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SOME SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING THE GENERALIZED SRIVASTAVA-ATTIYA OPERATOR

Zhi-Hong Liu*, Zhi-Gang Wang†‡, Feng-Hua Wen§ and Yong Sun¶

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Abstract

In the present paper, we introduce and investigate some new subclasses of multivalent analytic functions involving the generalized Srivastava-Attiya operator. Such results as inclusion relationships, subordination and superordination properties, integral-preserving properties and convolution properties are proved.

Keywords: Analytic functions, Multivalent functions, Differential subordination, Superordination, Hadamard product (or convolution), Generalized Srivastava-Attiya operator.

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1. Introduction

Let $\mathcal{A}_p(n)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \qquad (p,\ n \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z :\ z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

*Department of Mathematics, Honghe University, Mengzi 661100, Yunnan, People's Republic of China. E-mail: liuzhihongmath@163.com

†School of Mathematics and Statistics, Anyang Normal University, Anyang 455002, Henan, People's Republic of China. E-mail: wangmath@163.com

‡Corresponding Author.

§School of Econometrics and Management, Changsha University of Science and Technology, Changsha 410114, Hunan, People's Republic of China. E-mail: wfh@126.com

¶Department of Mathematics, Huaihua University, Huaihua 418008, Hunan, People's Republic of China. E-mail: sy785153@126.com

For simplicity, we write

$$\mathcal{A}_1(1) := \mathcal{A}.$$

Also let $\mathcal{H}[a,n]$ be the class of analytic functions of the form

$$h(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (z \in \mathbb{U}).$$

Let $f,\ g \in \mathcal{A}_p(n)$, where f is given by (1.1) and g is defined by

$$g(z) = z^p + \sum_{k=n}^{\infty} b_{p+k} z^{p+k}.$$

Then the Hadamard product (or convolution) $f*g$ of the functions f and g is defined by

$$(f*g)(z) := z^p + \sum_{k=n}^{\infty} a_{p+k} b_{p+k} z^{p+k} =: (g*f)(z).$$

Let $\mathcal{P}$ denote the class of functions of the form

$$p\,(z) = 1 + \sum_{k=n}^{\infty} p_k z^k \quad (n \in \mathbb{N}),$$

which are analytic and convex in $\mathbb{U}$ and satisfy the condition

$$\Re(p\,(z)) > 0 \quad (z \in \mathbb{U}).$$

For two functions f and g, analytic in $\mathbb{U}$, the function f is said to be subordinate to g in $\mathbb{U}$, or the function g is said to be superordinate to f in $\mathbb{U}$, and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g\big(\omega(z)\big) \quad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

The following we recall a general Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ defined by (*cf., e.g.,* [22, p. 121 *et sep.*])

$$\Phi(z,s,a) := \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s}$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \text{ when } |z| < 1;\ \Re(s) > 1 \text{ when } |z| = 1),$$

where, as usual,

$$\mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N} \quad (\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\};\ \mathbb{N} := \{1, 2, 3, \ldots\}).$$

Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ can be found in recent investigations by (for example) Choi and Srivastava [1], Ferreira and López [3], Garg *et al.* [4], Lin *et al.* [6], Luo and Srivastava [10], Wen and Liu [26], Wen and Yang [27] and others.

Recently, Srivastava and Attiya [21] (see also [2, 5, 8, 9, 14, 15, 16, 17, 18, 23, 24, 25, 28, 29]) introduced and investigated the linear operator

$$\mathcal{J}_{s,\ b}(f):\ \mathcal{A} \longrightarrow \mathcal{A}$$

defined, in terms of the Hadamard product (or convolution), by

$$\mathcal{J}_{s,\ b}f(z) := G_{s,\ b}(z) * f(z) \quad (z \in \mathbb{U};\ b \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C};\ f \in \mathcal{A}), \tag{1.2}$$

where, for convenience,

$$G_{s,\ b}(z) := (1+b)^s[\Phi(z,s,b) - b^{-s}] \quad (z \in \mathbb{U}). \tag{1.3}$$

It is easy to observe from (1.2) and (1.3) that

$$\mathcal{J}_{s,\ b}f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b}\right)^s a_k z^k.$$

By setting

$$f_{s,\ b}^{p,\ n}(z) := z^p + \sum_{k=n}^{\infty} \left(\frac{p+b}{p+k+b}\right)^s z^{p+k} \quad (z \in \mathbb{U};\ n \in \mathbb{N}).$$

Then, motivated essentially by the above-mentioned Srivastava-Attiya operator, we introduce the operator

$$\mathcal{J}_{s,\ b}^{p,\ n}(f):\ \mathcal{A}_p(n) \longrightarrow \mathcal{A}_p(n),$$

which is defined as

$$\mathcal{J}_{s,\ b}^{p,\ n}f(z) := f_{s,\ b}^{p,\ n}(z) * f(z) = z^p + \sum_{k=n}^{\infty} \left(\frac{p+b}{p+k+b}\right)^s a_{p+k} z^{p+k}, \tag{1.4}$$

where (and throughout this paper unless otherwise mentioned) the parameters $s,\ b,\ p$ and n are constrained as follows:

$$s \in \mathbb{C};\ \ b \in \mathbb{C} \setminus \mathbb{Z}_0^- \ \text{ and } \ p,\, n \in \mathbb{N}.$$

It is easily verified from (1.4) that

$$z\left(\mathcal{J}_{s+1,\ b}^{p,\ n}f\right)'(z) = (p+b)\mathcal{J}_{s,\ b}^{p,\ n}f(z) - b\mathcal{J}_{s+1,\ b}^{p,\ n}f(z). \tag{1.5}$$

In this paper, by making use of the operator $\mathcal{J}_{s,\ b}^{p,\ n}$ and the above-mentioned principle of subordination between analytic functions, we introduce and investigate the following subclasses of the class $\mathcal{A}_p(n)$ of p-valent analytic functions.

