac{1+i\sqrt{3}}{2}(t-b)\tilde{A}}x_3 + \frac{1}{3}A^{-2}\int_a^t e^{(s-t)A}f(s)\,ds \\ &\qquad + \frac{1-i\sqrt{3}}{6}A^{-2}\int_t^b e^{\frac{1-i\sqrt{3}}{2}(t-s)A}f(s)\,ds + \frac{1+i\sqrt{3}}{6}A^{-2}\int_t^b e^{\frac{1+i\sqrt{3}}{2}(t-s)A}f(s)\,ds, \end{aligned}$$

where

$$\left(W\hat{A}_1+\hat{A}_2\right)\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\left(W\hat{A}_3+\hat{A}_4\right)\begin{pmatrix}\int_a^b e^{(s-b)A}f(s)\,ds\\ \int_a^b e^{\frac{1-i\sqrt{3}}{2}(a-s)A}f(s)\,ds\\ \int_a^b e^{\frac{1+i\sqrt{3}}{2}(a-s)A}f(s)\,ds\end{pmatrix},$$

$$\hat{A}_3:=\begin{pmatrix}\frac{i}{3}\left(A^{-2}-E\right) & 0 & 0\\ \frac{i}{3}A^{-1} & \frac{-\sqrt{3}+i}{12}A^{-1} & \frac{\sqrt{3}+i}{12}A^{-1}\\ 0 & \frac{-\sqrt{3}+i}{6}A^{-2}+\frac{i}{3}E & \frac{\sqrt{3}-i}{6}A^{-2}+\frac{i}{3}E\end{pmatrix},$$

$$\hat{A}_4:=\begin{pmatrix}-\frac{i}{3}\left(A^{-2}+E\right) & 0 & 0\\ \frac{i}{6}A^{-1} & -\frac{\sqrt{3}+i}{4}A^{-1} & \frac{\sqrt{3}+i}{4}A^{-1}\\ 0 & -\frac{\sqrt{3}+i}{6}A^{-2}+\frac{i}{3}E & \frac{\sqrt{3}-i}{6}A^{-2}-\frac{i}{3}E\end{pmatrix}.$$

Because of $A^{-1}\in\mathfrak{S}_\infty(H)$ the operators $\int_a^t e^{(s-t)A}f(s)\,ds$, $\int_t^b e^{\frac{1-i\sqrt{3}}{2}(t-s)A}f(s)\,ds$, $\int_t^b e^{\frac{1+i\sqrt{3}}{2}(t-s)A}f(s)\,ds$ and $\int_a^b e^{(s-b)A}f(s)\,ds$ are compact [8]. Moreover, $0\in\rho(W\hat{A}_1+\hat{A}_2)$ implies that there exits a continuous inverse of $(W\hat{A}_1+\hat{A}_2)$ and so

$$\begin{aligned}L_W^{-1}f(t)\\ &=\left(e^{(a-t)\tilde{A}}\quad e^{\frac{1-i\sqrt{3}}{2}(t-b)\tilde{A}}\quad e^{\frac{1+i\sqrt{3}}{2}(t-b)\tilde{A}}\right)\Delta(W)\begin{pmatrix}\int_a^b e^{(s-b)A}f(s)\,ds\\ \int_a^b e^{\frac{1-i\sqrt{3}}{2}(a-s)A}f(s)\,ds\\ \int_a^b e^{\frac{1+i\sqrt{3}}{2}(a-s)A}f(s)\,ds\end{pmatrix}\\ &\quad+\frac{1}{3}A^{-2}\int_a^t e^{(s-t)A}f(s)\,ds+\frac{1-i\sqrt{3}}{6}A^{-2}\int_t^b e^{\frac{1-i\sqrt{3}}{2}(t-s)A}f(s)\,ds\\ &\quad+\frac{1+i\sqrt{3}}{6}A^{-2}\int_t^b e^{\frac{1+i\sqrt{3}}{2}(t-s)A}f(s)\,ds.\end{aligned}$$

is a compact operator in $L_2(H,(a,b))$, where $\Delta(W):=\left(W\hat{A}_1+\hat{A}_2\right)^{-1}\left(W\hat{A}_3+\hat{A}_4\right)$. □