1.1. Definition. A function $f \in \mathcal{A}_p(n)$ is said to be *in the class* $\mathcal{S}_{s,\ b}^{p,\ n}(\eta;\phi)$ if it satisfies the subordination condition

$$\frac{1}{p-\eta}\left(\frac{z\left(\mathcal{J}_{s,\ b}^{p,\ n}f\right)'(z)}{\mathcal{J}_{s,\ b}^{p,\ n}f(z)} - \eta\right) \prec \phi(z) \quad (z \in \mathbb{U};\ 0 \leqq \eta < p;\ \phi \in \mathcal{P}). \tag{1.6}$$

1.2. Definition. A function $f \in \mathcal{A}_p(n)$ is said to be *in the class* $\mathcal{K}_{s,\ b}^{p,\ n}(\lambda;\phi)$ if it satisfies the subordination condition

$$(1-\lambda)\frac{\mathcal{J}_{s+1,\ b}^{p,\ n}f(z)}{z^p} + \lambda\frac{\mathcal{J}_{s,\ b}^{p,\ n}f(z)}{z^p} \prec \phi(z) \quad (z \in \mathbb{U};\ \lambda \in \mathbb{C};\ \phi \in \mathcal{P}). \tag{1.7}$$

In the present paper, we aim at proving such results as inclusion relationships, subordination and superordination properties, integral-preserving properties and convolution properties for the classes $\mathcal{S}_{s,\ b}^{p,\ n}(\eta;\phi)$ and $\mathcal{K}_{s,\ b}^{p,\ n}(\lambda;\phi)$.

2. Preliminary results

In order to prove our main results, we need the following lemmas.

2.1. Lemma. (see [11]) *Let $\vartheta,\ \gamma \in \mathbb{C}$. Suppose that φ is convex and univalent in $\mathbb{U}$ with*

$$\varphi(0) = 1 \text{ and } \Re(\vartheta\varphi(z) + \gamma) > 0 \quad (z \in \mathbb{U}).$$

If $\mathfrak{p}$ is analytic in $\mathbb{U}$ with $\mathfrak{p}(0) = 1$, then the following subordination

$$\mathfrak{p}(z) + \frac{z\mathfrak{p}'(z)}{\vartheta\mathfrak{p}(z) + \gamma} \prec \varphi(z) \quad (z \in \mathbb{U})$$

implies that

$$\mathfrak{p}(z) \prec \varphi(z) \quad (z \in \mathbb{U}).$$ □

2.2. Lemma. (see [12]) *Let the function Ω be analytic and convex (univalent) in $\mathbb{U}$ with $\Omega(0) = 1$. Suppose also that the function Θ given by*

$$\Theta(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots$$

is analytic in $\mathbb{U}$. If

$$\Theta(z) + \frac{z\Theta'(z)}{\zeta} \prec \Omega(z) \quad (\Re(\zeta) > 0;\ \zeta \neq 0;\ z \in \mathbb{U}), \tag{2.1}$$

then

$$\Theta(z) \prec \chi(z) = \frac{\zeta}{n} z^{-\frac{\zeta}{n}} \int_0^z t^{\frac{\zeta}{n}-1} h(t)\, dt \prec \Omega(z) \quad (z \in \mathbb{U}),$$

and χ is the best dominant of (2.1). □

Denote by Q the set of all functions f that are analytic and injective on $\overline{\mathbb{U}} - E(f)$, where

$$E(f) = \left\{\varepsilon \in \partial\mathbb{U} :\ \lim_{z\to\varepsilon} f(z) = \infty\right\},$$

and such that $f'(\varepsilon) \neq 0$ for $\varepsilon \in \partial\mathbb{U} - E(f)$.

2.3. Lemma. (see [13]) *Let q be convex univalent in $\mathbb{U}$ and $\kappa \in \mathbb{C}$. Further assume that $\Re(\overline{\kappa}) > 0$. If*

$$p \in \mathcal{H}[q(0), 1] \cap Q,$$

and $p + \kappa z p'$ is univalent in $\mathbb{U}$, then

$$q(z) + \kappa z q'(z) \prec p(z) + \kappa z p'(z)$$

implies $q \prec p$ and q is the best subordinant. □

2.4. Lemma. (see [19]) *Let q be a convex univalent function in $\mathbb{U}$ and let $\sigma,\ \eta \in \mathbb{C}$ with*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0,\ -\Re\left(\frac{\sigma}{\eta}\right)\right\}.$$

If p is analytic in $\mathbb{U}$ and

$$\sigma p(z) + \eta z p'(z) \prec \sigma q(z) + \eta z q'(z),$$

then $p \prec q$ and q is the best dominant. □

2.5. Lemma. (see [20]) *Let the function Υ be analytic in $\mathbb{U}$ with*

$$\Upsilon(0) = 1 \text{ and } \Re(\Upsilon(z)) > \frac{1}{2} \quad (z \in \mathbb{U}).$$

Then, for any function Ψ analytic in $\mathbb{U}$, $(\Upsilon * \Psi)(\mathbb{U})$ is contained in the convex hull of $\Psi(\mathbb{U})$. □

3. Properties of the function class $\mathcal{S}^{p,\ n}_{s,\ b}(\eta;\phi)$

We begin by stating the following inclusion relationship for the function class $\mathcal{S}^{p,\ n}_{s,\ b}(\eta;\phi)$.

3.1. Theorem. *Let* $0 \leqq \eta < p$ *and* $\phi \in \mathcal{P}$ *with*

$$\Re\left(\phi(z)\right) > \max\left\{0,\ -\frac{\Re(b)+\eta}{p-\eta}\right\} \quad (z \in \mathbb{U}). \tag{3.1}$$

Then

$$\mathcal{S}^{p,\ n}_{s,\ b}(\eta;\phi) \subset \mathcal{S}^{p,\ n}_{s+1,\ b}(\eta;\phi). \tag{3.2}$$

Proof. Let $f \in \mathcal{S}^{p,\ n}_{s,\ b}(\eta;\phi)$ and suppose that

$$\psi(z) := \frac{1}{p-\eta}\left(\frac{z\left(\mathcal{J}^{p,\ n}_{s+1,\ b}f\right)'(z)}{\mathcal{J}^{p,\ n}_{s+1,\ b}f(z)} - \eta\right) \quad (z \in \mathbb{U}). \tag{3.3}$$

Then ψ is analytic in $\mathbb{U}$ with $\psi(0) = 1$. Combining (1.5) and (3.3), we easily find that

$$(p+b)\frac{\mathcal{J}^{p,\ n}_{s,\ b}f(z)}{\mathcal{J}^{p,\ n}_{s+1,\ b}f(z)} = (p-\eta)\psi(z) + b + \eta. \tag{3.4}$$

Differentiating both sides of (3.4) with respect to z logarithmically and using (3.3), we have

$$\frac{1}{p-\eta}\left(\frac{z\left(\mathcal{J}^{p,\ n}_{s,\ b}f\right)'(z)}{\mathcal{J}^{p,\ n}_{s,\ b}f(z)} - \eta\right) = \psi(z) + \frac{z\psi'(z)}{(p-\eta)\psi(z)+b+\eta} \prec \phi(z). \tag{3.5}$$

By noting that (3.1) holds, an application of Lemma 2.1 to (3.5) yields

$$\psi(z) = \frac{1}{p-\eta}\left(\frac{z\left(\mathcal{J}^{p,\ n}_{s+1,\ b}f\right)'(z)}{\mathcal{J}^{p,\ n}_{s+1,\ b}f(z)} - \eta\right) \prec \phi(z),$$

that is $f \in \mathcal{S}^{p,\ n}_{s+1,\ b}(\eta;\phi)$, which implies that the assertion (3.2) of Theorem 3.1 holds. □

Next, we prove some integral-preserving properties for the function class $\mathcal{S}^{p,\ n}_{s,\ b}(\eta;\phi)$.

3.2. Theorem. *Let* $f \in \mathcal{S}^{p,\ n}_{s,\ b}(\eta;\phi)$ *with*

$$\Re((p-\eta)\phi(z) + \mu + \eta) > 0 \quad (z \in \mathbb{U};\ \mu > -p).$$

Then the integral operator F *defined by*

$$F(z) := \frac{\mu+p}{z^{\mu}}\int_0^z t^{\mu-1}f(t)dt \quad (z \in \mathbb{U};\ \mu > -p) \tag{3.6}$$

belongs to the class $\mathcal{S}^{p,\ n}_{s,\ b}(\eta;\phi)$.

Proof. Let $f \in \mathcal{S}^{p,\ n}_{s,\ b}(\eta;\phi)$. Then, from (3.6), we find that

$$z\left(\mathcal{J}^{p,\ n}_{s,\ b}F\right)'(z) + \mu\mathcal{J}^{p,\ n}_{s,\ b}F(z) = (\mu+p)\mathcal{J}^{p,\ n}_{s,\ b}f(z). \tag{3.7}$$

By setting

$$q(z) := \frac{1}{p-\eta}\left(\frac{z\left(\mathcal{J}^{p,\ n}_{s,\ b}F\right)'(z)}{\mathcal{J}^{p,\ n}_{s,\ b}F(z)} - \eta\right), \tag{3.8}$$

we observe that q is analytic in $\mathbb{U}$ with $q(0)=1$. It follows from (3.7) and (3.8) that

$$\mu+\eta+(p-\eta)q(z)=(\mu+p)\frac{\mathcal{J}_{s,\ b}^{p,\ n}f(z)}{\mathcal{J}_{s,\ b}^{p,\ n}F(z)}. \tag{3.9}$$

Differentiating both sides of (3.9) with respect to z logarithmically and using (3.8), we get

$$q(z)+\frac{zq'(z)}{\mu+\eta+(p-\eta)q(z)}=\frac{1}{p-\eta}\left(\frac{z\left(\mathcal{J}_{s,\ b}^{p,\ n}f\right)'(z)}{\mathcal{J}_{s,\ b}^{p,\ n}f(z)}-\eta\right)\prec\phi(z). \tag{3.10}$$