5.8. Corollary. *If $A^{-1}\in\mathfrak{S}_\infty(H)$, $0\in\rho(W\hat{A}_1+\hat{A}_2)$, L_W is any normal operator and $\lambda\in\rho(L_W)$, then $R_\lambda(L_W)\in\mathfrak{S}_\infty\left(L^2\right)$.* □

5.9. Theorem. *Let A^{-1} be a compact operator in H and $0\in\rho(\hat{A}_1\pm\hat{A}_2)$. Then the relation*

$$\sigma\left(L_{\pm E}\right)=\sigma_p\left(\mathrm{Re}L_{\pm E}\right)+i\sigma_p\left(\mathrm{Im}\left(L_{\pm E}\right)\right)$$

is satisfied.

Proof. Under the assumptions of the theorem there exit compact inverses of L_E and L_{-E} from Theorem 5.7. Also, by using corollaries 5.3 and 5.5, the relation is correct. □

5.10. Theorem. *If $A^{-1} \in \mathfrak{S}_\infty(H)$, $0 \in \rho(\hat{A}_1 \pm \hat{A}_2)$ and $\lambda_n(A) \sim cn^\alpha$, $n \to +\infty$, $0 < c, \alpha < +\infty$, then the asymptotic behavior of $L_{\pm E}$ has the form*

$$|\lambda_n(L_{\pm E})| \sim \beta n^{\frac{3\alpha}{3+\alpha}},\ \beta > 0,\ n \to +\infty.$$

Proof. Firstly, assume that $\lambda_m(A) \sim cm^\alpha$, $m \to +\infty$, $0 < c, \alpha < +\infty$. For sufficiently large m we have

$$\left(c^2 m^{2\alpha} + \left(\frac{2n-1}{b-a}\pi\right)^6\right)^{1/2} \le |\lambda(L_{\pm E})| \le \left(c^2 m^{2\alpha} + \left(\frac{2n+1}{b-a}\pi\right)^6\right)^{1/2},$$
$$n \in \mathbb{Z},\ m \in \mathbb{N}.$$

from Corollary 5.3 and Corollary 5.5. Hence for sufficiently large m the relation

$$|\lambda(L_{\pm E})| \sim \left(c^2 m^{2\alpha} + d^6 (2n+1)^6\right)^{1/2},\ n \in \mathbb{Z},\ m \in \mathbb{N},\ d = \frac{\pi}{b-a}$$

is true. Now we define

$$N(\lambda; T) := \operatorname{card}\{n : |\lambda_n(T)| \leqslant |\lambda|\},$$

that is,

$$N(\lambda; T) := \sum_{0 \leqslant |\lambda_n(T)| \leqslant |\lambda|} 1,\ \lambda \in \mathbb{C},$$

which gives the number of eigenvalues of a linear closed operator T in any Hilbert space with the modules of the eigenvalues less than or equal to $|\lambda|$. This function takes values in the set of non-negative integers, and in the case where T is unbounded it is non-decreasing and tends to $+\infty$ as $|\lambda| \to \infty$.

In this case it is easy see that

$$N(\lambda, L_{\pm E}) = 2N_+(\lambda, L_{\pm E}) - N(\lambda, A),$$

where $N_+(\lambda, L_{\pm E}) := \sum_{\substack{|\lambda(L_{\pm E})| \le \lambda \\ \operatorname{Im}\lambda(L_{\pm E}) \ge 0}} 1,\ \lambda > 0.$

Moreover, using the methods established in [5] or [6] it can be obtained that

$$|\lambda_n(L_{\pm E})| \sim \beta n^{\frac{3\alpha}{3+\alpha}},\ \beta > 0,\ n \to +\infty. \qquad \square$$