Since

$$\Re((p-\eta)\phi(z)+\mu+\eta)>0 \quad (z\in\mathbb{U}),$$

an application of Lemma 2.1 to (3.10) yields

$$\frac{1}{p-\eta}\left(\frac{z\left(\mathcal{J}_{s,\ b}^{p,\ n}F\right)'(z)}{\mathcal{J}_{s,\ b}^{p,\ n}F(z)}-\eta\right)\prec\phi(z),$$

and we readily deduce that the assertion of Theorem 3.2 holds true. □

3.3. Theorem. *Let $f\in\mathcal{S}_{s,\ b}^{p,\ n}(\eta;\phi)$ with*

$$\Re((p-\eta)\delta\,\phi(z)+\mu+\eta\,\delta)>0 \quad (z\in\mathbb{U};\ \delta\neq 0).$$

Then the function $K\in\mathcal{A}_p(n)$ defined by

$$\mathcal{J}_{s,\ b}^{p,\ n}K(z):=\left(\frac{\mu+p\,\delta}{z^{\mu}}\int_0^z t^{\mu-1}\left(\mathcal{J}_{s,\ b}^{p,\ n}f(t)\right)^{\delta}dt\right)^{1/\delta} \quad (z\in\mathbb{U}) \tag{3.11}$$

belongs to the class $\mathcal{S}_{s,\ b}^{p,\ n}(\eta;\phi)$.

Proof. Let $f\in\mathcal{S}_{s,\ b}^{p,\ n}(\eta;\phi)$. We easily find from (3.11) that

$$z\left[\left(\mathcal{J}_{s,\ b}^{p,\ n}K(z)\right)^{\delta}\right]'+\mu\left(\mathcal{J}_{s,\ b}^{p,\ n}K(z)\right)^{\delta}=(\mu+p\,\delta)\left(\mathcal{J}_{s,\ b}^{p,\ n}f(z)\right)^{\delta}. \tag{3.12}$$

By putting

$$\varrho(z):=\frac{1}{p-\eta}\left(\frac{z\left(\mathcal{J}_{s,\ b}^{p,\ n}K\right)'(z)}{\mathcal{J}_{s,\ b}^{p,\ n}K(z)}-\eta\right) \quad (z\in\mathbb{U}), \tag{3.13}$$

in view of (3.12) and (3.13), we have

$$\mu+\eta\,\delta+(p-\eta)\delta\,\varrho(z)=(\mu+p\,\delta)\left(\frac{\mathcal{J}_{s,\ b}^{p,\ n}f(z)}{\mathcal{J}_{s,\ b}^{p,\ n}K(z)}\right)^{\delta}. \tag{3.14}$$

Making use of (3.11), (3.13) and (3.14), we get

$$\varrho(z)+\frac{z\varrho'(z)}{\mu+\eta\,\delta+(p-\eta)\delta\,\varrho(z)}=\frac{1}{p-\eta}\left(\frac{z\left(\mathcal{J}_{s,\ b}^{p,\ n}f\right)'(z)}{\mathcal{J}_{s,\ b}^{p,\ n}f(z)}-\eta\right)\prec\phi(z). \tag{3.15}$$

Since

$$\Re((p-\eta)\delta\,\phi(z)+\mu+\eta\,\delta)>0 \quad (z\in\mathbb{U}),$$

it follows from (3.15) and Lemma 2.1 that

$$\varrho(z)\prec\phi(z) \quad (z\in\mathbb{U}),$$

that is $K \in \mathcal{S}_{s,\ b}^{p,\ n}(\eta;\phi)$. This completes the proof of Theorem 3.3. □

Now, we derive certain convolution properties for the class $\mathcal{S}_{s,\ b}^{p,\ n}(\eta;\phi)$.

3.4. Theorem. *Let $f \in \mathcal{S}_{s,\ b}^{p,\ n}(\eta;\phi)$. Then*

$$f(z) = \left[z^p \cdot \exp\left((p-\eta)\int_0^z \frac{\phi(\omega(\xi))-1}{\xi}\,d\xi\right)\right] * \left(z^p + \sum_{k=n}^{\infty}\left(\frac{p+k+b}{p+b}\right)^s z^{p+k}\right), \tag{3.16}$$

where ω is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

Proof. Suppose that $f \in \mathcal{S}_{s,\ b}^{p,\ n}(\eta;\phi)$. We know that the subordination condition (1.6) can be written as follows:

$$\frac{z\left(\mathcal{J}_{s,\ b}^{p,\ n}f\right)'(z)}{\mathcal{J}_{s,\ b}^{p,\ n}f(z)} = (p-\eta)\phi(\omega(z)) + \eta, \tag{3.17}$$

where ω is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

We now find from (3.17) that

$$\frac{\left(\mathcal{J}_{s,\ b}^{p,\ n}f\right)'(z)}{\mathcal{J}_{s,\ b}^{p,\ n}f(z)} - \frac{p}{z} = (p-\eta)\frac{\phi(\omega(z))-1}{z}, \tag{3.18}$$

which, upon integration, yields

$$\log\left(\frac{\mathcal{J}_{s,\ b}^{p,\ n}f(z)}{z^p}\right) = (p-\eta)\int_0^z \frac{\phi(\omega(\xi))-1}{\xi}\,d\xi. \tag{3.19}$$

It follows from (3.19) that

$$\mathcal{J}_{s,\ b}^{p,\ n}f(z) = z^p \cdot \exp\left((p-\eta)\int_0^z \frac{\phi(\omega(\xi))-1}{\xi}\,d\xi\right). \tag{3.20}$$

The assertion (3.16) of Theorem 3.4 can now easily be derived from (1.4) and (3.20). □

3.5. Theorem. *Let $f \in \mathcal{A}_p(n)$ and $\phi \in \mathcal{P}$. Then $f \in \mathcal{S}_{s,\ b}^{p,\ n}(\eta;\phi)$ if and only if*

$$\begin{aligned} &\frac{1}{z}\left\{f * \left\{p\,z^p + \sum_{k=n}^{\infty}(p+k)\left(\frac{p+b}{p+k+b}\right)^s z^{p+k}\right.\right.\\ &\qquad \left.\left. - \left[(p-\eta)\phi\left(e^{i\theta}\right)+\eta\right]\left(z^p + \sum_{k=n}^{\infty}\left(\frac{p+b}{p+k+b}\right)^s z^{p+k}\right)\right\}\right\} \neq 0 \\ &\qquad\qquad \left(z \in \mathbb{U};\ 0 \leqq \theta < 2\pi\right). \end{aligned} \tag{3.21}$$

Proof. Suppose that $f \in \mathcal{S}_{s,\ b}^{p,\ n}(\eta;\phi)$. We know that (1.6) is equivalent to

$$\frac{1}{p-\eta}\left(\frac{z\left(\mathcal{J}_{s,\ b}^{p,\ n}f\right)'(z)}{\mathcal{J}_{s,\ b}^{p,\ n}f(z)} - \eta\right) \neq \phi\left(e^{i\theta}\right) \quad (z \in \mathbb{U};\ 0 \leqq \theta < 2\pi). \tag{3.22}$$

It is easy to see that the condition (3.22) can be written as follows:

$$\frac{1}{z}\left\{z\left(\mathcal{J}_{s,\ b}^{p,\ n}f\right)'(z) - \left[(p-\eta)\phi\left(e^{i\theta}\right)+\eta\right]\mathcal{J}_{s,\ b}^{p,\ n}f(z)\right\} \neq 0 \quad (z \in \mathbb{U};\ 0 \leqq \theta < 2\pi). \tag{3.23}$$

On the other hand, we find from (1.4) that

$$z\left(\mathcal{J}_{s,\ b}^{p,\ n}f\right)'(z)=p\,z^p+\sum_{k=n}^{\infty}(p+k)\left(\frac{p+b}{p+k+b}\right)^s a_{p+k}z^{p+k}. \tag{3.24}$$

Combining (1.4), (3.23) and (3.24), we readily get the convolution property (3.21) asserted by Theorem 3.5. □

4. Properties of the function class $\mathcal{K}_{s,\ b}^{p,\ n}(\lambda;\phi)$

In this section, we first derive the following subordination property.

4.1. Theorem. *Let* $f\in\mathcal{K}_{s,\ b}^{p,\ n}(\lambda;\phi)$ *with* $\Re(\lambda)>0$. *Then*

$$\frac{\mathcal{J}_{s+1,\ b}^{p,\ n}f(z)}{z^p}\prec\frac{p+b}{n\lambda}z^{-\frac{p+b}{n\lambda}}\int_0^z t^{\frac{p+b}{n\lambda}-1}\phi(t)\,dt\prec\phi(z). \tag{4.1}$$

Proof. Let $f\in\mathcal{K}_{s,\ b}^{p,\ n}(\lambda;\phi)$ and suppose that

$$h(z):=\frac{\mathcal{J}_{s+1,\ b}^{p,\ n}f(z)}{z^p}\quad(z\in\mathbb{U}). \tag{4.2}$$

Then h is analytic in $\mathbb{U}$. By virtue of (1.5), (1.7) and (4.2), we find that

$$h(z)+\frac{\lambda}{p+b}zh'(z)=(1-\lambda)\frac{\mathcal{J}_{s+1,\ b}^{p,\ n}f(z)}{z^p}+\lambda\frac{\mathcal{J}_{s,\ b}^{p,\ n}f(z)}{z^p}\prec\phi(z). \tag{4.3}$$

Thus, an application of Lemma 2.2 to (4.3) yields the assertion (4.1) of Theorem 4.1. □

In view of Theorem 4.1, we easily get the following inclusion relationship.

4.2. Corollary. *Let* $\Re(\lambda)>0$. *Then*

$$\mathcal{K}_{s,\ b}^{p,\ n}(\lambda;\phi)\subset\mathcal{K}_{s,\ b}^{p,\ n}(0;\phi).$$ □

Now, we give another inclusion relationship for the function class $\mathcal{K}_{s,\ b}^{p,\ n}(\lambda;\phi)$.

4.3. Theorem. *Let* $\lambda_2>\lambda_1\geqq 0$. *Then*

$$\mathcal{K}_{s,\ b}^{p,\ n}(\lambda_2;\phi)\subset\mathcal{K}_{s,\ b}^{p,\ n}(\lambda_1;\phi).$$

Proof. Suppose that $f\in\mathcal{K}_{s,\ b}^{p,\ n}(\lambda_2;\phi)$. It follows that

$$(1-\lambda_2)\frac{\mathcal{J}_{s+1,\ b}^{p,\ n}\ f(z)}{z^p}+\lambda_2\frac{\mathcal{J}_{s,\ b}^{p,\ n}f(z)}{z^p}\prec\phi(z)\quad(z\in\mathbb{U}). \tag{4.4}$$