5.11. Theorem. *Suppose that $A^* = A \ge E$, $A^{-1} \in \mathfrak{S}_\infty(H)$, $\Delta(W) - \Delta(E) \in \mathfrak{S}_\infty(H^3)$ and $s_n(\Delta(W) - \Delta(E)) = O\left(n^{-\frac{3\alpha}{3+\alpha}}\right)$, $n \to +\infty$. Then for an eigenvalue of the operator L_W^{-1} the relation*

$$\left|\lambda_n\left(L_W^{-1}\right)\right| = O\left(n^{-\frac{3\alpha}{3+\alpha}}\right),\ n \to +\infty$$

is true.

Proof. From the proof of Theorem 5.7

$$(L_W^{-1} - L_E^{-1})f(t) = \begin{pmatrix} e^{(a-t)\tilde{A}} & e^{\frac{1-i\sqrt{3}}{2}(t-b)\tilde{A}} & e^{\frac{1+i\sqrt{3}}{2}(t-b)\tilde{A}} \end{pmatrix}$$
$$\times (\Delta(W) - \Delta(E)) \begin{pmatrix} \int_a^b e^{(s-b)A} f(s)\, ds \\ \int_a^b e^{\frac{1-i\sqrt{3}}{2}(a-s)A} f(s)\, ds \\ \int_a^b e^{\frac{1+i\sqrt{3}}{2}(a-s)A} f(s)\, ds \end{pmatrix}$$

is valid. This implies that for all $n \in \mathbb{N}$ the equality

$$s_n\left(L_W^{-1} - L_E^{-1}\right) \leq d\, s_n\left(\Delta\left(W\right)\right),\ d > 0$$

holds. Because of this and a known result for the numbers s [6],

$$\begin{aligned} s_{2n-1}\left(L_W^{-1}\right) &= s_{2n-1}\left(L_W^{-1} - L_E^{-1} + L_E^{-1}\right) \\ &\leq s_n\left(L_W^{-1} - L_E^{-1}\right) + s_n\left(L_E^{-1}\right) \\ &\leq d\, s_n\left(\Delta\left(W\right)\right) + s_n\left(L_E^{-1}\right),\ n \geq 1 \end{aligned}$$

is obtained. Also $\lambda_n(A) \sim cn^{\alpha},\ n \to +\infty,\ 0 < c, \alpha < +\infty,\ s_n(\Delta(W) - \Delta(E)) = O\left(n^{-\frac{3\alpha}{3+\alpha}}\right)$, $n \to +\infty$; and from the theorem $s_n\left(L_E^{-1}\right) \sim \beta n^{-\frac{3\alpha}{3+\alpha}}$, $n \to +\infty$, $\beta > 0$, so that

$$0 \leq \frac{s_{2n-1}\left(L_W^{-1}\right)}{n^{-\frac{3\alpha}{3+\alpha}}} \leq d\,\frac{s_n\left(\Delta\left(W\right)\right)}{n^{-\frac{3\alpha}{3+\alpha}}} + \frac{s_n\left(L_E^{-1}\right)}{n^{-\frac{3\alpha}{3+\alpha}}} \leq h,\ \ h > 0,\ n \to +\infty$$

i.e.,

$$s_{2n-1}\left(L_W^{-1}\right) = O\left(n^{-\frac{3\alpha}{3+\alpha}}\right),\ n \to +\infty.$$