Since

$$0\leqq\frac{\lambda_1}{\lambda_2}<1$$

and the function ϕ is convex and univalent in $\mathbb{U}$, we deduce from (4.1) and (4.4) that

$$\begin{aligned}
&(1-\lambda_1)\frac{\mathcal{J}_{s+1,\ b}^{p,\ n}\ f(z)}{z^p}+\lambda_1\frac{\mathcal{J}_{s,\ b}^{p,\ n}f(z)}{z^p}\\
&\quad=\frac{\lambda_1}{\lambda_2}\left[(1-\lambda_2)\frac{\mathcal{J}_{s+1,\ b}^{p,\ n}\ f(z)}{z^p}+\lambda_2\frac{\mathcal{J}_{s,\ b}^{p,\ n}f(z)}{z^p}\right]+\left(1-\frac{\lambda_1}{\lambda_2}\right)\frac{\mathcal{J}_{s+1,\ b}^{p,\ n}\ f(z)}{z^p}\\
&\quad\prec\phi(z)\quad(z\in\mathbb{U}),
\end{aligned}$$

which implies that $f\in\mathcal{K}_{s,\ b}^{p,\ n}(\lambda_1;\phi)$. The proof of Theorem 4.3 is evidently completed. □

4.4. Theorem. *Let* $f\in\mathcal{K}_{s,\ b}^{p,\ n}(\lambda;\phi)$. *If the function* $F\in\mathcal{A}_p(n)$ *is defined by* (3.6), *then*

$$\frac{\mathcal{J}_{s+1,\ b}^{p,\ n}\ F(z)}{z^p}\prec\phi(z)\quad(z\in\mathbb{U}). \tag{4.5}$$

Proof. Let $f \in \mathcal{K}_{s,\ b}^{p,\ n}(\lambda;\phi)$. Suppose also that

$$G(z) := \frac{\mathcal{J}_{s+1,\ b}^{p,\ n} F(z)}{z^p} \quad (z \in \mathbb{U}). \tag{4.6}$$

From (3.6), we deduce that

$$z\left(\mathcal{J}_{s+1,\ b}^{p,\ n} F\right)'(z) + \mu \mathcal{J}_{s+1,\ b}^{p,\ n} F(z) = (\mu+p)\mathcal{J}_{s+1,\ b}^{p,\ n} f(z). \tag{4.7}$$

Combining(4.1), (4.6) and (4.7), we have

$$G(z) + \frac{1}{\mu+p} zG'(z) = \frac{\mathcal{J}_{s+1,\ b}^{p,\ n} f(z)}{z^p} \prec \phi(z). \tag{4.8}$$

Thus, by Lemma 2.2 and (4.8), we conclude that the assertion (4.5) of Theorem 4.4 holds true. □

4.5. Theorem. *Let* $f \in \mathcal{K}_{s,\ b}^{p,\ 1}(\lambda;\phi)$ *and* $g \in \mathcal{A}_p(1)$ *with* $\Re\left(\frac{g(z)}{z^p}\right) > \frac{1}{2}$. *Then*

$$(f * g)(z) \in \mathcal{K}_{s,\ b}^{p,\ 1}(\lambda;\phi).$$

Proof. Let $f \in \mathcal{K}_{s,\ b}^{p,\ 1}(\lambda;\phi)$ and $g \in \mathcal{A}_p(1)$ with $\Re\left(\frac{g(z)}{z^p}\right) > \frac{1}{2}$. Suppose also that

$$H(z) := (1-\lambda)\frac{\mathcal{J}_{s+1,\ b}^{p,\ 1} f(z)}{z^p} + \lambda \frac{\mathcal{J}_{s,\ b}^{p,\ 1} f(z)}{z^p} \prec \phi(z). \tag{4.9}$$

It follows from (4.9) that

$$(1-\lambda)\frac{\mathcal{J}_{s+1,\ b}^{p,\ 1}(f*g)(z)}{z^p} + \lambda \frac{\mathcal{J}_{s,\ b}^{p,\ 1}(f*g)(z)}{z^p} = H(z) * \frac{g(z)}{z^p}. \tag{4.10}$$

Since the function ϕ is convex and univalent in $\mathbb{U}$, by virtue of (4.9), (4.10) and Lemma 2.5, we conclude that

$$(1-\lambda)\frac{\mathcal{J}_{s+1,\ b}^{p,\ 1}(f*g)(z)}{z^p} + \lambda \frac{\mathcal{J}_{s+1,\ b}^{p,\ 1}(f*g)(z)}{z^p} \prec \phi(z), \tag{4.11}$$

which implies that the assertion of Theorem 4.5 holds true. □

4.6. Theorem. *Let* $f \in \mathcal{K}_{s,\ b}^{p,\ 1}(\lambda;\phi)$ *and suppose that* F *is defined by* (3.6) *with* $f \in \mathcal{A}_p(1)$ *and* $\mu > -p$. *Then* $F \in \mathcal{K}_{s,\ b}^{p,\ 1}(\lambda;\phi)$.