In the same way, the relation

$$s_{2n}\left(L_W^{-1}\right) = O\left(n^{-\frac{3\alpha}{3+\alpha}}\right),\ n \to +\infty$$

is found. On the other hand the relations

$$\frac{s_{2n-1}\left(L_W^{-1}\right)}{(2n-1)^{-\frac{3\alpha}{3+\alpha}}} = \frac{s_{2n-1}\left(L_W^{-1}\right)}{n^{-\frac{3\alpha}{3+\alpha}}}\left(\frac{2n-1}{n}\right)^{\frac{3\alpha}{3+\alpha}},$$

$$\frac{s_{2n}\left(L_W^{-1}\right)}{(2n)^{-\frac{3\alpha}{3+\alpha}}} = \frac{s_{2n}\left(L_W^{-1}\right)}{n^{-\frac{3\alpha}{3+\alpha}}}\left(\frac{2n}{n}\right)^{\frac{3\alpha}{3+\alpha}},$$

are true. These may be correlated as follows:

$$s_{2n-1}\left(L_W^{-1}\right) = O\left((2n-1)^{-\frac{3\alpha}{3+\alpha}}\right),\ n \to +\infty,$$

$$s_{2n}\left(L_W^{-1}\right) = O\left((2n)^{-\frac{3\alpha}{3+\alpha}}\right),\ n \to +\infty.$$

Since L_W^{-1} is a normal operator,

$$\left|\lambda_n\left(L_W^{-1}\right)\right| = s_n\left(L_W^{-1}\right) = O\left(n^{-\frac{3\alpha}{3+\alpha}}\right),\ n \to +\infty$$

is valid. □

5.12. Example. The operator $Nu\left(t,x\right) := \frac{\partial^3 u(t,x)}{\partial t^3} - \frac{\partial^6 u(t,x)}{\partial x^6}$ with the following boundary conditions

$$\begin{aligned} &u\left(a,x\right) = u\left(b,x\right) = 0,\ c \leq x \leq d, \\ &\frac{\partial u\left(a,x\right)}{\partial t} + \frac{\partial u\left(b,x\right)}{\partial t} = 0,\ c \leq x \leq d, \\ &u\left(t,c\right) = u\left(t,d\right) = 0,\ a \leq t \leq b, \\ &\frac{\partial^2 u\left(t,c\right)}{\partial x^2} = \frac{\partial^2 u\left(t,d\right)}{\partial x^2} = 0,\ a \leq t \leq b, \\ &\frac{\partial^4 u\left(t,c\right)}{\partial x^4} = \frac{\partial^4 u\left(t,d\right)}{\partial x^4} = 0,\ a \leq t \leq b \end{aligned}$$

is an extension of the minimal operator L_0 generated by the differential expression

$$l(\cdot) = \frac{\partial^3}{\partial t^3} - \frac{\partial^6}{\partial x^6}$$

in the Hilbert space $L^2((a,b)\times(c,d))$, $a < b$, $c < d$, $a, b, c, d \in \mathbb{R}$. In this case, the extension N can be written in the form

$$Nu(t,x) = \frac{\partial^3 u(t,x)}{\partial t^3} + A^3 u(t,x),$$
$$u\,(a,x) = u\,(b,x) = 0, \quad c \le x \le d,$$
$$\frac{\partial u(a,x)}{\partial t} + \frac{\partial u(b,x)}{\partial t} = 0,\; c \le x \le d,$$

in $L^2(H,(a,b))$, where $H = L^2(c,d)$, $A : W_2^2(c,d) \cap \overset{0}{W}{}_2^1(c,d) \subset L^2(c,d) \to L^2(c,d)$,

$$Au(t,x) = -\frac{\partial^2 u\,(t,x)}{\partial x^2},$$
$$u\,(t,c) = u\,(t,d) = 0,\; a \le t \le b.$$

On the other hand by the Theorem 4.1 ($W = E$) the extension N is a normal operator. It is know that $A = A^* \ge 2E$, $A^{-1} \in \mathfrak{S}_\infty(L^2(c,d))$ and that the eigenvalues of A have the following asymptotic at infinity [5]

$$\lambda_n\,(A) \sim \frac{\pi^2 n^2}{(d-c)^2},\; n \to +\infty.$$

Hence, according to Theorem 5.10 the asymptotic behavior of the eigenvalues of the operator N is obtained as

$$|\lambda_n\,(N)| \sim \alpha n^{\frac{6}{5}},\; \alpha > 0,\; n \to +\infty.$$

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