Proof. Let $f \in \mathcal{K}_{s,\ b}^{p,\ 1}(\lambda;\phi)$ and suppose that F is defined by (3.6) with $\mu > -p$. We easily find that

$$F(z) = \frac{\mu+p}{z^\mu}\int_0^z t^{\mu-1} f(t)\,dt = (f * \mathfrak{h})(z),$$

where

$$\mathfrak{h}(z) = \frac{\mu+p}{z^\mu}\int_0^z \frac{t^{\mu+p-1}}{1-t}\,dt \in \mathcal{A}_p(1).$$

Moreover, for $\mu > -p$, we have

$$\begin{aligned} \Re\left(\frac{\mathfrak{h}(z)}{z^p}\right) &= \Re\left(\frac{\mu+p}{z^{\mu+p}}\int_0^z \frac{t^{\mu+p-1}}{1-t}dt\right) \\ &= (\mu+p)\int_0^1 u^{\mu+p-1}\Re\left(\frac{1}{1-uz}\right)du \\ &> (\mu+p)\int_0^1 \frac{u^{\mu+p-1}}{1+u}du > \frac{1}{2} \quad (z \in \mathbb{U}). \end{aligned} \tag{4.12}$$

Combining (4.12) and Theorem 4.5, we conclude that $F \in \mathcal{K}_{s,\ b}^{p,\ 1}(\lambda;\phi)$. The proof of Theorem 4.6 is thus completed. □

4.7. Theorem. *Let* $f \in \mathcal{K}_{s,\ b}^{p,\ 1}(\lambda;\phi)$ *and*

$$S_j(z) := z^p + \sum_{k=1}^{j-1} a_{p+k} z^{p+k} \quad (z \in \mathbb{U};\ j \in \mathbb{N} \setminus \{1\}). \tag{4.13}$$

Then the function W_j *defined by*

$$W_j(z) := z^{p-1} \int_0^z \frac{S_j(t)}{t^p}\, dt \quad (z \in \mathbb{U};\ j \in \mathbb{N} \setminus \{1\})$$

belongs to the class $\mathcal{K}_{s,\ b}^{p,\ 1}(\lambda;\phi)$.

Proof. Let $f \in \mathcal{K}_{s,\ b}^{p,\ 1}(\lambda;\phi)$ and let S_j be defined by (4.13). We readily get

$$W_j(z) = z^{p-1} \int_0^z \frac{S_j(t)}{t^p}\, dt = (f * g_j)(z) \quad (z \in \mathbb{U};\ j \in \mathbb{N} \setminus \{1\}),$$

where

$$g_j(z) = z^p + \sum_{k=1}^{j-1} \frac{1}{k+1} z^{p+k} \in \mathcal{A}_p(1).$$

For $j \in \mathbb{N} \setminus \{1\}$, we know from [20] that

$$\Re\left(\frac{g_j(z)}{z^p}\right) = \Re\left(1 + \sum_{k=1}^{j-1} \frac{1}{k+1} z^k\right) > \frac{1}{2}. \tag{4.14}$$

Combining (4.14) and Theorem 4.5, we deduce that $W_j \in \mathcal{K}_{s,\ b}^{p,\ 1}(\lambda;\phi)$. We thus complete the proof of Theorem 4.7. □

4.8. Theorem. *Let* $f \in \mathcal{K}_{s,\ b}^{p,\ n}(\lambda;\phi)$. *Then*

$$\frac{1}{z}\left[\left(z^p + \sum_{k=n}^{\infty} \left(\frac{p+b}{p+k+b}\right)^{s+1} z^{p+k}\right) * f(z) - z^p \phi\left(e^{i\theta}\right)\right] \neq 0 \tag{4.15}$$

$$\big(z \in \mathbb{U};\ 0 \leqq \theta < 2\pi\big).$$

Proof. Suppose that $f \in \mathcal{K}_{s,\ b}^{p,\ n}(\lambda;\phi)$. By virtue of Theorem 4.1, we know that

$$\frac{\mathcal{J}_{s+1,\ b}^{p,\ n} f(z)}{z^p} \prec \phi(z) \quad (z \in \mathbb{U}). \tag{4.16}$$

Thus, by similarly applying the method of Theorem 3.5, we easily get the convolution property (4.15) asserted by Theorem 4.8. □

4.9. Theorem. *Let* q_1 *be univalent in* $\mathbb{U}$ *and* $\Re(\lambda) > 0$. *Suppose also that* q_1 *satisfies*

$$\Re\left(1 + \frac{z q_1''(z)}{q_1'(z)}\right) > \max\left\{0,\ -\Re\left(\frac{p+b}{\lambda}\right)\right\}. \tag{4.17}$$

If $f \in \mathcal{A}_p(n)$ *satisfies the following subordination*

$$(1-\lambda)\frac{\mathcal{J}_{s+1,\ b}^{p,\ n} f(z)}{z^p} + \lambda \frac{\mathcal{J}_{s,\ b}^{p,\ n} f(z)}{z^p} \prec q_1(z) + \frac{\lambda}{p+b} z q_1'(z), \tag{4.18}$$

then

$$\frac{\mathcal{J}_{s+1,\ b}^{p,\ 1} f(z)}{z^p} \prec q_1(z),$$

and q_1 *is the best dominant.*

Proof. Let the function h be defined by (4.2). We know that (4.3) holds. Combining (4.3) and (4.18), we find that

$$h(z)+\frac{\lambda}{p+b}zh'(z)\prec q_1(z)+\frac{\lambda}{p+b}zq_1'(z). \tag{4.19}$$

By Lemma 2.4 and (4.19), we easily get the assertion of Theorem 4.9. □

Taking $q_1(z)=\frac{1+Az}{1+Bz}$ in Theorem 4.9, we get the following result.

4.10. Corollary. *Let $\Re(\lambda)>0$ and $-1\leqq B<A\leqq 1$. Suppose also that $\frac{1+Az}{1+Bz}$ satisfies the condition (4.17). If $f\in\mathcal{A}_p(n)$ satisfies the following subordination*

$$(1-\lambda)\frac{\mathfrak{J}^{p,\ n}_{s+1,\ b}f(z)}{z^p}+\lambda\frac{\mathfrak{J}^{p,\ n}_{s,\ b}f(z)}{z^p}\prec\frac{1+Az}{1+Bz}+\frac{\lambda}{p+b}\frac{(A-B)z}{(1+Bz)^2},$$

then

$$\frac{\mathfrak{J}^{p,\ n}_{s+1,\ b}f(z)}{z^p}\prec\frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant. □

We now derive the following superordination result for the class $\mathcal{K}_{p,n}(m,\lambda,l;\beta;\phi)$.

4.11. Theorem. *Let q_2 be convex univalent in $\mathbb{U}$, $\lambda\in\mathbb{C}$ with $\Re(\lambda)>0$. Also let*

$$\frac{\mathfrak{J}^{p,\ n}_{s+1,\ b}f(z)}{z^p}\in\mathcal{H}[q_2(0),1]\cap Q$$

and

$$(1-\lambda)\frac{\mathfrak{J}^{p,\ n}_{s+1,\ b}f(z)}{z^p}+\lambda\frac{\mathfrak{J}^{p,\ n}_{s,\ b}f(z)}{z^p}$$

be univalent in $\mathbb{U}$. If

$$q_2(z)+\frac{\lambda}{p+b}zq_2'(z)\prec(1-\lambda)\frac{\mathfrak{J}^{p,\ n}_{s+1,\ b}f(z)}{z^p}+\lambda\frac{\mathfrak{J}^{p,\ n}_{s,\ b}f(z)}{z^p},$$

then

$$q_2(z)\prec\frac{\mathfrak{J}^{p,\ n}_{s+1,\ b}f(z)}{z^p},$$

and q_2 is the best subordinant.

Proof. Let the function h be defined by (4.2). Then

$$q_2(z)+\frac{\lambda}{p+b}zq_2'(z)\prec(1-\lambda)\frac{\mathfrak{J}^{p,\ n}_{s+1,\ b}f(z)}{z^p}+\lambda\frac{\mathfrak{J}^{p,\ n}_{s,\ b}f(z)}{z^p}=h(z)+\frac{\lambda}{p+b}zh'(z).$$

Thus, an application of Lemma 2.3 yields the assertion of Theorem 4.11. □

Taking $q_2(z)=\frac{1+Az}{1+Bz}$ in Theorem 4.11, we get the following corollary.

4.12. Corollary. *Let q_2 be convex univalent in $\mathbb{U}$ and $-1\leqq B<A\leqq 1$, $\lambda\in\mathbb{C}$ with $\Re(\lambda)>0$. Also let*

$$\frac{\mathfrak{J}^{p,\ n}_{s+1,\ b}f(z)}{z^p}\in\mathcal{H}[q_2(0),1]\cap Q$$

and

$$(1-\lambda)\frac{\mathfrak{J}^{p,\ n}_{s+1,\ b}f(z)}{z^p}+\lambda\frac{\mathfrak{J}^{p,\ n}_{s,\ b}f(z)}{z^p}$$

be univalent in $\mathbb{U}$*. If*

$$\frac{1+Az}{1+Bz}+\frac{\lambda}{p+b}\frac{(A-B)z}{(1+Bz)^2}\prec(1-\lambda)\frac{\mathcal{J}_{s+1,\ b}^{p,\ n}f(z)}{z^p}+\lambda\frac{\mathcal{J}_{s,\ b}^{p,\ n}f(z)}{z^p},$$

then

$$\frac{1+Az}{1+Bz}\prec\frac{\mathcal{J}_{s+1,\ b}^{p,\ n}f(z)}{z^p},$$

and $\frac{1+Az}{1+Bz}$ *is the best subordinant.* □

Finally, combining the above results of subordination and superordination, we easily get the following "sandwich-type result".

4.13. Corollary. *Let* q_3 *be convex univalent and let* q_4 *be univalent in* $\mathbb{U}$*,* $\lambda\in\mathbb{C}$ *with* $\Re(\lambda)>0$*. Suppose also that* q_4 *satisfies*

$$\Re\left(1+\frac{zq_4''(z)}{q_4'(z)}\right)>\max\left\{0,\ -\Re\left(\frac{p+b}{\lambda}\right)\right\}.$$

If

$$0\neq\frac{\mathcal{J}_{s+1,\ b}^{p,\ n}f(z)}{z^p}\in\mathcal{H}[q_3(0),1]\cap Q,$$

and

$$(1-\lambda)\frac{\mathcal{J}_{s+1,\ b}^{p,\ n}f(z)}{z^p}+\lambda\frac{\mathcal{J}_{s,\ b}^{p,\ n}f(z)}{z^p}$$

is univalent in $\mathbb{U}$*, also*

$$q_3(z)+\frac{\lambda}{p+b}zq_3'(z)\prec(1-\lambda)\frac{\mathcal{J}_{s+1,\ b}^{p,\ n}f(z)}{z^p}+\lambda\frac{\mathcal{J}_{s,\ b}^{p,\ n}f(z)}{z^p}\prec q_4(z)+\frac{\lambda}{p+b}zq_4'(z),$$

then

$$q_3(z)\prec\frac{\mathcal{J}_{s+1,\ b}^{p,\ n}f(z)}{z^p}\prec q_4(z),$$

and q_3 *and* q_4 *are, respectively, the best subordinant and the best dominant.* □

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